Linear Algebra (Math 2890) Solution to Final Review Problems

1. Let $A = \begin{bmatrix} -1 & 6 & 6 \\ 3 & -8 & 3 \\ 1 & -2 & 6 \\ 1 & -4 & -3 \end{bmatrix}$.

(a) What is the column space of A?

(b) Describe the subspace $col(A)^{\perp}$ and find an basis for $col(A)^{\perp}$.

(c) Use Gram-Schmidt process to find an orthogonal basis for the column of the matrix A.

(d) Find an orthonormal basis for the column of the matrix A.

(e) Find the orthogonal projection of $y = \begin{bmatrix} -1 \\ 8 \\ -6 \\ 4 \end{bmatrix}$ onto the column space of A and write $y = \widehat{y} + z$ where $\widehat{y} \in col(A)$ and $z \in col(A)^{\perp}$. Also find the shortest distance from y to $Col(A)$.

Solution: (a) The column space is the subspace spanned by the column vectors. So $Col(A) = span\{ \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ -8 \\ -2 \\ -4 \end{bmatrix}, \begin{bmatrix} 6 \\ 3 \\ 6 \\ -3 \end{bmatrix} \}$.

(b) $col(A)^{\perp} = \{x | x \cdot y = 0 \ for \ all \ y \in col(A)\}$

$= \{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} | \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} = 0, \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \cdot \begin{bmatrix} 6 \\ -8 \\ -2 \\ -4 \end{bmatrix} = 0, \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \cdot \begin{bmatrix} 6 \\ 3 \\ 6 \\ -3 \end{bmatrix} = 0\}$

$= \{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} | -x_1 + 3x_2 + x_3 + x_4 = 0, 6x_1 - 8x_2 - 2x_3 - 4x_4 = 0, 6x_1 + 3x_2 + 6x_3 - 3x_4 = 0\}$

Consider $\left[\begin{array}{cccc|c} -1 & 3 & 1 & 1 & 0 \\ 6 & -8 & -2 & -4 & 0 \\ 6 & 3 & 6 & -3 & 0 \end{array}\right] \overset{6r_1 + r_2, 6r_1 + r_3}{\sim} \left[\begin{array}{cccc|c} -1 & 3 & 1 & 1 & 0 \\ 0 & 10 & 4 & 2 & 0 \\ 0 & 21 & 12 & 3 & 0 \end{array}\right]$

$$\overset{-\frac{21}{10}r_2+r_3}{\sim} \begin{bmatrix} -1 & 3 & 1 & 1 & 0 \\ 0 & 10 & 4 & 2 & 0 \\ 0 & 0 & \frac{18}{5} & -6/5 & 0 \end{bmatrix}$$

$$\overset{-\frac{20}{18}r_3+r_2,\,-\frac{5}{18}r_3+r_1}{\sim} \left[\begin{array}{cccc|c} -1 & 3 & 0 & 4/3 & 0 \\ 0 & 10 & 0 & 10/3 & 0 \\ 0 & 0 & \frac{18}{5} & -6/5 & 0 \end{array}\right]$$

$$\overset{-\frac{21}{10}r_2+r_3}{\sim} \left[\begin{array}{cccc|c} -1 & 3 & 1 & 1 & 0 \\ 0 & 10 & 4 & 2 & 0 \\ 0 & 0 & \frac{18}{5} & -6/5 & 0 \end{array}\right]$$

$$\overset{-\frac{3}{10}r_2+r_1}{\sim} \left[\begin{array}{cccc|c} -1 & 0 & 0 & 1/3 & 0 \\ 0 & 10 & 0 & 10/3 & 0 \\ 0 & 0 & \frac{18}{5} & -6/5 & 0 \end{array}\right]$$

$$\overset{-r_1,\,\frac{1}{10}r_2,\,\frac{5}{18}r_3}{\sim} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1/3 & 0 \\ 0 & 1 & 0 & 1/3 & 0 \\ 0 & 0 & 1 & -1/3 & 0 \end{array}\right]$$

So $x_1 - \frac{1}{3}x_4 = 0$, $x_2 + \frac{1}{3}x_4 = 0$ and $x_3 - \frac{1}{3}x_4 = 0$. This implies that $x_1 = \frac{1}{3}x_4$, $x_2 = -\frac{1}{3}x_4$, $x_3 = \frac{1}{3}x_4$ and $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \frac{1}{3}x_4 \\ -\frac{1}{3}x_4 \\ \frac{1}{3}x_4 \\ x_4 \end{bmatrix} = x_4 \begin{bmatrix} \frac{1}{3} \\ -\frac{1}{3} \\ \frac{1}{3} \\ 1 \end{bmatrix}$. Hence $col(A)^{\perp} = span\{\begin{bmatrix} \frac{1}{3} \\ -\frac{1}{3} \\ \frac{1}{3} \\ 1 \end{bmatrix}\}$ and $\{\begin{bmatrix} \frac{1}{3} \\ -\frac{1}{3} \\ \frac{1}{3} \\ 1 \end{bmatrix}\}$ is a basis for $col(A)^{\perp}$.

Let $w_1 = \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix}$, $w_2 = \begin{bmatrix} 6 \\ -8 \\ -2 \\ -4 \end{bmatrix}$ and $w_3 = \begin{bmatrix} 6 \\ 3 \\ 6 \\ -3 \end{bmatrix}$.

Gram-Schmidt process is
$v_1 = w_1$, $v_2 = w_2 - \frac{w_2 \cdot v_1}{v_1 \cdot v_1} v_1$ and $v_3 = w_3 - \frac{w_3 \cdot v_1}{v_1 \cdot v_1} v_1 - \frac{w_3 \cdot v_2}{v_2 \cdot v_2} v_2$.

So $v_1 = \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix}$. Compute $w_2 \cdot v_1 = \begin{bmatrix} 6 \\ -8 \\ -2 \\ -4 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} = -36$, $v_1 \cdot v_1 = \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} = 12$ and $v_2 = \begin{bmatrix} 6 \\ -8 \\ -2 \\ -4 \end{bmatrix} - \frac{(-36)}{12} \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix}$.

Compute $w_3 \cdot v_1 = \begin{bmatrix} 6 \\ 3 \\ 6 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} = 6$, $w_3 \cdot v_2 = \begin{bmatrix} 6 \\ 3 \\ 6 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix} = 30$, $v_2 \cdot v_2 = \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix} = 12$ and

$v_3 = w_3 - \frac{w_3 \cdot v_1}{v_1 \cdot v_1} v_1 - \frac{w_3 \cdot v_2}{v_2 \cdot v_2} v_2 = \begin{bmatrix} 6 \\ 3 \\ 6 \\ -3 \end{bmatrix} - \frac{6}{12} \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} - \frac{30}{12} \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 3 \\ -1 \end{bmatrix}$.

Hence $\{ \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 3 \\ -1 \end{bmatrix} \}$ is an orthogonal basis for $Col(A)$.

$\{ \frac{v_1}{||v_1||}, \frac{v_2}{||v_2||}, \frac{v_3}{||v_3||} \} = \{ \begin{bmatrix} -\frac{1}{\sqrt{12}} \\ \frac{3}{\sqrt{12}} \\ \frac{1}{\sqrt{12}} \\ \frac{1}{\sqrt{12}} \end{bmatrix}, \begin{bmatrix} \frac{3}{\sqrt{12}} \\ \frac{1}{\sqrt{12}} \\ \frac{1}{\sqrt{12}} \\ -\frac{1}{\sqrt{12}} \end{bmatrix}, \begin{bmatrix} -\frac{1}{\sqrt{12}} \\ -\frac{1}{\sqrt{12}} \\ \frac{3}{\sqrt{12}} \\ -\frac{1}{\sqrt{12}} \end{bmatrix} \}$ is an orthonormal basis for $Col(A)$.

(e) $y = \begin{bmatrix} -1 \\ 8 \\ -6 \\ 4 \end{bmatrix}$.

Since $\{ v_1 = \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix}, v_3 = \begin{bmatrix} -1 \\ -1 \\ 3 \\ -1 \end{bmatrix} \}$ is an orthogonal basis for $Col(A)$, $y = \widehat{y} + z$ where $\widehat{y} = \frac{y \cdot v_1}{v_1 \cdot v_1} v_1 + \frac{y \cdot v_2}{v_2 \cdot v_2} v_2 + \frac{y \cdot v_3}{v_3 \cdot v_3} v_3 \in Col(A)$ and $z = y - \widehat{y} \in Col(A)^{\perp}$. Compute $y \cdot v_1 = \begin{bmatrix} -1 \\ 8 \\ -6 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} = 1 + 24 - 6 + 4 = 23$, $v_1 \cdot v_1 = \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} = 1 + 9 + 1 + 1 = 12$, $y \cdot v_2 = \begin{bmatrix} -1 \\ 8 \\ -6 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix} = -3 + 8 - 6 - 4 = -5$, $v_2 \cdot v_2 = \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix} =$

$9+1+1+1=12$,

$y \cdot v_3 = \begin{bmatrix} -1 \\ 8 \\ -6 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ -1 \\ 3 \\ -1 \end{bmatrix} = 1-8-18-4 = -29$, $v_3 \cdot v_3 = \begin{bmatrix} -1 \\ -1 \\ 3 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ -1 \\ 3 \\ -1 \end{bmatrix} =$ $1+1+9+1=12$.

So $\widehat{y} = \frac{23}{12} \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} + \frac{(-5)}{12} \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix} + \frac{(-29)}{12} \begin{bmatrix} -1 \\ -1 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} -3/4 \\ 31/4 \\ -23/4 \\ 19/4 \end{bmatrix}$ and $z =$

$y - \widehat{y} = \begin{bmatrix} -1 \\ 8 \\ -6 \\ 4 \end{bmatrix} - \begin{bmatrix} -3/4 \\ 31/4 \\ -23/4 \\ 19/4 \end{bmatrix} = \begin{bmatrix} -1/4 \\ 1/4 \\ -1/4 \\ -3/4 \end{bmatrix}$.

The shortest distance from y to $Col(A) = ||y - \widehat{y}|| = ||z|| = \sqrt{(-1/4)^2 + (1/4)^2 + (-1/4)^2 + (-3/4)^2} = \sqrt{12/16} = \sqrt{3/4}$

2. (a) Show that the set of vectors

$$B = \left\{ u_1 = \left(-\frac{3}{5}, \frac{4}{5}, 0 \right), \ u_2 = \left(\frac{4}{5}, \frac{3}{5}, 0 \right), \ u_3 = (0,0,1) \right\}$$

is an **orthonormal basis** of $\mathbb{R}^3$.

Solution: Compute $u_1 \cdot u_2 = \left(-\frac{3}{5}, \frac{4}{5}, 0\right) \cdot \left(\frac{4}{5}, \frac{3}{5}, 0\right) = \frac{-12}{5} + \frac{12}{5} = 0$, $u_1 \cdot u_3 = \left(-\frac{3}{5}, \frac{4}{5}, 0\right) \cdot (0,0,1) = 0$, $u_2 \cdot u_3 = \left(\frac{4}{5}, \frac{3}{5}, 0\right) \cdot (0,0,1) = 0$, $u_1 \cdot u_1 = \left(-\frac{3}{5}, \frac{4}{5}, 0\right) \cdot \left(-\frac{3}{5}, \frac{4}{5}, 0\right) = \frac{9}{25} + \frac{16}{25} = 1$, $u_3 \cdot u_3 = (0,0,1) \cdot (0,0,1) = 1$, $u_2 \cdot u_2 = \left(\frac{4}{5}, \frac{3}{5}, 0\right) \cdot \left(\frac{4}{5}, \frac{3}{5}, 0\right) = \frac{16}{25} + \frac{9}{25} = 1$

(b) Find the coordinates of the vector $(1,-1,2)$ with respect to the basis in (a).

Solution: Let $y = (1,-1,2)$. So $y = \frac{y \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{y \cdot u_2}{u_2 \cdot u_2} u_2 + \frac{y \cdot u_3}{u_3 \cdot u_3} u_3 = (y \cdot u_1)u_1 + (y \cdot u_2)u_2 + (y \cdot u_3)u_3$. Compute $y \cdot u_1 = (1,-1,2) \cdot \left(-\frac{3}{5}, \frac{4}{5}, 0\right) = -\frac{3}{5} - \frac{4}{5} = -\frac{7}{5}$, $y \cdot u_2 = (1,-1,2) \cdot \left(\frac{4}{5}, \frac{3}{5}, 0\right) = \frac{4}{5} - \frac{3}{5} = \frac{1}{5}$, $y \cdot u_3 = (1,-1,2) \cdot (0,0,1) = 2$.

So the coordinate of y with respect to the basis in (a) is $(-\frac{7}{5}, \frac{1}{5}, 2)$.

3. Let $A = \begin{bmatrix} 1 & 3 & 4 & 0 \\ -3 & -6 & -7 & 2 \\ 3 & 3 & 0 & -4 \\ -5 & -3 & 2 & 9 \end{bmatrix}$

(a) Find an LU decomposition of A.

Solution: $A = \begin{bmatrix} 1 & 3 & 4 & 0 \\ -3 & -6 & -7 & 2 \\ 3 & 3 & 0 & -4 \\ -5 & -3 & 2 & 9 \end{bmatrix}$

$\widetilde{3r_1 + r_2, -3r_1 + r_2, 5r_1 + r_4} \begin{bmatrix} 1 & 3 & 4 & 0 \\ 0 & 3 & 5 & 2 \\ 0 & -6 & -12 & -4 \\ 0 & 12 & 22 & 9 \end{bmatrix}$

$\widetilde{2r_2 + r_3, -4r_2 + r_4} \begin{bmatrix} 1 & 3 & 4 & 0 \\ 0 & 3 & 5 & 2 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 2 & 1 \end{bmatrix}$

$\widetilde{r_3 + r_4} \begin{bmatrix} 1 & 3 & 4 & 0 \\ 0 & 3 & 5 & 2 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$

So $U = \begin{bmatrix} 1 & 3 & 4 & 0 \\ 0 & 3 & 5 & 2 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$

Consider the matrix $\begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 3 & -6 & -2 & 0 \\ \underbrace{-5}_{divide\ by\ 1} & \underbrace{12}_{divide\ by\ 3} & \underbrace{2}_{divide\ by\ -2} & \underbrace{1}_{divide\ by\ 1} \end{bmatrix}.$

We get $L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 \\ 3 & -2 & 1 & 0 \\ -5 & 4 & -1 & 1 \end{bmatrix}$ with $A = LU$

(b) Use LU factorization to solve $Ax = \begin{bmatrix} 1 \\ -2 \\ -1 \\ 2 \end{bmatrix}$

Solution: $Ax = b \Leftrightarrow L\underbrace{Ux}_{y} = b \Leftrightarrow Ly = b$ and $Ux = y$.

So we have to solve $Ly = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 \\ 3 & -2 & 1 & 0 \\ -5 & 4 & -1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ -1 \\ 2 \end{bmatrix}$ first, that is

$y_1 = 1$, $-3y_1 + y_2 = -2$, $3y_1 - 2y_2 + y_3 = -1$, $-5y_1 + 4y_2 - y_3 + y_4 = 2$.

Thus $y_1 = 1$, $y_2 = -2 + 3y_1 = -2 + 3 = 1$, $y_3 = -1 - 3y_1 + 2y_2 = -1 - 3 + 2 = -2$ and $y_4 = 2 + 5y_1 - 4y_2 + y_3 = 2 + 5 - 4 - 2 = 1$.

Now we solve $Ux = y$, i.e $\begin{bmatrix} 1 & 3 & 4 & 0 \\ 0 & 3 & 5 & 2 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -2 \\ 1 \end{bmatrix}$. So

$x_4 = 1$, $-2x_3 = -2$, $3x_2 + 5x_3 + 2x_4 = 1$ and $x_1 + 3x_2 + 4x_3 = 1$.

Finally, we get $x_4 = 1$, $x_3 = -2/-2 = 1$, $x_2 = (1 - 5x_3 - 2x_4)/3 = (1 - 5 - 2)/3 = -2$ and $x_1 = 1 - 3x_2 - 4x_3 = 1 - 3(-2) - 4 = 3$.

So $x = \begin{bmatrix} 3 \\ -2 \\ 1 \\ 1 \end{bmatrix}$

(c) Find the inverse matrix of A if possible.

Consider $[A|I] = \left[\begin{array}{cccc|cccc} 1 & 3 & 4 & 0 & 1 & 0 & 0 & 0 \\ -3 & -6 & -7 & 2 & 0 & 1 & 0 & 0 \\ 3 & 3 & 0 & -4 & 0 & 0 & 1 & 0 \\ -5 & -3 & 2 & 9 & 0 & 0 & 0 & 1 \end{array}\right]$

$$\underset{3r_1+r_2,\,-3r_1+r_3,\,5r_1+r_4}{\sim}\left[\begin{array}{cccc|cccc} 1 & 3 & 4 & 0 & 1 & 0 & 0 & 0 \\ 0 & 3 & 5 & 2 & 3 & 1 & 0 & 0 \\ 0 & -6 & -12 & -4 & -3 & 0 & 1 & 0 \\ 0 & 12 & 22 & 9 & 5 & 0 & 0 & 1 \end{array}\right]$$

$$\underset{2r_2+r_3,\,-4r_2+r_4}{\sim}\left[\begin{array}{cccc|cccc} 1 & 3 & 4 & 0 & 1 & 0 & 0 & 0 \\ 0 & 3 & 5 & 2 & 3 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 & 3 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 & -7 & -4 & 0 & 1 \end{array}\right]$$

$$\underset{r_3+r_4}{\sim}\left[\begin{array}{cccc|cccc} 1 & 3 & 4 & 0 & 1 & 0 & 0 & 0 \\ 0 & 3 & 5 & 2 & 3 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 & 3 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 & -4 & -2 & 1 & 1 \end{array}\right]$$

$$\underset{(-1/2)r_3}{\sim}\left[\begin{array}{cccc|cccc} 1 & 3 & 4 & 0 & 1 & 0 & 0 & 0 \\ 0 & 3 & 5 & 0 & 11 & 5 & -2 & -2 \\ 0 & 0 & 1 & 0 & -3/2 & -1 & -1/2 & 0 \\ 0 & 0 & 0 & 1 & -4 & -2 & 1 & 1 \end{array}\right]$$

$$\underset{-5r_3+r_2,\,-4r_3+r_1}{\sim}\left[\begin{array}{cccc|cccc} 1 & 3 & 0 & 0 & 7 & 4 & 2 & 0 \\ 0 & 3 & 0 & 0 & \frac{37}{2} & 10 & 1/2 & -2 \\ 0 & 0 & 1 & 0 & -3/2 & -1 & -1/2 & 0 \\ 0 & 0 & 0 & 1 & -4 & -2 & 1 & 1 \end{array}\right]$$

$$\underset{(1/3)r_2}{\sim}\left[\begin{array}{cccc|cccc} 1 & 3 & 0 & 0 & 7 & 4 & 2 & 0 \\ 0 & 1 & 0 & 0 & \frac{37}{6} & 10/3 & 1/6 & -2/3 \\ 0 & 0 & 1 & 0 & -3/2 & -1 & -1/2 & 0 \\ 0 & 0 & 0 & 1 & -4 & -2 & 1 & 1 \end{array}\right]$$

$$\underset{-3r_2+r_1}{\sim}\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -23/2 & -6 & 3/2 & 2 \\ 0 & 1 & 0 & 0 & \frac{37}{6} & 10/3 & 1/6 & -2/3 \\ 0 & 0 & 1 & 0 & -3/2 & -1 & -1/2 & 0 \\ 0 & 0 & 0 & 1 & -4 & -2 & 1 & 1 \end{array}\right].$$

So $A^{-1} = \begin{bmatrix} -23/2 & -6 & 3/2 & 2 \\ \frac{37}{6} & 10/3 & 1/6 & -2/3 \\ -3/2 & -1 & -1/2 & 0 \\ -4 & -2 & 1 & 1 \end{bmatrix}$

(d) Use the inverse of A to solve $Ax = \begin{bmatrix} 1 \\ -2 \\ -1 \\ 2 \end{bmatrix}$.

Solution: We get $x = A^{-1} \begin{bmatrix} 1 \\ -2 \\ -1 \\ 2 \end{bmatrix}$

$$= \begin{bmatrix} -23/2 & -6 & 3/2 & 2 \\ \frac{37}{6} & 10/3 & 1/6 & -2/3 \\ -3/2 & -1 & -1/2 & 0 \\ -4 & -2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 1 \\ 1 \end{bmatrix}.$$

4. Let A be the matrix

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$

Suppose the characteristic polynomial of $det(A - \lambda)$ is $(\lambda - 1)^2(\lambda - 4)$.

(a) Orthogonally diagonalizes the matrix A, giving an orthogonal matrix P and a diagonal matrix D such that $A = PDP^t$

Solution: We know that the eigenvalues are 1 ,1 and 4.

When $\lambda = 1$, $A - (1)I = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$x \in Null(A - I)$ if $x_1 + x_2 + x_3 = 0$. So $x_1 = -x_2 - x_3$ and

$x = \begin{bmatrix} -x_2 - x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$. Thus $\{w_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, w_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\}$ is a basis for $Null(A - (-1)I)$.

Now we use Gram-Schmidt process to find an orthogonal basis for $Null(A - I)$.

Let $v_1 = w_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and $v_2 = w_2 - \frac{w_2 \cdot v_1}{v_1 \cdot v_1} v_1$. Compute $w_2 \cdot v_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = 1$ and $v_1 \cdot v_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = 2$.

So $v_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - (\frac{1}{2}) \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$.

Hence $\{v_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}\}$is an orthogonal basis for $Null(A - I)$.

When $\lambda = 4$, $A - 4I = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \overset{\frown}{interchange}\ r_1 \ and\ r_2$,

$$\begin{bmatrix} 1 & -2 & 1 \\ -2 & 1 & 1 \\ 1 & 1 & -2 \end{bmatrix}$$

$$-2\ r_1 \overset{\frown}{+ r_2}, -r_1 + r_3 \begin{bmatrix} 1 & -2 & 1 \\ 0 & -3 & 3 \\ 0 & 3 & -3 \end{bmatrix}$$

$r_2 \overset{\frown}{+ r_3}, r_2/(-3) \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \overset{\frown}{2r_2 + r_1} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ $x \in Null(A - 4I)$ if $x_1 - x_3 = 0$ and $x_2 - x_3 = 0$. So $x = \begin{bmatrix} x_3 \\ x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Thus $\{v_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\}$ is a basis for $Null(A - 4I)$.

So $\{v_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}, v_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\}$ is an orthogonal basis for R^3 which are eigenvectors corresponding to $\lambda = 1$, $\lambda = 1$ and $\lambda = 4$. Compute $||v_1|| = \sqrt{2}$, $||v_2|| = \sqrt{\frac{1}{4} + \frac{1}{4} + 1} = \sqrt{\frac{6}{4}} = \sqrt{\frac{3}{2}}$ and $||v_3|| = \sqrt{3}$.

Thus $\{\frac{v_1}{||v_1||} = \begin{bmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}, \frac{v_2}{||v_2||} = \begin{bmatrix} -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{bmatrix}, \frac{v_3}{||v_3||} = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}\}$ is an orthonormal basis for R^3 which are eigenvectors corresponding to $\lambda = 1$, $\lambda = 1$ and $\lambda = 4$.

Finally, we have $A = P \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix} P^T$ where $P = [\frac{v_1}{||v_1||}\ \frac{v_2}{||v_2||}\ \frac{v_3}{||v_3||}] = \begin{bmatrix} \frac{-1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$.

(b) Find A^{10} and e^A.

So $A^1 0 = P \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4^1 0 \end{bmatrix} P^T$ and $e^A = P \begin{bmatrix} e & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & e^4 \end{bmatrix} P^T$

5. Classify the quadratic forms for the following quadratic forms. Make a change of variable $x = Py$, that transforms the quadratic form into one with no cross term. Also write the new quadratic form.

(a) $9x_1^2 - 8x_1x_2 + 3x_2^2$.
Let $Q(x_1, x_2) = 9x_1^2 - 8x_1x_2 + 3x_2^2 = x^T \begin{bmatrix} 9 & -4 \\ -4 & 3 \end{bmatrix} x$ and $A = \begin{bmatrix} 9 & -4 \\ -4 & 3 \end{bmatrix}$. We want to orthogonally diagonalizes A.
Compute $A - \lambda I = \begin{bmatrix} 9-\lambda & -4 \\ -4 & 3-\lambda \end{bmatrix}$ and $det(A - \lambda I) = (9-\lambda)(3-\lambda) - 16 = \lambda^2 - 12\lambda + 27 - 16 = \lambda^2 - 12\lambda + 11 = (\lambda - 1)(\lambda - 11)$. So $\lambda = 1$ or $\lambda = 11$. Since the eigenvalues of A are all positive, we know that the quadratic form is positive definite.

Now we diagonalize A.
$\lambda = 1$: $A - 1 \cdot I = \begin{bmatrix} 9-1 & -4 \\ -4 & 3-1 \end{bmatrix} = \begin{bmatrix} 8 & -4 \\ -4 & 2 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix}$. So $x \in Null(A - 1 \cdot I)$ iff $2x_1 - x_2 = 0$. So $x_2 = 2x_1$ and $x = \begin{bmatrix} x_1 \\ 2x_1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. So $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is an eigenvector corresponding to eigenvalue $\lambda = 1$.
$\lambda = 11$: $A - 11 \cdot I = \begin{bmatrix} 9-11 & -4 \\ -4 & 3-11 \end{bmatrix} = \begin{bmatrix} -2 & -4 \\ -4 & -8 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$. So $x \in Null(A - 11 \cdot I)$ iff $x_1 + 2x_2 = 0$. So $x_1 = -2x_2$ and $x = \begin{bmatrix} -2x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix}$. So $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ is an eigenvector corresponding to eigenvalue $\lambda = 11$.
Now $\{v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, v_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}\}$ is an orthogonal basis. Compute $||v_1|| = \sqrt{5}$ and $||v_2|| = \sqrt{5}$. Thus $\{\frac{v_1}{||v_1||} = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix}, \frac{v_2}{||v_2||} = \begin{bmatrix} \frac{-2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix}\}$ is an orthonormal basis of eigenvectors. So we have $A = Q \begin{bmatrix} 1 & 0 \\ 0 & 11 \end{bmatrix} Q^T$ where $Q = \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}$.
Now $Q(x) = x^T A x = x^T Q \begin{bmatrix} 1 & 0 \\ 0 & 11 \end{bmatrix} Q^T x = y^T \begin{bmatrix} 1 & 0 \\ 0 & 11 \end{bmatrix} y = y_1^1 + 11y_2^2$ if $y = Q^T x$. So $Qy = QQ^T x$, $x = Qy$ and $P = Q = \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}$. Note that we have used the fact that $QQ^T = I$.

(b) $-5x_1^2 + 4x_1x_2 - 2x_2^2$.
Let $Q(x_1, x_2) = -5x_1^2 + 4x_1x_2 - 2x_2^2 = x^T \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix} x$ and $A = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}$. We want to orthogonally diagonalizes A.
Compute $A - \lambda I = \begin{bmatrix} -5-\lambda & 2 \\ 2 & -2-\lambda \end{bmatrix}$ and $det(A - \lambda I) = (-5-\lambda)(-2-\lambda) - 4 = \lambda^2 + 7\lambda + 10 - 4 = \lambda^2 + 7\lambda + 6 = (\lambda + 1)(\lambda + 6)$. So $\lambda = -1$ or $\lambda = -6$. Since the eigenvalues of A are all negative, we know that the quadratic form is negative definite.
Now we diagonalize A.

$\lambda=-1$: $A-(-1)\cdot I=\begin{bmatrix} -5-(-1) & 2 \\ 2 & -2-(-1)\end{bmatrix}=\begin{bmatrix} -4 & 2 \\ 2 & -1\end{bmatrix}\frown\begin{bmatrix} 2 & -1 \\ 0 & 0\end{bmatrix}$. So $x\in Null(A-1\cdot I)$ iff $2x_1-x_2=0$. So $x_2=2x_1$ and $x=\begin{bmatrix} x_1 \\ 2x_1\end{bmatrix}=x_1\begin{bmatrix} 1 \\ 2\end{bmatrix}$. So $\begin{bmatrix} 1 \\ 2\end{bmatrix}$ is an eigenvector corresponding to eigenvalue $\lambda=-1$.

$\lambda=-6$: $A-(-6)\cdot I=\begin{bmatrix} -5-(-6) & 2 \\ 2 & (-2)-(-6)\end{bmatrix}=\begin{bmatrix} 1 & 2 \\ 2 & 4\end{bmatrix}\frown\begin{bmatrix} 1 & 2 \\ 0 & 0\end{bmatrix}$. So $x\in Null(A-11\cdot I)$ iff $x_1+2x_2=0$. So $x_1=-2x_2$ and $x=\begin{bmatrix} -2x_2 \\ x_2\end{bmatrix}=x_2\begin{bmatrix} -2 \\ 1\end{bmatrix}$. So $\begin{bmatrix} -2 \\ 1\end{bmatrix}$ is an eigenvector corresponding to eigenvalue $\lambda=-6$.

Now $\{v_1=\begin{bmatrix} 1 \\ 2\end{bmatrix}, v_2=\begin{bmatrix} -2 \\ 1\end{bmatrix}\}$ is an orthogonal basis. Compute $||v_1||=\sqrt{5}$ and $||v_2||=\sqrt{5}$. Thus $\{\frac{v_1}{||v_1||}=\begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}}\end{bmatrix}, \frac{v_2}{||v_2||}=\begin{bmatrix} \frac{-2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}}\end{bmatrix}\}$ is an orthonormal basis of eigenvectors. So we have $A=Q\begin{bmatrix} -1 & 0 \\ 0 & -6\end{bmatrix}Q^T$ where $Q=\begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}}\end{bmatrix}$.

Now $Q(x)=x^TAx=x^TQ\begin{bmatrix} -1 & 0 \\ 0 & -6\end{bmatrix}Q^Tx=y^T\begin{bmatrix} 1 & 0 \\ 0 & 11\end{bmatrix}y=-y_1^1-6y_2^2$ if $y=Q^Tx$. So $Qy=QQ^Tx$, $x=Qy$ and $P=Q=\begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}}\end{bmatrix}$.

(c) $8x_1^2+6x_1x_2$.

Let $Q(x_1,x_2)=8x_1^2+6x_1x_2=x^T\begin{bmatrix} 8 & 3 \\ 3 & 0\end{bmatrix}x$ and $A=\begin{bmatrix} 8 & 3 \\ 3 & 0\end{bmatrix}$. We want to orthogonally diagonalizes A.

Compute $A-\lambda I=\begin{bmatrix} 8-\lambda & 3 \\ 3 & 0-\lambda\end{bmatrix}$ and $det(A-\lambda I)=(8-\lambda)-\lambda-9=\lambda^2-8\lambda-9=(\lambda+1)(\lambda-9)$. So $\lambda=-1$ or $\lambda=8$. Since A has positive and negative eigenvalues, we know that the quadratic form is indefinite.

Now we diagonalize A.

$\lambda=-1$ $A-(-1)\cdot I=\begin{bmatrix} 8-(-1) & 3 \\ 3 & 0-(-1)\end{bmatrix}=\begin{bmatrix} 9 & 3 \\ 3 & 1\end{bmatrix}\frown\begin{bmatrix} 3 & 1 \\ 0 & 0\end{bmatrix}$. So $x\in Null(A-1\cdot I)$ iff $3x_1+x_2=0$. So $x_2=-3x_1$ and $x=\begin{bmatrix} x_1 \\ -3x_1\end{bmatrix}=x_1\begin{bmatrix} 1 \\ -3\end{bmatrix}$. So $\begin{bmatrix} 1 \\ -3\end{bmatrix}$ is an eigenvector corresponding to eigenvalue $\lambda=-1$.

$\lambda=9$: $A-9\cdot I=\begin{bmatrix} 8-9 & 3 \\ 3 & 0-9\end{bmatrix}=\begin{bmatrix} -1 & 3 \\ 3 & -9\end{bmatrix}\frown\begin{bmatrix} 1 & -3 \\ 0 & 0\end{bmatrix}$. So $x\in Null(A-9\cdot I)$ iff $x_1-3x_2=0$. So $x_1=3x_2$ and $x=\begin{bmatrix} 3x_2 \\ x_2\end{bmatrix}=x_2\begin{bmatrix} 3 \\ 1\end{bmatrix}$. So $\begin{bmatrix} 3 \\ 1\end{bmatrix}$ is an eigenvector corresponding to eigenvalue $\lambda=9$.

Now $\{v_1=\begin{bmatrix} 1 \\ -3\end{bmatrix}, v_2=\begin{bmatrix} 3 \\ 1\end{bmatrix}\}$ is an orthogonal basis. Compute $||v_1||=\sqrt{10}$ and $||v_2||=\sqrt{10}$. Thus $\{\frac{v_1}{||v_1||}=\begin{bmatrix} \frac{1}{\sqrt{10}} \\ \frac{-3}{\sqrt{10}}\end{bmatrix}, \frac{v_2}{||v_2||}=$

$\begin{bmatrix} \frac{3}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} \end{bmatrix}\}$ is an orthonormal basis of eigenvectors. So we have $A = Q \left[\begin{smallmatrix} -1 & 0 \\ 0 & 9 \end{smallmatrix}\right] Q^T$ where $Q = \begin{bmatrix} \frac{1}{\sqrt{10}} & \frac{-3}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \end{bmatrix}$.
Now $Q(x) = x^T Ax = x^T Q \left[\begin{smallmatrix} -1 & 0 \\ 0 & 9 \end{smallmatrix}\right] Q^T x = y^T \left[\begin{smallmatrix} -1 & 0 \\ 0 & 9 \end{smallmatrix}\right] y = -y_1^2 + 9y_2^2$ if $y = Q^T x$. So $Qy = QQ^T x$, $x = Qy$ and $P = Q = \begin{bmatrix} \frac{1}{\sqrt{10}} & \frac{-3}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \end{bmatrix}$.

6. Find an SVD of $A = \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}$. This problem is not covered. This will not be in the final exam.

7. Let $A = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 \\ 2 & -6 & 9 & -1 & 8 \\ 2 & -6 & 9 & -1 & 9 \\ -1 & 3 & -4 & 2 & -5 \end{bmatrix}$.

(a) Find a basis for the column space of A
(b) Find a basis for the nullspace of A
(c) Find the rank of the matrix A
(d) Find the dimension of the nullspace of A.

(e) Is $\begin{bmatrix} 1 \\ 4 \\ 3 \\ 1 \end{bmatrix}$ in the range of A?

(e) Does $Ax = \begin{bmatrix} 0 \\ 3 \\ 2 \\ 0 \end{bmatrix}$ have any solution? Find a solution if it's solvable.

Solution: Consider the matrix $\left[\begin{array}{ccccc|c|c} 1 & -3 & 4 & -2 & 5 & 1 & 0 \\ 2 & -6 & 9 & -1 & 8 & 4 & 3 \\ 2 & -6 & 9 & -1 & 9 & 3 & 2 \\ -1 & 3 & -4 & 2 & -5 & 1 & 0 \end{array}\right]$

$$\underset{-2r_1+r_2,\ \widetilde{-2r_1}+r_3,\ r_1+r_4}{\left[\begin{array}{ccccc|c|c} 1 & -3 & 4 & -2 & 5 & 1 & 0 \\ 0 & 0 & 1 & 3 & -2 & 2 & 3 \\ 0 & 0 & 1 & 3 & -1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 \end{array}\right]}$$

$\widetilde{-r_2+r_3}$

$$\begin{bmatrix} 1 & -3 & 4 & -2 & 5 & | & 1 & | & 0 \\ 0 & 0 & 1 & 3 & -2 & | & 2 & | & 3 \\ 0 & 0 & 0 & 0 & 1 & | & -1 & | & -1 \\ 0 & 0 & 0 & 0 & 0 & | & 2 & | & 0 \end{bmatrix}$$

$\widetilde{2r_3 + r_2, -5r_3 + r_1}$

$$\begin{bmatrix} 1 & -3 & 4 & -2 & 0 & | & 6 & | & 5 \\ 0 & 0 & 1 & 3 & 0 & | & 0 & | & 1 \\ 0 & 0 & 0 & 0 & 1 & | & -1 & | & -1 \\ 0 & 0 & 0 & 0 & 0 & | & 2 & | & 0 \end{bmatrix}$$

$\widetilde{-4r_2 + r_1}$

$$\begin{bmatrix} 1 & -3 & 0 & -14 & 0 & | & 6 & | & 1 \\ 0 & 0 & 1 & 3 & 0 & | & 0 & | & 1 \\ 0 & 0 & 0 & 0 & 1 & | & -1 & | & -1 \\ 0 & 0 & 0 & 0 & 0 & | & 2 & | & 0 \end{bmatrix}.$$

So the first, third and fifth vector forms a basis for Col(A), i.e $\{ \begin{matrix} 1 & 4 & 5 \\ 2 & 9 & 8 \\ 2 & 9 & 9 \\ -1 & -4 & -5 \end{matrix} \}$ is a basis for Col(A). The rank of A is 3 and the dimension of the null space is $5 - 3 = 2$.

$x \in Null(A)$ if $x_1 - 3x_2 - 14x_4 = 0$, $x_3 + 3x_4 = 0$ and $x_5 = 0$. So $x = \begin{bmatrix} 3x_2 + 14x_4 \\ x_2 \\ -x_4 \\ x_4 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 14 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$. Thus$\{ \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 14 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$ is a basis for $NULL(A)$.

From the result of row reduction, we can see that $Ax = \begin{bmatrix} 1 \\ 4 \\ 3 \\ 1 \end{bmatrix}$ is inconsistent (not solvable) and $\begin{bmatrix} 1 \\ 4 \\ 3 \\ 1 \end{bmatrix}$ is not in the range of A.

From the result of row reduction, we can see that $Ax = \begin{bmatrix} 0 \\ 3 \\ 2 \\ 0 \end{bmatrix}$ is solvable.

8. Determine if the columns of the matrix form a linearly independent set. Justify your answer.

$$\begin{bmatrix} 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 \\ 2 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} -4 & -3 & 0 \\ 0 & -1 & 4 \\ 1 & 0 & 3 \\ 5 & 4 & 6 \end{bmatrix}, \begin{bmatrix} -4 & -3 & 1 & 5 & 1 \\ 2 & -1 & 4 & -1 & 2 \\ 1 & 2 & 3 & 6 & -3 \\ 5 & 4 & 6 & -3 & 2 \end{bmatrix}.$$

Solution:

$$\begin{bmatrix} 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 \\ 2 & 0 & 0 & 0 \end{bmatrix} \; move \; the \; last \; \widetilde{row} \; to \; the \; first \; row \; \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

This matrix has four pivot vectors. So the columns of the matrix form a linearly independent set.

$$\begin{bmatrix} -4 & -3 & 0 \\ 0 & -1 & 4 \\ 1 & 0 & 3 \\ 5 & 4 & 6 \end{bmatrix} \quad \widetilde{interchange\ first\ and\ third\ row} \begin{bmatrix} 1 & 0 & 3 \\ 0 & -1 & 4 \\ -4 & -3 & 0 \\ 5 & 4 & 6 \end{bmatrix}$$

$$\widetilde{r_3 + 4r_1, r_4 + (-5)r_1} \begin{bmatrix} 1 & 0 & 3 \\ 0 & -1 & 4 \\ 0 & -3 & 12 \\ 0 & 4 & -9 \end{bmatrix} \quad \widetilde{(-1)r_2} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -4 \\ 0 & -3 & 12 \\ 0 & 4 & -9 \end{bmatrix}$$

$$\widetilde{r_3 + 3r_2, r_4 + (-4)r_2} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -4 \\ 0 & 0 & 0 \\ 0 & 0 & 7 \end{bmatrix} \quad \widetilde{interchange\ 3rd\ and\ 4th\ row, \frac{1}{7}r_4} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

This matrix has three pivot vectors. So the columns of the matrix form a linearly independent set.

The column vectors of

$$\begin{bmatrix} -4 & -3 & 1 & 5 & 1 \\ 2 & -1 & 4 & -1 & 2 \\ 1 & 2 & 3 & 6 & -3 \\ 5 & 4 & 6 & -3 & 2 \end{bmatrix}$$

form a dependent set since we have five column vectors in R^4.

9. Circle True or False:

T **F** The matrix $\begin{bmatrix} 3 & 5 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 4 \end{bmatrix}$ is diagonalizable

T Because $\begin{bmatrix} 3 & 5 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 4 \end{bmatrix}$ has three distinct eigenvalues.

T **F** The matrix $\begin{bmatrix} 3 & 5 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 4 \end{bmatrix}$ is orthogonally diagonalizable

F Because $\begin{bmatrix} 3 & 5 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 4 \end{bmatrix}$ is not symmetric. Recall that a matrix is orthogonally diagonalizable if and only if it's symmetric.

T **F** An orthogonal $n \times n$ matrix times an orthogonal $n \times n$ matrix is orthogona

T Suppose A and B are orthogonal. Then $AA^T = A^T A = I$,

$BB^T = B^T B = I$, $(AB) \cdot (AB)^T = ABB^T A^T = AIA^T = AA^T = I$.

Similarly, we have $(AB)^T AB = B^T A^T AB = I$. Not that we have used the

fact that $(AB)^T = B^T A^T$.

T **F** A 5×5 orthogonally diagonalizable matrix has an orthonormal set of 5 eigenvectors

T A is orthogonally diagonalizable if $A = PDP^T$. Recall that the column

vectors of P are eigenvectors and it is an orthornormal basis.

T **F** A square matrix that has the zero eigenvalue is not invertible

T A matrix A has the zero eigenvalue if there exists a nonzero vector x

such that $Ax = 0x = 0$. So $Ax = 0$ has nonzero solution and A is not

invertible.

T **F** A subspace of dimension 3 can not have a spanning set of 4 vectors

F Let $S = Span\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\}$ Then $dim(S) = 3$ and

it is spanned by 4 vectors.

T **F** A subspace of dimension 3 can not have a linearly independent set of 4 vectors

T A subspace of dimension 3 have at most three linearly independent set of

vectors

T **F** The characteristic polynomial of a 2×2 matrix is always a polynomial of degree 2

T The characteristic polynomial of a $n \times n$ matrix is always a polynomial of degree n.

T **F** If the characteristic polynomial of a matrix is $(\lambda - 4)^3(\lambda - 1)^2$ and the eigenspace associated to $\lambda = 4$ has dimension 3, than the matrix is diagonalizable

F Because the eigenspace associated to $\lambda = 4$ has dimension 3 and the

eigenspace associated to $\lambda = 1$ could have dimension 1, then we may not

have five independent eigenvectors. So the matrix is not necessarily

diagonizable.

T **F** If the characteristic polynomial of a matrix is $(\lambda - 4)^3(\lambda - 1)\lambda - 2)$ and the eigenspace associated to $\lambda = 4$ has dimension 3, than the matrix is diagonalizable

T Because the eigenspace associated to $\lambda = 4$ has dimension 3, the eigenspace

associated to $\lambda = 1$ have dimension 1 and the eigenspace associated

to $\lambda = 2$ have dimension 1, then we

may not have five independent eigenvectors. So the matrix is diagonizable.

T **F** The columns of an orthogonal matrix are orthonormal vectors

T This is true by the definition of an orthogonal matrix.

T **F** $AB = BA$ for any $n \times n$ matrices A and B

F The matrix multiplication is not necessarily commutative.

T **F** $\det(A+B) = \det A + \det B$ for any $n \times n$ matrices A and B

F This is false. For example, $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Then $A + B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$,

$det(A) = det(B) = 0$ and $det(A+B) = det(I) = 1$.

T **F** Any upper triangular matrix is always diagonalizable.

F It may not have enough eigenvectors. For example, $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is upper triangular matrix. But it has only one eigenvector. So it is not diagonizable.