**1.2 Congruence of Triangles**
**September 5, 2012**

Last time, we discussed SAS axiom.

**Axiom 1** ***(The Side, Angle, Side (SAS) Correspondence Condition)*** *If two sides and the angle included between these sides are congruent to two sides and that the included angles of the second triangle, then the triangles are congruent*

We used it to prove "The Isosceles Triangle Theorem".

**Theorem 0.1** ***The Isosceles Triangle Theorem*** *If two sides of a triangle are congruent, then the angles opposite to these sides are congruent.*

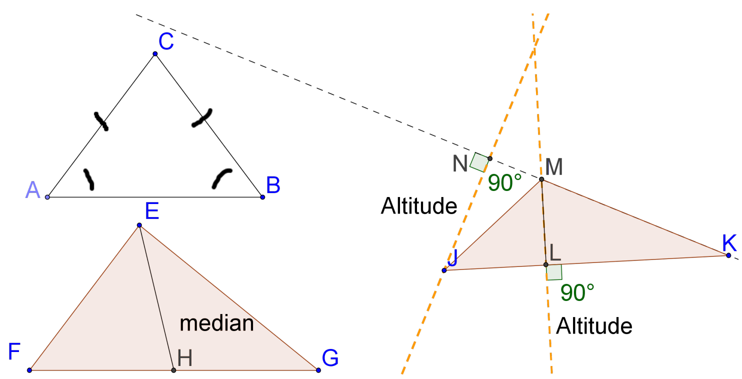

Figure 1:

Now we introduce the definition of median and altitude of a triangle.

**Definition 1** *A segment joining a vertex of a triangle to the midpoint of the opposite side is called a* ***median*** *of the triangle.*

**Definition 2** *An* ***altitude*** *for a triangle is a line through one vertex that is perpendicular to the line determined by the other two vertices.*

We will continue to use "SAS" to prove other theorems.

**Theorem 0.2** *The median to the base of an isosceles triangle is the perpendicular bisector as well as the angle bisector of the angle opposite the base.*

*Proof.* Let $\Delta ABC$ be an isosceles triangle with $\overline{AB} \cong \overline{AC}$. Let M be the midpoint of $\overline{BC}$. We want to prove that $\overline{AM}$ is the angle bisector of the angle $\angle A$ and $\overline{AM}$ is perpendicular to $\overline{BC}$. Since $\Delta ABC$ is an isosceles triangle, we have $\angle B \cong \angle C$. Because $M$ is the midpoint of $\overline{BC}$, we know that $\overline{BM} \cong \overline{CM}$. Thus $\triangle ABM \cong \triangle ACM$ by SAS ($\overline{AB} \cong \overline{AC}$, $\angle B \cong \angle C$ and $\overline{BM} \cong \overline{CM}$). This implies that $\angle MAB \cong \angle MAC$ and $\overline{AM}$ is the angle bisector of the angle $\angle A$. Also $\angle BMA \cong \angle CMA$. Since $\angle BMA$ and $\angle CMA$ form a linear pair and $\angle BMA \cong \angle CMA$, we conclude that $m(\angle BMA) + m(\angle CMA) = 180°$. and $m(\angle BMA) = m(\angle CMA) = 90°$. Thus $\overline{AM}$ is perpendicular to $\overline{BC}$. $\square$

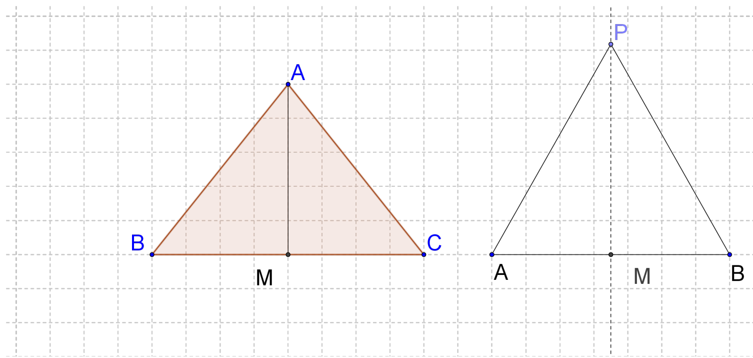


Figure 2:

**Theorem 0.3** *Every point on the perpendicular bisector of a segment is equidistant from the endpoints of the segment.*

*Proof.* Let $\overline{AB}$ be a given segment, $M$ is the midpoint of $\overline{AB}$ and the line $L$ is the perpendicular bisector of $\overline{AB}$. Let $P$ be a point on the perpendicular bisector of $AB$. We want to prove that $\overline{PA} \cong \overline{PB}$. There are two possible cases. Case 1. $P$ lies on the segment $\overline{AB}$ If this happens, then $P$ is the midpoint of $\overline{AB}$ and $\overline{PA} \cong \overline{PB}$. Case 2. $P$ doesn't lies on the segment $\overline{AB}$ If this happens, then $\triangle PAB$ is an triangle. Since $PM$ is an perpendicular bisector of $\overline{AB}$, we have $\angle PMA \cong \angle PMB = 90°$ and $\overline{AM} \cong \overline{BM}$. Using SAS ($\overline{PM} \cong \overline{PM}$, $\angle PMA \cong \angle PMB$ and $\overline{AM} \cong \overline{BM}$), we get $\triangle PAM \cong \triangle PBM$. In particular, this implies $\overline{PA} \cong \overline{PB}$ □

**Remark 1** *The proof also implies that $\overline{PM}$ is the angle bisector of $\triangle PAB$ in case 2.*

**Remark 2** *This Theorem (0.3) tells us that every point on the perpendicular bisector of a segment is equidistant from the end points. Previous Theorem (0.2) tells us that if $P$ is equidistant from $A$ and $B$, it must lie on the perpendicular bisector of $\overline{AB}$.*

Combining these two remarks, we get the following corollary.

**Corollary 0.1** *A point is equidistant from the endpoints of the segment if and only if it is on the perpendicular bisector of the segment. Equivalently, the locus of all points equidistant from the endpoints of the segment is the perpendicular bisector of the segment.*

**Remark 3** *In geometry, the set of points satisfy a certain property is often called a locus.*
*For example, the locus of all points that are equidistant (nonzero distance) from a fixed point on the plane is a circle.*
*For example, the locus of all points that are equidistant (nonzero distance) from a fixed point in $R^3$ is a sphere.*

**Corollary 0.2** *A point is equidistant from the endpoints of the segment if and only if it is on the perpendicular bisector of the segment. Equivalently, the locus of all points equidistant from the endpoints of the segment is the perpendicular bisector of the segment.*

Suppose two distinct points $P$ and $Q$ are equidistant from the endpoints of a segment $\overline{AB}$. From Corollary ).1, we know that $P$ and $Q$ are on the perpendicular bisector of the segment $\overline{AB}$. Because a line is determine by two points. Hence $\overline{PQ}$ is the perpendicular bisector of the segment $\overline{AB}$. Thus we have the following Corollary.

**Corollary 0.3** *If two distinct points are equidistant from the endpoints of the segment, then the line through these two points is the perpendicular bisector of the segment.*

**Remark 4** *This Corollary tells us that we can construct the perpendicular bisector of a segment by using ruler and compass. Use the end points of a segment and create a circle of radius which is greater than half of the length of the segment. These two circles intersect at two points. The line determine by these two points is the perpendicular bisector of the segment.*

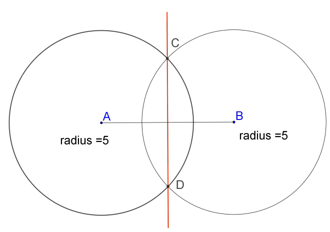


Figure 3:

Now we want to use $SAS$ prove another $ASA$ congruent theorem.

**Theorem 0.4** ***The Angle, Side, Angle (ASA) Condition***
*Given a one-to-one correspondence between the vertices of two triangles. if two angles and the included side of one triangle are congruent to the corresponding parts of the second triangle, the two triangles are congruent*

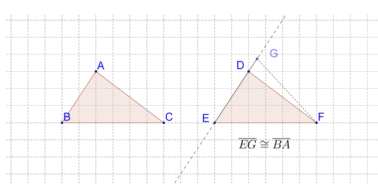


Figure 4:

*Proof.* Given two triangles $\triangle ABC$ and $\triangle DEF$. Suppose $\angle B \cong \angle E$, $\angle C \cong \angle F$ and $\overline{BC} \cong \overline{EF}$. We want to prove that $\triangle ABC \cong \triangle DEF$.

First we construct a point $G$ on the line $\overline{ED}$ such that $\overline{EG} \cong \overline{BA}$ Then $\triangle ABC \cong \triangle GEF$ by SAS ($\overline{AB} \cong \overline{GE}$, $\angle B \cong \angle E$ and $\overline{BC} \cong \overline{EF}$. This implies that $\angle BCA \cong \angle EFG$. Since $\angle BCA \cong \angle AFE$, we have $\angle EFG \cong \angle AFE$. Thus the line $FG$ and line $FD$ must overlap and $G$ must overlap with $D$. Hence $\triangle ABC \cong \triangle GEF$. $\square$