Limits of Solutions to a Parabolic Monge-Ampère Equation

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November 13, 2007

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1 Introduction

In [10], we study solutions to the affine normal flow for an initial hypersurface $\mathcal{L} \subset \mathbb{R}^{n+1}$ which is a convex, properly embedded, noncompact hypersurface. The method we used was to consider an exhausting sequence $\mathcal{L}_i$ of smooth, strictly convex, compact hypersurfaces so that each $\mathcal{L}_i$ is contained in the convex hull of $\mathcal{L}_{i+1}$ for each i, and so that $\mathcal{L}_i \to \mathcal{L}$ locally uniformly. If the compact $\mathcal{L}_i$ is the initial hypersurface, the affine normal flow $\mathcal{L}_i(t)$ is well-defined for all time t from 0 to the extinction time T_i [7]. Then for all positive t, we define the affine normal flow for initial hypersurface $\mathcal{L}$ as a limit $\mathcal{L}(t) = \lim_{i\to\infty} \mathcal{L}_i(t)$. Ben Andrews extensively studies the affine normal flow for compact initial hypersurfaces [1, 2].

The method of proof in [10] is to consider the support functions $s_{\mathcal{L}_i} = s_i$ and to take the limit as $i \to \infty$. For each $Y \in \mathbb{R}^{n+1}$, the support function is defined by

$$s(Y) = s_{\mathcal{L}}(Y) = \sup_{x\in\mathcal{L}} \langle x, Y\rangle,$$

for $\langle\cdot,\cdot\rangle$ the Euclidean inner produce on $\mathbb{R}^{n+1}$. It is immediate that s is a convex function of homogeneity one on $\mathbb{R}^{n+1}$. The homogeneity property means that it suffices to study the behavior of s when restricted to the unit sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$. Also, s restricted to an affine hyperplane not touching the origin in $\mathbb{R}^{n+1}$ determines s on a half-space of $\mathbb{R}^{n+1}$. We consider s in this setting primarily: If $Y = (y, -1)$ for $y \in \mathbb{R}^n$, then s evolves under the affine normal flow by

$$\frac{\partial s}{\partial t} = -\left(\det \frac{\partial^2 s}{\partial y^i \partial y^j}\right)^{-\frac{1}{n+2}}. \tag{1.1}$$

Note that this setting of considering the restriction of s to the hyperplane $\{y^{n+1} = 1\}$ has its roots in the Minkowski problem (see Cheng-Yau [4]).

In the present paper, we consider our previous result primarily the point of view of Equation (1.1)—in other words, from more of a classical PDE point of view as opposed to the largely tensorial point of view in [10]. Also, to the extent possible, we phrase the proofs in analytic terms, and try not to rely too much on the affine geometry. In particular, consider the support function s_i of $\mathcal{L}_i$. Then as $i \to \infty$, $s_i(Y)$ increases to the limit $s(Y)$ for all $Y \in \mathbb{R}^{n+1}$ (this follows by the exhaustion property of $\mathcal{L}_i \to \mathcal{L}$). The noncompactness of $\mathcal{L}$ implies that s is equal to $+\infty$ on at least a half-space of $\mathbb{R}^{n+1}$. Let $\mathcal{D}^\circ(s)$ be the largest open subset of $\mathbb{R}^{n+1}$ on which $s < \infty$. ($\mathcal{D}^\circ(s)$ is then the interior of the domain of s, which is defined by $\mathcal{D}(s) = \{Y : s(Y) < +\infty\}$.) Since $\mathcal{D}^\circ(s)$ is contained in an open half-space of $\mathbb{R}^{n+1}$, we may (by choosing new coordinates if necessary) restrict to the affine hyperplane $\{Y = (y, -1) : y \in \mathbb{R}^n\}$ and consider the limit $s_i \nearrow s$.

We make the following nondegeneracy assumptions about $\mathcal{L}$ and thus s. First, assume that $\mathcal{L}$ does not contain any lines. This is equivalent to

$$\mathcal{D}^\circ(s) \neq \emptyset \tag{1.2}$$

(see e.g. Rockafellar [12]). Also assume that $\mathcal{L}$ is a hypersurface, and not a lower-dimensional set. So, in particular, the convex hull $\hat{\mathcal{L}}$ has nonempty interior, and thus contains a small ball $B_\epsilon(P)$. Thus $s = s_{\mathcal{L}} = s_{\hat{\mathcal{L}}} \geq s_{B_\epsilon(P)}$, and there are $P \in \mathbb{R}^{n+1}$ and $\epsilon > 0$ so that for all $Y \in \mathbb{R}^{n+1}$,

$$s(Y) \geq \epsilon|Y| + \langle P, Y\rangle \tag{1.3}$$

For $Y = (y, -1)$, this assumption becomes that there are $\epsilon > 0$, $p \in \mathbb{R}^n$ and $c \in R$ so that for all $y \in \mathbb{R}^n$,

$$s(y) \geq \epsilon\sqrt{|y|^2 + 1} + \langle p, y\rangle - c \tag{1.4}$$

Also note that equation (1.3) may be computed using the following useful transformation law for the support function: If $A \in \mathbf{GL}(n+1,\mathbb{R})$ and $b \in \mathbb{R}^{n+1}$, then

$$s_{A\mathcal{L}+b}(Y) = s_{\mathcal{L}}(A^\top Y) + \langle b, Y\rangle. \tag{1.5}$$

This rule is particularly useful, since the affine normal flow is invariant under all affine volume-preserving maps of $\mathbb{R}^{n+1}$. Note also that (1.5) is equivalent to a projective transformation of s when restricted to $\{y^{n+1} = -1\}$.

In terms of the support function functions, we consider $s_i \nearrow s$, where the $s_i \colon \mathbb{R}^{n+1} \to \mathbb{R}$ are all convex functions of homogeneity one on $\mathbb{R}^{n+1} \setminus \{0\}$ which are smooth and strictly convex on each affine hyperplane in $\mathbb{R}^{n+1}$ which does not pass through the origin. Then the affine normal flow $s_i(t)$ may be defined by solving (1.1) on affine coordinate hyperplanes $\{y^i = \pm 1\}$ and patching together the solutions. More simply, $s_i|_{\mathbb{S}^n}$ solves a parabolic equation, and thus we have existence and uniqueness for a short time (as noted by Chow [7] originally). Then we let $s_i \to s$ pointwise everywhere in $\mathbb{R}^{n+1}$, given the nondegeneracy assumptions (1.2) and (1.4) and as well that the interior of the domain $\mathcal{D}^\circ(s)$ is contained in the half-space $\{y^{n+1} < 0\}$.

Now for the affine normal flow, $s_i(t) \nearrow s(t)$ as $i \to \infty$. On $\mathcal{D}^\circ(s)$, this is an increasing limit of smooth strictly convex functions (and so $s(t)$ is Lipschitz a priori). Our problem is then to examine which properties of the solutions $s_i(t)$ to (1.1) survive in the limit $s_i(t) \nearrow s(t)$ on $\mathcal{D}^\circ(s)$. This will determine the regularity properties of $s(t)$. In particular, there are locally uniform spacelike $C^{0,1}$ estimates on s_i on $\mathcal{D}^\circ(s)$ just by convexity. Uniform spacelike C^2 and ellipticity estimates follow by a global speed estimate of Andrews [2] which survives in the limit as $s_i \to s$ and a local Pogorelov-type estimate of Gutiérrez-Huang [8]. We also use a barrier due to Calabi [3] to ensure we can apply Gutiérrez-Huang's estimate to get locally uniform spacelike C^2 estimates on s_i for all positive t. Then Evans-Krylov theory applies to get locally uniform parabolic $C^{2+\alpha,1+\alpha/2}$ estimates and standard bootstrapping implies local C^∞ convergence of $s_i \to s$ for positive time t.

There is also an important estimate of Ben Andrews [1] on $|C|^2$ associated to s_i the support function a compact, smooth, strictly convex hypersurfaces $\mathcal{L}_i$, for a tensor C called the cubic form. This estimate shows that for any ancient solution to the affine normal flow, $|C|^2 = 0$, which implies by a classical theorem of Berwald that $\mathcal{L}$ is a quadratic hypersurface. In Section 7 below, we reproduce this classical theorem from the point of view of the support function s.

The first author is grateful for the support of the NSF under Grant DMS0405873, and to the organizers of the 2007 Conference on Geometric Analysis in Taipei, for the opportunity to speak and for their kind hospitality during the conference. The second author wishes to express his gratitude to Taida Institute for Mathematical Sciences for providing an excellent research environment and the support of C.S. Lin and Y.I. Lee. We would both like to thank D.H. Phong for his support and for his suggestion to write up our results in this way.

2 Support Function

In this section, we compute some of the basic quantities of affine differential geometry in terms of the support function s. In the end of this section, we show that (1.1) is equivalent to affine normal flow.

Let F be a smooth embedding of a strictly convex hypersurface in terms of an extended Gauss map. This means $F = F(Y)$ for any vector Y equal to a negative multiple of the inward-pointing unit normal vector ν to the image of F. So F is a function from an collection of open rays in $\mathbb{R}^{n+1} \setminus \{0\}$ to $\mathbb{R}^{n+1}$ which is homogeneous of degree 0. In particular, we have

$$s(Y) = \langle F, Y \rangle.$$

The affine normal ξ is a transverse vector field to the image of F which is invariant under the action of all volume-preserving affine maps in $\mathbb{R}^{n+1}$. We recall the basic tensors and structure equations of affine differential geometry: For each y in the domain of F, consider the basis $F_1, \ldots, F_n, \xi$ of $\mathbb{R}^{n+1}$, write the derivatives of these basis elements in terms of the same basis:

$$\begin{aligned} F_{ij} &= (\Gamma^k_{ij} + C^k_{ij})F_k + g_{ij}\xi, \\ \xi_i &= -A^j_i F_j. \end{aligned}$$

Here g_{ij} the affine metric, or affine second fundamental form, is positive definite for strictly convex hypersurfaces; Γ^k_{ij} is its Levi-Civita connection; C^k_{ij} is the cubic form; and A^j_i is the affine curvature, or affine shape operator.

Now we derive the formula for the cubic form C^k_{ij} in terms of the support function s:

Under the extended Gauss map, the inward-pointing Euclidean unit normal ν satisfies

$$\nu = -\frac{Y}{|Y|} = \frac{(-y^1, \ldots, -y^n, 1)}{\sqrt{1+|y|^2}}, \tag{2.1}$$

and the (Euclidean) second fundamental form is given by

$$h_{ij} = \frac{s_{ij}}{\sqrt{1+|y|^2}}, \qquad h^{ij} = \sqrt{1+|y|^2}s^{ij}. \tag{2.2}$$

The scalar function ϕ is defined to be $(\det h_{ij})^{\frac{1}{n+2}}(\det \bar{g}_{ij})^{-\frac{1}{n+2}}$ for $\bar{g}_{ij} = \langle F_i, F_j \rangle$ the induced metric from Euclidean $\mathbb{R}^{n+1}$. We compute (using the formula for $\bar{g}_{ij}$ below)

$$\phi = (1+|y|^2)^{-\frac{1}{2}} D^{-\frac{1}{n+2}},$$

for $D = \det s_{ij}$, and

$$\phi_k = -y^k(1+|y|^2)^{-\frac{3}{2}} D^{-\frac{1}{n+2}} - \frac{1}{n+2}(1+|y|^2)^{-\frac{1}{2}} D^{-\frac{1}{n+2}} (\ln D)_k,$$

where $(\ln D)_k = s^{pq}s_{pqk}$. We also define the vector field Z^i by

$$\begin{aligned}
&Z^i \\
=& - h^{ik}\phi_k \\
=& - \sqrt{1+|y|^2}s^{ik}[-y^k(1+|y|^2)^{-\frac{3}{2}} D^{-\frac{1}{n+2}} - \frac{1}{n+2}(1+|y|^2)^{-\frac{1}{2}} D^{-\frac{1}{n+2}} (\ln D)_k] \\
=& D^{-\frac{1}{n+2}}[(1+|y|^2)^{-1}s^{ik}y^k + \frac{1}{n+2}s^{ik}(\ln D)_k]
\end{aligned} \tag{2.3}$$

$$\begin{aligned}
F &= (s_1, \ldots, s_n, (s_l y^l) - s), \\
F_i &= (s_{1i}, \ldots, s_{ni}, s_{li}y^l)
\end{aligned} \tag{2.4}$$

$$F_{ij} = (s_{1ij}, \ldots, s_{nij}, s_{lij}y^l + s_{ij}) \tag{2.5}$$

In terms of the scalar function ϕ and the vector field Z^i defined above, we define the affine normal ξ as

$$\begin{aligned}
&\xi \\
&= \phi\nu + Z^i F_i \\
&= (1+|y|^2)^{-\frac{1}{2}} D^{-\frac{1}{n+2}} \cdot \frac{(-y^1,\ldots,-y^n,1)}{\sqrt{1+|y|^2}} \\
&+ D^{-\frac{1}{n+2}}[(1+|y|^2)^{-1} s^{ik} y^k + \frac{1}{n+2} s^{ik} (\ln D)_k](s_{1i},\ldots,s_{ni},s_{li}y^l) \\
&= D^{-\frac{1}{n+2}}(1+|y|^2)^{-1}\cdot(-y^1,\ldots,-y^n,1) \\
&+ D^{-\frac{1}{n+2}}(1+|y|^2)^{-1}(y^1,\ldots,y^n,|y|^2) \\
&+ \frac{D^{-\frac{1}{n+2}}}{n+2}((\ln D)_1,\ldots,(\ln D)_n,(\ln D)_i y^i) \\
&= \frac{1}{n+2} D^{-\frac{1}{n+2}}((\ln D)_1,\ldots,(\ln D)_n,(n+2)+(\ln D)_i y^i).
\end{aligned} \tag{2.6}$$

The affine normal ξ is invariant under volume-preserving affine actions on $\mathbb{R}^{n+1}$. The affine metric (also called the affine second fundamental form) g_{ij} is invariant under the same group, and is given by $g_{ij} = \phi^{-1} h_{ij}$. So compute

$$g_{ij} = D^{\frac{1}{n+2}} s_{ij}.$$

In terms of $s = s(y,-1) = s(Y)$, the embedding F is given by

$$F = (s_1,\ldots,s_n,(s_i y^i) - s),$$

$$g_{ij} = (\det s_{k\ell})^{\frac{1}{n+2}} s_{ij}$$

$$\partial_m g_{ij} = (\det s_{k\ell})^{\frac{1}{n+2}} (\frac{1}{n+2} s^{pq} s_{pqm} s_{ij} + s_{ijm})$$

$$g^{ij} = (\det s_{k\ell})^{-\frac{1}{n+2}} s^{ij}$$

$$
\begin{aligned}
&\Gamma^k_{ij}\\
&= \tfrac{1}{2} g^{kl}(\partial_i g_{j\ell} + \partial_j g_{i\ell} - \partial_\ell g_{ij})\\
&= \tfrac{1}{2} s^{kl}(\frac{1}{n+2} s^{mp} s_{mpi} s_{jl} + s_{jli} + \frac{1}{n+2} s^{mp} s_{mpj} s_{il} + s_{ilj} - \frac{1}{n+2} s^{mp} s_{mpl} s_{ij} - s_{ijl})\\
&= \frac{1}{2}(\frac{1}{n+2} s^{mp} s_{mpj} \delta^k_i + \frac{1}{n+2} s^{mp} s_{mpi} \delta^k_j + s^{k\ell} s_{ij\ell} - \frac{1}{n+2} s^{k\ell} s^{mp} s_{mp\ell} s_{ij})\\
&= \frac{1}{2}(\frac{1}{n+2}(\ln D)_j \delta^k_i + \frac{1}{n+2}(\ln D)_i \delta^k_j + s^{k\ell} s_{ij\ell} - \frac{1}{n+2} s^{k\ell}(\ln D)_\ell s_{ij}),
\end{aligned} \tag{2.7}
$$

where we define $D = \det s_{ij}$. Now compute the metric induced from the Euclidean metric $\bar{g}_{ij}$.

$$
\begin{aligned}
\bar{g}_{ij} &= \left\langle \frac{\partial F}{\partial y^i}, \frac{\partial F}{\partial y^j} \right\rangle\\
&= \sum_{k,l=1}^{n} \frac{\partial^2 s}{\partial y^i \partial y^k}(y^k y^l + \delta^{kl}) \frac{\partial^2 s}{\partial y^j \partial y^l},\\
\bar{g}^{ij} &= \sum_{k,l=1}^{n} s^{ik}(-\frac{y^k y^l}{1+|y|^2} + \delta_{kl}) s^{lj}, \text{where } s^{mn} \text{ is the inverse of } s_{ij},\\
\det \bar{g}_{ij} &= \det\left(\frac{\partial^2 s}{\partial y^i \partial y^k}\right) \det(y^k y^l + \delta^{kl}) \det\left(\frac{\partial^2 s}{\partial y^j \partial y^l}\right)\\
&= (1+|y|^2) \det\left(\frac{\partial^2 s}{\partial y^i \partial y^j}\right)^2.
\end{aligned}
$$

Recall that $\xi = \phi\nu + Z^k F_k$, $h_{ij} = \phi g_{ij}$ and

$$
F_{ij} = g_{ij}(\phi\nu + Z^k F_k) + (\Gamma^k_{ij} + C^k_{ij}) F_k.
$$

So

$$
\begin{aligned}
\langle F_{ij}, F_l \rangle &= g_{ij} Z^k \overline{g}_{kl} + \Gamma^k_{ij} \overline{g}_{kl} + C^k_{ij} \overline{g}_{kl},\\
\langle F_{ij}, F_l \rangle \overline{g}^{lm} &= g_{ij} Z^m + \Gamma^m_{ij} + C^m_{ij},\\
C^m_{ij} &= \langle F_{ij}, F_l \rangle \overline{g}^{lm} - g_{ij} Z^m - \Gamma^m_{ij}
\end{aligned}
$$

and, lowering the index by the affine metric $C_{ijk} = C^l_{ij} g_{lk}$,

$$
C_{ijk} = \langle F_{ij}, F_l \rangle \overline{g}^{lm} g_{mk} - g_{ij} Z^m g_{mk} - \Gamma^m_{ij} g_{mk}. \tag{2.8}
$$

First, we compute

$$
\begin{aligned}
& -g_{ij}Z^k \\
= & -(\det s_{k\ell})^{\frac{1}{n+2}} s_{ij} (\det s_{rs})^{-\frac{1}{n+2}} [(1+|y|^2)^{-1} s^{kl} y^l + \frac{1}{n+2} s^{kl} (\sum_{p,q} s^{pq} s_{pql})] \\
= & -s_{ij} [(1+|y|^2)^{-1} s^{kl} y^l + \frac{1}{n+2} s^{kl} (\ln D)_l]
\end{aligned}
$$

So

$$
\begin{aligned}
& -g_{ij}Z^l g_{lk} \\
= & -s_{ij}[(1+|y|^2)^{-1} s^{lm} y^m + \frac{1}{n+2} s^{lm} (\sum_{p,q} s^{pq} s_{pqm})] (\det s_{rs})^{\frac{1}{n+2}} s_{lk} \\
= & -s_{ij} (\det s_{rs})^{\frac{1}{n+2}} [(1+|y|^2)^{-1} y^k + \frac{1}{n+2} (\sum_{p,q} s^{pq} s_{pqk})] \\
= & -(\det s_{rs})^{\frac{1}{n+2}} [s_{ij}(1+|y|^2)^{-1} y^k + \frac{s_{ij}}{n+2} (\ln D)_k]
\end{aligned}
\tag{2.9}
$$

Now, we compute

$$
\begin{aligned}
& \overline{g}^{lm} g_{mk} \\
& = s^{lp}(-y^p y^q (1+y^2)^{-1} + \delta_{pq}) s^{qm} D^{\frac{1}{n+2}} s_{mk} \\
& = D^{\frac{1}{n+2}} (-s^{lp} y^p y^k (1+y^2)^{-1} + s^{lk})
\end{aligned}
$$

and

$$
\begin{aligned}
& \langle F_{ij}, F_l \rangle \\
= & \langle (s_{1ij}, \ldots, s_{nij}, s_{rij} y^r + s_{ij}), (s_{1k}, \ldots, s_{nk}, s_{mk} y^m) \rangle \\
= & \sum_p s_{pij} s_{pl} + s_{rij} y^r s_{ml} y^m + s_{ij} s_{ml} y^m.
\end{aligned}
$$

So

$$
\begin{aligned}
&\langle F_{ij}, F_l\rangle \overline{g}^{lm} g_{mk} \\
= &D^{\frac{1}{n+2}}(\sum_p s_{pij}s_{pl} + s_{rij}y^r s_{ml}y^m + s_{ij}s_{ml}y^m)(-s^{lq}y^q y^k(1+y^2)^{-1} + s^{lk}) \\
= &D^{\frac{1}{n+2}}(-\sum_p s_{pij}y^p y^k(1+y^2)^{-1} - s_{rij}|y|^2(1+y^2)^{-1}y^r y^k - s_{ij}|y|^2(1+y^2)^{-1}y^k \\
+ &s_{kij} + s_{rij}y^r y^k + s_{ij}y^k)) \\
= &D^{\frac{1}{n+2}}(-\sum_p s_{pij}y^p y^k - s_{ij}|y|^2(1+y^2)^{-1}y^k + s_{kij} + s_{rij}y^r y^k + s_{ij}y^k))
\end{aligned} \tag{2.10}
$$

$$
\begin{aligned}
&\Gamma_{ij}^m g_{mk} \\
= &\frac{1}{2}(\frac{1}{n+2}(\ln D)_j\delta_i^m + \frac{1}{n+2}(\ln D)_i\delta_j^m + s^{m\ell}s_{ij\ell} - \frac{1}{n+2}s^{m\ell}(\ln D)_\ell s_{ij})D^{\frac{1}{n+2}}s_{mk} \\
= &\frac{D^{\frac{1}{n+2}}}{2}(\frac{1}{n+2}(\ln D)_j s_{ik} + \frac{1}{n+2}(\ln D)_i s_{jk} + s_{ijk} - \frac{1}{n+2}(\ln D)_k s_{ij})
\end{aligned} \tag{2.11}
$$

From (2.10), (2.9), (2.11) and (2.8), we have

$$
\begin{aligned}
&C_{ijk} \\
= &D^{\frac{1}{n+2}}\Big(-\sum_p s_{pij}y^p y^k - s_{ij}|y|^2(1+y^2)^{-1}y^k \\
+ &s_{kij} + s_{rij}y^r y^k + s_{ij}y^k)\Big) - (\det s_{rs})^{\frac{1}{n+2}}[s_{ij}(1+|y|^2)^{-1}y^k + \frac{s_{ij}}{n+2}(\ln D)_k] \\
- &\frac{D^{\frac{1}{n+2}}}{2}(\frac{1}{n+2}(\ln D)_j s_{ik} + \frac{1}{n+2}(\ln D)_i s_{jk} + s_{ijk} - \frac{1}{n+2}(\ln D)_k s_{ij}) \\
= &D^{\frac{1}{n+2}}\Big[\frac{1}{2}s_{ijk} - \frac{1}{2(n+2)}s_{ki}(\ln D)_j - \frac{1}{2(n+2)}s_{kj}(\ln D)_i - \frac{s_{ij}}{2(n+2)}(\ln D)_k\Big]
\end{aligned} \tag{2.12}
$$

Now we prove that (1.1) is equivalent to the affine normal flow.

Proposition 2.1 *The affine normal flow*

$$\frac{\partial}{\partial t}F = \xi$$

is equivalent to the evolution of the support function

$$\frac{\partial s}{\partial t} = -\left(\det \frac{\partial^2 s}{\partial y^i \partial y^j}\right)^{-\frac{1}{n+2}}.$$

Proof We first compute s_t from $F_t = \xi$: Recall that $\nu = -\frac{Y}{|Y|} = \frac{(-y^1,\ldots,-y^n,1)}{\sqrt{1+|y|^2}}$ which is independent of time in our coordinate system, since $\partial_t \nu = 0$ (see [10]). Using the definition $s = \langle F, Y\rangle$, $\xi = (1+|y|^2)^{-\frac{1}{2}} D^{-\frac{1}{n+2}} \nu + Z^i F_i$ and $\partial_t Y = 0$, we have

$$\begin{aligned}
\partial_t s &= \langle \partial_t F, Y\rangle + \langle F, \partial_t Y\rangle \\
&= \left\langle \det(1+|y|^2)^{-\frac{1}{2}} D^{-\frac{1}{n+2}} \nu + Z^i F_i, -(1+|y|^2)^{\frac{1}{2}} \nu \right\rangle \\
&= -D^{-\frac{1}{n+2}}.
\end{aligned}$$

For good measure, we also compute F_t from $s_t = -D^{-\frac{1}{n+2}}$: Recall that the position function F can be expressed by the support function

$$F = (s_1, \ldots, s_n, (s_l y^l) - s).$$

Recall $D = \det(\frac{\partial^2 s}{\partial y^i \partial y^j})$. Note that

$$\begin{aligned}
s_t &= -D^{-\frac{1}{n+2}}, \\
s_{it} &= s_{ti} = (-D^{-\frac{1}{n+2}})_i \\
&= \left[\frac{1}{n+2}(\ln D)_i D^{-\frac{1}{n+2}}\right]
\end{aligned}$$

Compute

$$\begin{aligned}
\frac{\partial}{\partial t} F &= (s_{t1}, \ldots, s_{tn}, s_{tl} y^l - s_t) \\
&= \frac{1}{n+2} D^{-\frac{1}{n+2}} ((\ln D)_1, \ldots, (\ln D)_n, (\ln D)_l y^l + n + 2).
\end{aligned} \tag{2.13}$$

Recall that

$$\xi = \frac{1}{n+2} D^{-\frac{1}{n+2}} ((\ln D)_1, \ldots, (\ln D)_n, (n+2) + (\ln D)_i y^i)$$

from (2.6). Therefore $\frac{\partial}{\partial t} F = \xi$.

□

3 Andrews's Speed Estimate

In this section, we repeat, for the reader's convenience, our version of a speed estimate of Andrews [2].

Proposition 3.1 *Let s be the support function of a smooth strictly convex compact hypersurface evolving under affine normal flow. If $s(Y,t) \geq r > 0$ for all $Y \in \mathbb{S}^n$ and $t \in [0,T]$, then*

$$|\partial_t s| \leq \left(C + C' t^{-\frac{n}{2n+2}} \right) s$$

on $\mathbb{S}^n \times [0,T]$, where C and C' are constants only depending on r and n.

Proof Consider the function

$$q = \frac{-\partial_t s}{s - r/2}.$$

We apply the maximum principle to $\log q = \log |\partial_t s| - \log(s - r/2)$. In particular, at a fixed time $t \in [0,T]$, consider a point $Y \in \mathbb{S}^n$ at which q attains its maximum. By changing coordinates, we may assume that this point $Y = (0,\ldots,0,-1)$ is the south pole. Then, as in Tso [13], consider the coordinates $y = (y^1,\ldots,y^n)$ for s restricted to the hyperplane $\{(y^1,\ldots,y^n,-1)\}$. At $y = 0$, we have for $i = 1,\ldots,n$

$$(\log q)_i = 0 \quad \Longleftrightarrow \quad \frac{s_{ti}}{s_t} = \frac{s_i}{s - r/2} \tag{3.1}$$

The condition for $(\log q)|_{\mathbb{S}^n}$ to have a maximum at the south pole is

$$(\log q)_{ij} + (\log q)_{n+1}\delta_{ij} \leq 0 \tag{3.2}$$

as a symmetric matrix. Here we use subscripts to denote ordinary differentiation $f_i = \partial_{y^i} f$ and $f_t = \partial_t f$.

To compute the second term in (3.2), use Euler's identities for a function of homogeneity one

$$\sum_{i=1}^{n+1} y^i s_{ti} = s_t, \qquad \sum_{i=1}^{n+1} y^i s_i = s$$

at the point $Y = (0, \ldots, 0, -1)$ to conclude $s_{tn+1} = -s_t$, $s_{n+1} = -s$, and

$$(\log q)_{n+1} = \frac{r/2}{s - r/2}.$$

For the first term in (3.2), compute

$$\begin{aligned}(\log q)_{ij} &= \frac{s_{tij}}{s_t} - \frac{s_{ti}s_{tj}}{s_t^2} - \frac{s_{ij}}{s - r/2} + \frac{s_i s_j}{(s - r/2)^2} \\ &= \frac{s_{tij}}{s_t} - \frac{s_{ij}}{s - r/2}\end{aligned}$$

at $y = 0$ by (3.1). Thus (3.2) becomes at $y = 0$

$$\frac{r/2}{s - r/2}\delta_{ij} + \frac{s_{tij}}{s_t} - \frac{s_{ij}}{s - r/2} \leq 0. \tag{3.3}$$

Now, we compute using the flow equation (1.1)

$$\begin{aligned}(\log q)_t &= \partial_t \log|\partial_t s| - \partial_t \log(s - r/2) \\ &= -\frac{1}{n+2}\,\partial_t \log\det(s_{ij}) - \frac{s_t}{s - r/2} \\ &= -\frac{1}{n+2}\,s^{ij}s_{tij} - \frac{s_t}{s - r/2}\end{aligned}$$

for s^{ij} the inverse matrix of s_{ij}. Then (3.3) implies that

$$\begin{aligned}(\log q)_t &\leq \frac{r/2}{n+2} \cdot \frac{s_t}{s - r/2}\delta_{ij}s^{ij} - \frac{2n}{n+2} \cdot \frac{s_t}{s - r/2} \\ &= -\frac{r/2}{n+2}\,q\,\delta_{ij}s^{ij} + \frac{2n}{n+2}\,q, \\ q_t &\leq -\frac{r/2}{n+2}\,q^2\,\delta_{ij}s^{ij} + \frac{2n}{n+2}\,q^2.\end{aligned}$$

Now if we let μ_i be the eigenvalues of s^{ij}, or equivalently the reciprocals of the eigenvalues of s_{ij}, then we see

$$|s_t| = (\det s_{ij})^{-\frac{1}{n+2}} = \left(\prod_{i=1}^{n}\mu_i\right)^{\frac{1}{n+2}} \leq \left(\frac{1}{n}\sum_{i=1}^{n}\mu_i\right)^{\frac{n}{n+2}} = \left(\frac{1}{n}\,\delta_{ij}s^{ij}\right)^{\frac{n}{n+2}}$$

by the arithmetic-geometric mean inequality. Therefore,

$$\delta_{ij} s^{ij} \geq n|s_t|^{\frac{n+2}{n}} = nq^{\frac{n+2}{n}}(s - r/2)^{\frac{n+2}{n}} \geq nq^{\frac{n+2}{n}}(r/2)^{\frac{n+2}{n}}$$

since $s \geq r$. And so finally, at $y = 0$, and thus at any maximum point of $q|_{\mathbb{S}^n}$,

$$q_t \leq -\frac{n(r/2)^{\frac{2n+2}{n}}}{n+2} q^{\frac{3n+2}{n}} + \frac{2n}{n+2} q^2. \tag{3.4}$$

Now define $Q(t) = \max_{Y \in \mathbb{S}^n} q(Y, t)$. Then (3.4) implies that

$$Q_t \leq -Q^2 \left(c_n r^{\frac{2n+2}{n}} Q^{\frac{n+2}{n}} - c_n' \right)$$

for constants c_n, c_n' depending only on n. Therefore,

$$Q \leq \max \left\{ c_n r^{-\frac{2n+2}{n+2}}, c_n' r^{-1} t^{-\frac{n}{2n+2}} \right\} \tag{3.5}$$

for c_n, c_n' new constants depending only on n. The result easily follows.

Remark 1 *Q may not be differentiable as a function of t, but the above estimate (3.5) still holds—see e.g. Hamilton [9, Section 3].*

□

4 Gutiérrez-Huang's Hessian Estimate

Again, for the convenience of the reader, we reproduce our version of Gutiérrez-Huang's Pogorelov-type estimate [8] for solutions to the Monge-Ampère equation.

First we define a *bowl-shaped domain* in spacetime and its *parabolic boundary.* A set $\Omega \subset \mathbb{R}^n \times \mathbb{R}$ is bowl-shaped if there are constants $t_0 < T$ so that

$$\Omega = \bigcup_{t_0 \leq t \leq T} \Omega_t \times \{t\},$$

where each Ω_t is convex and $\Omega_{t_1} \subset \Omega_{t_2}$ whenever $t_1 < t_2$. The parabolic boundary of Ω is then $\partial\Omega \setminus (\Omega_T \times \{T\})$.

Proposition 4.1 *Let s be a smooth solution to (1.1) which is convex in y, and let Ω be a bowl-shaped domain in space-time $\mathbb{R}^n \times \mathbb{R}$ so that $s = 0$ on the parabolic boundary of Ω. Let β be any unit direction in space.*

Then at the maximum point P of the function

$$w = |s|\ \partial^2_{\beta\beta} s\ e^{\frac{1}{2}(\partial_\beta s)^2},$$

w is bounded by a constant depending on only $s(P)$, $\nabla s(P)$ and n.

Proof Choose coordinates so that $\beta = (1, 0, \ldots, 0)$ and so that at a maximum point P of w, s_{ij} is diagonal (in order to bound all second derivatives $s_{\beta\beta}$, it suffices to focus only on the eigendirections of the Hessian of s).

Since w is positive in Ω and 0 on the parabolic boundary, there is a point P outside the parabolic boundary of Ω at which w assumes its maximum value. We work with $\log w$ instead of w. Then at P,

$$(\log w)_i = 0, \qquad (\log w)_t \geq 0, \qquad (\log w)_{ij} \leq 0.$$

Here we use i, j, t subscripts for partial derivatives in y^i, y^j and t, and the last inequality is as a symmetric matrix. These equations become, at P,

$$\frac{s_i}{s} + \frac{s_{11i}}{s_{11}} + s_1 s_{1i} = 0, \tag{4.1}$$

$$\frac{s_t}{s} + \frac{s_{11t}}{s_{11}} + s_1 s_{1t} \geq 0, \tag{4.2}$$

$$\frac{s_{ij}}{s} - \frac{s_i s_j}{s^2} + \frac{s_{11ij}}{s_{11}} - \frac{s_{11i}s_{11j}}{s_{11}^2} + s_{1j}s_{1i} + s_1 s_{1ij} \leq 0. \tag{4.3}$$

To use (4.2), we compute, for $D = \det s_{ij}$,

$$\begin{aligned} s_{1t} &= \left(D^{-\frac{1}{n+2}}\right)_1 = \frac{1}{n+2}\, D^{-\frac{1}{n+2}} s^{ij} s_{ij1}, \\ s_{11t} &= D^{-\frac{1}{n+2}} \left[-\frac{1}{(n+2)^2} (s^{ij}s_{ij1})^2 - \frac{1}{n+2}\, s^{ik}s^{jl}s_{kl1}s_{ij1} + \frac{1}{n+2}\, s^{ij}s_{ij11} \right]. \end{aligned}$$

Now plug into (4.2) and divide out by $D^{-\frac{1}{n+2}}$ to find

$$\begin{aligned} \frac{1}{s_{11}} &\left[-\frac{1}{(n+2)^2}(s^{ij}s_{ij1})^2 - \frac{1}{n+2}\, s^{ik}s^{jl}s_{kl1}s_{ij1} + \frac{1}{n+2}\, s^{ij}s_{ij11} \right] \\ &\qquad - \frac{1}{s} + s_1 \left(\frac{1}{n+2}\, s^{ij}s_{ij1}\right) \geq 0 \end{aligned} \tag{4.4}$$

The last term of the first line of (4.4) leads us to contract (4.3) with the positive-definite matrix s^{ij} so that at P:

$$
\begin{aligned}
0 &\geq s^{ij}\left(\frac{s_{ij}}{s} - \frac{s_i s_j}{s^2} + \frac{s_{11ij}}{s_{11}} - \frac{s_{11i}s_{11j}}{s_{11}^2} + s_{1j}s_{1i} + s_1 s_{1ij}\right)\\
&= \frac{n}{s} - \frac{2s^{ij}s_i s_j}{s^2} + \frac{s^{ij}s_{11ij}}{s_{11}} - \frac{s^{ij}s_i s_1 s_{1j}}{s} - \frac{s^{ij}s_j s_1 s_{1i}}{s}\\
&\quad - s^{ij}s_1^2 s_{1i}s_{1j} + s^{ij}s_{1j}s_{1i} + s^{ij}s_1 s_{1ij} \qquad \text{(by (4.1))}\\
&= \frac{n}{s} - \frac{2s^{ij}s_i s_j}{s^2} + \frac{s^{ij}s_{11ij}}{s_{11}} - \frac{2s_1^2}{s} - s_1^2 s_{11} + s_{11} + s^{ij}s_1 s_{1ij}\\
&\qquad \text{(since } s_{ij} \text{ is diagonal at } P\text{)}\\
&\geq \frac{n}{s} - \frac{2s^{ij}s_i s_j}{s^2} - \frac{2s_1^2}{s} - s_1^2 s_{11} + s_{11} + s^{ij}s_1 s_{1ij} + \frac{n+2}{s}\\
&\quad - s_1 s^{ij}s_{ij1} + \frac{(s^{ij}s_{ij1})^2}{(n+2)s_{11}} + \frac{s^{ik}s^{jl}s_{kl1}s_{ij1}}{s_{11}}\\
&\qquad \text{(by (4.4))}\\
&\geq \frac{2n+2}{s} - 2\sum_{i=1}^{n}\frac{s_i^2}{s^2 s_{ii}} - \frac{2s_1^2}{s} - s_1^2 s_{11} + s_{11} + \sum_{i,j=1}^{n}\frac{s_{ij1}^2}{s_{11}s_{ii}s_{jj}}
\end{aligned}
$$

by collecting terms, completing the square, and since s_{ij} is diagonal at P. Continue computing

$$
\begin{aligned}
0 &\geq \frac{2n+2}{s} - 2\sum_{i=1}^{n}\frac{s_i^2}{s^2 s_{ii}} - \frac{2s_1^2}{s} - s_1^2 s_{11} + s_{11} + \frac{s_{111}^2}{s_{11}^3} + 2\sum_{i=2}^{n}\frac{s_{11i}^2}{s_{11}^2 s_{ii}}\\
&= \frac{2n+2}{s} - \frac{2s_1^2}{s^2 s_{11}} - \frac{2s_1^2}{s} - s_1^2 s_{11} + s_{11} + \frac{s_1^2}{s_{11}s^2} + \frac{2s_1^2}{s} + s_1^2 s_{11}
\end{aligned}
$$

by (4.1) and since s_{ij} is diagonal at P. Finally, collect terms so that

$$
0 \geq s_{11} + \frac{2n+2}{s} + \frac{1}{s_{11}}\left(-\frac{s_1^2}{s^2}\right)
$$

and multiply each side of the inequality by $s^2 s_{11} e^{s_1^2}$ to find a quadratic inequality

$$
w^2 + aw + b \leq 0
$$

for $w = |s| s_{11} e^{\frac{1}{2}s_1^2}$ at P the point in Ω at which the maximum of w is achieved. The coefficients a and b involve only n, $s(P)$ and $s_1(P)$, and so there is an upper bound of w on Ω depending on only these quantities. □

This bounds s_{ij} away from infinity, which, together with Andrews's speed estimate, shows that the ellipticity is locally uniformly controlled in the interior of appropriate bowl-shaped domains. In the next section, we use barriers essentially due to Calabi [3] to ensure that appropriate bowl-shaped domains exist, and so Gutiérrez-Huang's estimate applies.

5 Barriers

We will use two soliton solutions to the affine normal flow as inner and outer barriers. First of all, the unit sphere is a shrinking soliton, and we use its affine images, ellipsoids, as inner barriers. Since the ellipsoids are compact, their support functions are finite and smooth on all $\mathbb{R}^{n+1}$, and the usual maximum principle applies: If for an ellipsoid E, $s_E \leq s_i$ on all $\mathbb{R}^{n+1}$ (which is equivalent to the inclusion of convex hulls $\hat{E} \subset \widehat{\mathcal{L}_i}$ for $\mathcal{L}_i$ the hypersurface whose support function is s_i), then the maximum principle for parabolic equations on $\mathbb{S}^n$ shows that $s_E(t) \leq s_i(t)$ for all positive t before the extinction time of $s_E(t)$.

The outer barrier we use is an expanding soliton due to Calabi [3]. Upon taking an affine transformation, its support function $s_{\mathcal{C}}$ has $\mathcal{D}^\circ(s_{\mathcal{C}})$ an open cone over a simplex, and has the value of a linear function there. (Outside its domain, recall the support function is $+\infty$.) Moreover, under the affine normal flow, $s_{\mathcal{C}}(t)$ satisfies Dirichlet conditions on the boundary, and is continuous and finite on the closure of its domain. These properties make Calabi's example very useful as an outer barrier (as exploited by Cheng-Yau [5, 6] for the elliptic real Monge-Ampère equation).

Recall that $s_i \nearrow s$, where s_i are the support functions of strictly convex smooth compact hypersurfaces $\mathcal{L}_i$ which approach $\mathcal{L}$. On $\mathcal{D}^\circ(s)$, as $s_i \nearrow s$ uniformly on compact subsets, and since the s_i are convex, we automatically have uniform C^0 and C^1 estimates on compact subsets of $\mathcal{D}^\circ(s)$. We define $s(t) = \lim_{i\to\infty} s_i(t)$ for positive t also, and so we have locally uniform C^0 and C^1 estimates for positive t as well.

To get similar uniform local ellipticity bounds for small positive t, we need to check the hypotheses of Propositions 3.1 and 4.1 as well. For Proposition 3.1, we must ensure that $s_i(Y) \geq r$ for all large i, $t \in [0, T]$, and $Y \in \mathbb{S}^n$. The affine normal flow of a sphere provides a lower barrier to show this. In particular, we have the solution corresponding to the affine normal flow of a sphere centered at the origin. For any $r_0 > 0$, let

$$u(Y,t) = r(t)|Y|, \qquad r(t) = \left(r_0^{\frac{2n+2}{n+2}} - \frac{2n+2}{n+2}\, t \right)^{\frac{n+2}{2n+2}}. \tag{5.1}$$

Then u satisfies the affine normal flow equation for a support function. Now the nondegeneracy assumption (1.3) shows that we can use the transformation law (1.5) with A the identity matrix and $b = -P$ to show $s(Y) \geq \epsilon$ for all $Y \in \mathbb{S}^n$. Thus (5.1) and the maximum principle show that for $r = \epsilon/2$ there is a $T > 0$ so that for all $t \in [0,T]$ and $Y \in \mathbb{S}^n$, and large i, we have $s_i(Y,t) \geq r$. Thus we can apply Andrews's estimate for all time in $t \in [0,T]$.

Proposition 5.1 *Let $\mathcal{L}$ be a noncompact convex properly embedded hypersurface in $\mathbb{R}^{n+1}$ which contains no lines. Then the affine normal flow $\mathcal{L}(t)$ exists for all positive time $t > 0$.*

Proof We will phrase this in terms of the support function. Since $\mathcal{L}$ is noncompact, there is a ray $R = \{v + tw : t \geq 0\}$ contained in the convex hull $\hat{\mathcal{L}}$. We may choose coordinates so that $w = (0,1) \in \mathbb{R}^n \times \mathbb{R}$. Therefore, the support function

$$s(Y) = s_{\mathcal{L}}(Y) \geq s_R(Y) = \begin{cases} +\infty & \text{for} \quad y^{n+1} > 0 \\ \langle v, Y \rangle & \text{for} \quad y^{n+1} \leq 0 \end{cases}$$

We will use this estimate, together with the nondegeneracy assumption (1.3) to provide a lower barrier. In particular, there is an $\epsilon > 0$ so that $s(Y) = +\infty$ for $y^{n+1} > 0$ and

$$s(Y) \geq \epsilon |Y| + \langle v, Y \rangle \quad \text{for} \quad y^{n+1} \leq 0.$$

The barrier we will use is, for $y = (y^1, \ldots, y^n)$ and $Y = (y, y^{n+1})$,

$$s_{E_j}(Y) = \epsilon \sqrt{|y|^2 + (j y^{n+1})^2} + \langle v, Y \rangle + j y^{n+1}.$$

This is the support function of an ellipsoid centered at $P + (0,j)$ with n minor axes of length ϵ and one major axis of length ϵj. Clearly for all $j > 1$, $s_{E_j}(Y) \leq s(Y)$. As $j \to \infty$, the ellipsoid is equivalent, under a volume-preserving affine map, to a sphere of radius $\epsilon j^{\frac{1}{n+1}}$, which also goes to infinity. Now (5.1) shows that the extinction time of the ellipsoid under the affine normal flow goes to infinity as $j \to \infty$. Since the s_{E_j} are all lower barriers to s (which is equivalent to the ellipsoids E_j being inside the convex hull $\hat{\mathcal{L}}$), we have that the affine normal flow applied to s must exist for all time. $\square$

Now to find appropriate bowl-shaped domains to apply Proposition 4.1, we use an upper barrier due to Calabi. This barrier is first used in the real elliptic Monge-Ampère equation by Cheng-Yau [5, 6]. Calabi's example is based on the fact that the hypersurface

$$\mathcal{C}(t) = \left\{ (x^1, \ldots, x^{n+1}) \in \mathbb{R}^{n+1} : x_i > 0, \prod_{j=1}^{n+1} x^j = k > 0 \right\}$$

is an expanding soliton for the affine normal flow (which evolves by setting the parameter $k = k(t)$ for an appropriate function). At time $t = 0$, we set the hypersurface

$$\mathcal{C}(0) = \left\{ (x^1, \ldots, x^{n+1}) \in \mathbb{R}^{n+1} : x_i \geq 0, \prod_{j=1}^{n+1} x^j = 0 \right\}$$

the boundary of the first orthant in $\mathbb{R}^{n+1}$. The support function of this example is given for $c_n = (n+1)^{\frac{1}{2}} (\frac{2}{n+2})^{\frac{n+2}{2}}$:

$$s_{\mathcal{C}}(Y, t) = \begin{cases} +\infty & \text{if any } y^i > 0 \\ -(n+1) \left(c_n t^{\frac{n+2}{n}} \prod_{i=1}^{n+1} |y^i| \right)^{\frac{1}{n+1}} & \text{if all } y^i \leq 0 \end{cases} \tag{5.2}$$

Note in particular that for time $t = 0$, $s_{\mathcal{C}}(Y, 0)$ is 0 on the closed orthant on which all the $y^i \leq 0$ and is $+\infty$ elsewhere. In order to find a more flexible class of barriers, we can apply (1.5) to transform $s_{\mathcal{C}}$ by a volume-preserving affine map $\Phi \colon x \mapsto Ax + b$ to be any linear function $\langle b, Y \rangle$ on any linear image $(A^\top)^{-1}\mathcal{C}$, and $+\infty$ elsewhere. In our standard affine coordinates $Y = (y, -1)$, we find that the support function of $\mathcal{C}(0)$ can be transformed to have its domain be a simplex (this is a projective image of the first orthant in $\mathbb{R}^n$), and the value of $s_{\Phi\mathcal{C}}(0)$ is any affine function of y on this domain. The graphs of these functions will give us the flexibility to create upper barriers for the support function which ensure that the function s does move by a certain amount under the affine normal flow. This in turn gives a bowl-shaped domain in which to apply Gutiérrez-Huang's interior estimates for the Hessian of s.

Assume that the domain $\mathcal{D}^\circ(s)$ is contained in the lower half-space of $\mathbb{R}^{n+1}$. So since s has homogeneity one, s can be described by its behavior on the affine hyperplane $\mathcal{H} = \{(y, -1) : y \in \mathbb{R}^n\}$. For the remainder of this section, we consider the domain $\mathcal{D}^\circ(s)$ to be a subset of $\mathbb{R}^n$, as identified with the affine plane $\mathcal{H}$.

Each $x \in \mathcal{D}^\circ(s)$ has a convex neighborhood $\mathcal{N}$ on which $s_i \to s$ uniformly as an increasing sequence of convex functions, and so that the Lipschitz norms $\|s_i\|_{C^{0,1}}(\mathcal{N})$ are bounded by a constant C independent of i. By adding linear functions (constant in t) to the s_i, we may assume $s_i(x) = 0$ and $\nabla s_i(x) = 0$. This normalization does not affect the Monge-Ampère equation (1.1) or the Hessian of s_i (and so the C^2 estimates we derive apply to the original s_i as well). We can choose points $p_1, \dots p_{n+1}$ so that $0 \le s_i(y) \le C'$ for C' a constant independent of i and y in the convex hull $\mathcal{Q}$ of the p_j. We may also assume that x is in the interior of $\mathcal{Q}$. Now consider the simplices $\mathcal{S}_j$ to be the convex hull in $\mathbb{R}^n$ of the points

$$x, p_1, \dots, p_{j-1}, \widehat{p_j}, p_{j+1}, \dots, p_{n+1},$$

where p_j is omitted from the list. Define P_j to be an affine function on each $\mathcal{S}_j$ which is equal to C' on each of the $p_k \in \mathcal{S}_j$ and is equal to 0 at x, and define P_j to be $+\infty$ outside $\mathcal{S}_j$. Then define $P(y) = \min_j P_j(y)$. Then it is clear that P is satisfies $P_j(y) \ge P(y) \ge s_i(y)$ for all i and for all $y \in \mathbb{R}^n$.

We do not know the explicit solution to the Monge-Ampère equation (1.1) with initial value P, but all we need to show to produce uniformly large bowl-shaped domains centered at x for each of the s_i is that $P(x, t) < 0$ for positive t. This can be verified as follows: By the discussion above, P_j is the image of Calabi's example $\mathcal{C}(0)$ under an affine transformation $z \mapsto Az + b$ of $\mathbb{R}^{n+1}$. By the explicit solution (5.2) and the transformation law (1.5), we see that $P(t, y) \le P_j(t, y) < 0$ for small $t > 0$ and all y near x on the ray from x to the barycenter of $\mathcal{S}_j$. Therefore, since $P(t, y)$ is convex in y and x is in the convex hull of the barycenters of the $\mathcal{S}_j$, we have shown that $P(t, x) < 0$ for all small positive t.

By the maximum principle, each sub-level set of each s_i contains a sub-level set of P, which shows that $x \in \mathcal{D}^\circ(s)$ has a uniformly large bowl-shaped domain around it for each s_i independently of i. So Gutiérrez-Huang's Hessian estimates are uniform in every compact subset of $\mathcal{D}^\circ(s) \times (0, T]$ for small T.

By standard techniques, both Gutiérrez-Huang's and Andrews's estimates can be extended in time to be uniform in compact subsets of $\mathcal{D}^\circ(s) \times (0, \infty)$. These estimates uniformly control the spacelike C^2 norm and the ellipticity of s_i. Then the Monge-Ampère equation allows us to apply Krylov's regularity theory to get local uniform $C^{2+\alpha,1+\alpha/2}$ estimates, which can then be bootstrapped to show

Theorem 5.1 *On $\mathcal{D}^\circ(s) \times (0,\infty)$, $s_i \to s$ in the C^∞_{loc} topology.*

Also note that in [10] we use the same inner and outer barriers to show

Proposition 5.2 *Under the affine normal flow, s satisfies a Dirichlet boundary condition on $\partial\mathcal{D}(s)$.*

This proposition holds regardless of the boundary regularity—s can be infinite or finite and discontinuous on the boundary $\partial\mathcal{D}(s)$ [12]. We also use the barriers to show

Proposition 5.3 *For every $t > 0$, $F = F(y,t)$ is properly embedded as a function of y for $(y,-1) \in \mathcal{D}^\circ(s)$. In other words, as $(y,-1) \to \partial\mathcal{D}^\circ(s)$, at least one coordinate of*

$$F(y) = (s_1(y), \ldots, s_n(y), s_k(y)y^k - s(y))$$

goes to $\pm\infty$.

6 The evolution of $|C|^2$

Here we recall an estimate of Andrews [1] on the evolution of $|C|^2 = g^{il}g^{jm}g^{kp}C_{ijk}C_{lmp}$. For a compact strictly convex initial hypersurface evolving under the affine normal flow,

$$\left(\partial_t - \frac{1}{n+2}\Delta\right)|C|^2 \le -\frac{2}{n(n+2)}|C|^4.$$

Then the maximum principle shows that for all $t \in (0,T)$ for T the extinction time,

$$|C|^2 \le \frac{n(n+2)}{2t} \tag{6.1}$$

independently of initial conditions.

Since Theorem 5.1 above shows that $s_i \to s$ in C^∞_{loc} on $\mathcal{D}^\circ(s) \times (0,\infty)$, the pointwise bound (6.1) survives in the limit for any solution to the affine normal flow beginning at time $t = 0$. If the flow begins at time τ instead, then of course we have

$$|C|^2 \leq \frac{n(n+2)}{2(t-\tau)},$$

and for an ancient solution ($\tau \to -\infty$), we must have $|C|^2 = 0$. In the following section, we give a proof of the classical theorem of Berwald that says that $C_{ijk} = 0$ implies the hypersurface is quadric. Thus any ancient solution to the affine normal flow must be a quadric hypersurface. Since a hyperboloid cannot form part of an ancient solution, we have

Theorem 6.1 *Any ancient solution to the affine normal flow is a paraboloid or an ellipsoid.*

7 Quadric Hypersurfaces

Now we prove a classical theorem of Berwald, that the cubic form $C_{ijk} = 0$ implies that the hypersurface is a quadric. The first step is to show that the hypersurface is an affine sphere (i.e., that $\xi = aF + V$ for a constant scalar a and a constant vector V).

Compute for $C_{ijk} = 0$

$$s_{ijk} = \frac{1}{n+2}\Big(s_{ij}(\ln D)_k + s_{jk}(\ln D)_i + s_{ki}(\ln D)_j\Big) \tag{7.1}$$

and differentiate to find

$$\begin{aligned} &(n+2)s_{ijkl} \\ =\ &\Big(s_{ijl}(\ln D)_k + s_{ij}(\ln D)_{kl} + s_{jkl}(\ln D)_i + s_{jk}(\ln D)_{il} + s_{kil}(\ln D)_j + s_{ki}(\ln D)_{jl}\Big) \\ =\ &\frac{1}{n+2}(s_{ij}(\ln D)_l(\ln D)_k + s_{jl}(\ln D)_i(\ln D)_k + s_{li}(\ln D)_j(\ln D)_k) + s_{ij}(\ln D)_{kl} \\ +\ &\frac{1}{n+2}(s_{jk}(\ln D)_l(\ln D)_i + s_{kl}(\ln D)_j(\ln D)_i + s_{lj}(\ln D)_k(\ln D)_i) + s_{jk}(\ln D)_{il} \\ +\ &\frac{1}{n+2}(s_{ki}(\ln D)_l(\ln D)_j + s_{il}(\ln D)_k(\ln D)_j + s_{lk}(\ln D)_i(\ln D)_j) + s_{ki}(\ln D)_{jl} \end{aligned} \tag{7.2}$$

$$
\begin{aligned}
&(n+2)s_{ilkj} \\
=& \frac{1}{n+2}(s_{il}(\ln D)_j(\ln D)_k + s_{lj}(\ln D)_i(\ln D)_k + s_{ji}(\ln D)_l(\ln D)_k) + s_{il}(\ln D)_{kj} \\
+& \frac{1}{n+2}(s_{lk}(\ln D)_j(\ln D)_i + s_{kj}(\ln D)_l(\ln D)_i + s_{jl}(\ln D)_k(\ln D)_i) + s_{lk}(\ln D)_{ij} \\
+& \frac{1}{n+2}(s_{ki}(\ln D)_j(\ln D)_l + s_{ij}(\ln D)_k(\ln D)_l + s_{jk}(\ln D)_i(\ln D)_l) + s_{ki}(\ln D)_{lj}
\end{aligned}
\tag{7.3}
$$

Using $s_{ijkl} = s_{ilkj}$, we have

$$
\begin{aligned}
&s_{ij}((\ln D)_{kl} - \frac{1}{n+2}(\ln D)_k(\ln D)_l) + s_{jk}((\ln D)_{li} - \frac{1}{n+2}(\ln D)_l(\ln D)_i) \\
=\ & s_{il}((\ln D)_{kj} - \frac{1}{n+2}(\ln D)_k(\ln D)_j) + s_{lk}((\ln D)_{ij} - \frac{1}{n+2}(\ln D)_i(\ln D)_l)
\end{aligned}
\tag{7.4}
$$

Multiplying s^{ij} to previous equation, we get

$$
\begin{aligned}
&n((\ln D)_{kl} - \frac{1}{n+2}(\ln D)_k(\ln D)_l) + ((\ln D)_{lk} - \frac{1}{n+2}(\ln D)_l(\ln D)_k) \\
=\ & ((\ln D)_{kl} - \frac{1}{n+2}(\ln D)_k(\ln D)_l) + s_{lk}s^{ij}((\ln D)_{ij} - \frac{1}{n+2}(\ln D)_i(\ln D)_l)
\end{aligned}
\tag{7.5}
$$

So

$$
n((\ln D)_{kl} - \frac{1}{n+2}(\ln D)_k(\ln D)_l) = s_{lk}s^{ij}((\ln D)_{ij} - \frac{1}{n+2}(\ln D)_i(\ln D)_l)
$$

Let S be the matrix (s_{ij}) and T be the matrix with $T_{ij} = (\ln D)_{ij} - \frac{(\ln D)_i(\ln D)_j}{n+2}$. So we have $T = \frac{g^{ij}T_{ij}}{n}S$. Denote $\operatorname{tr} T = g^{ij}T_{ij}$.

From $(n+2)\xi = D^{-\frac{1}{n+2}}((\ln D)_1, \ldots, (\ln D)_n, (n+2) + (\ln D)_i y^i)$. So for ξ^i the i^{th} component of ξ,

$$
\begin{aligned}
&(n+2)\partial_j(\xi^i) \\
=\ & \partial_j(D^{-\frac{1}{n+2}}(\ln D)_i) \\
=\ & -\frac{1}{n+2}D^{-\frac{1}{n+2}}(\ln D)_j(\ln D)_i + D^{-\frac{1}{n+2}}(\ln D)_{ij} \\
=\ & D^{-\frac{1}{n+2}}T_{ij} = \frac{D^{-\frac{1}{n+2}}\operatorname{tr} T}{n}s_{ij}
\end{aligned}
\tag{7.6}
$$

for $1 \leq i \leq n$. Similarly,

$$(n+2)\partial_j(\xi^{n+1}) = D^{-\frac{1}{n+2}}((\ln D)_{ij} - \frac{1}{n+2}(\ln D)_i(\ln D)_j)y^i = D^{-\frac{1}{n+2}}T_{ij}y^i$$
$$= \frac{D^{-\frac{1}{n+2}}\operatorname{tr} T}{n} s_{ij}y^i$$

Therefore $\xi_{,i} = \frac{D^{-\frac{1}{n+2}}\operatorname{tr} T}{n}F_i$

Recall that $F_i = (s_{1i}, \ldots, s_{ni}, s_{li}y^l)$. We have $\xi_{,i} = \frac{D^{-\frac{1}{n+2}}\operatorname{tr} T}{n}F_i$. Affine curvature is defined by $\xi_{,i} = -A_i^k F_{,k}$. So $-A_i^k = \frac{D^{-\frac{1}{n+2}}\operatorname{tr} T}{n}\delta_i^k = a\delta_i^k$ where $a = \frac{D^{-\frac{1}{n+2}}\operatorname{tr} T}{n}$

Now the affine structure equations, applied to the second ordinary derivative ξ_{ij}, shows

$$\begin{aligned}
\xi_{ij} &= (aF_i)_j \\
&= a_jF_i + aF_{ij} \\
&= a_jF_i + a(g_{ij}\xi + (\Gamma_{ij}^k + C_{ij}^k)F_k \\
&= (a_j\delta_i^k + a\Gamma_{ij}^k)F_k + ag_{ij}\xi.
\end{aligned}$$

So $a_j\delta_i^k$ must be symmetric in i, j, and in particular, $a_i\delta_k^k = a_k\delta_i^k$. Since $n \geq 2$, we have $a_i = 0$ for all i. So a is constant and $\xi_k = aF_k$ implies that $\xi = aF + V$, where V is a constant vector.

So far, we have shown

Proposition 7.1 *Let $n \geq 2$. If $C_{ijk} = 0$ then $\xi = aF + V$ for V a constant vector and a a constant scalar.*

The rest of the proof of the following theorem follows Nomizu-Sasaki [11].

Theorem 7.1 *Assume $n \geq 2$. If the cubic form $C_{ijk} = 0$, then the hypersurface given by the image of F is a quadric hypersurface. In other words, there is a second-degree polynomial map $\mathcal{P}\colon \mathbb{R}^{n+1} \to \mathbb{R}$ so that $\mathcal{L}$ is an open subset of $\{\mathcal{P} = 0\}$.*

Proof Let $\mathcal{L}$ denote our hypersurface with is (locally) the image of the embedding F. For each $x = F(y) \in \mathcal{L}$, since $\{F_1, \ldots, F_n, \xi\}$ is a basis of $\mathbb{R}^{n+1}$, we can write each point $P \in \mathbb{R}^n$ uniquely as

$$P = F(y) + U_P^i(y)F_i(y) + \mu_P(y)\xi(y). \tag{7.7}$$

Then the *Lie quadric* of $\mathcal{L}$ at $x = F(y)$ is defined as the locus

$$\mathcal{F}_y = \{P \in \mathbb{R}^{n+1} : g_{ij}U^iU^j - a\mu^2 - 2\mu = 0\},$$

where a is the constant determined in Proposition 7.1 above and $g_{ij} = g_{ij}(y)$. For each y, $\mathcal{F}_y$ is clearly a quadric hypersurface in $\mathbb{R}^{n+1}$.

Now we will show that for each $x \in \mathcal{L}$, that $\mathcal{L} \subset \mathcal{F}_x$. By dimension considerations, this show that $\mathcal{L}$ is an open subset of the quadric $\mathcal{F}_x$, and we are done. Now consider y_0 for $F(y_0) = P \in \mathcal{L}$, and consider U^i and μ defined in (7.7) above as functions of y with y_0 fixed. Now differentiate (7.7) to find for $k = 1, \ldots, n$ and $U^i_k = \partial_k U^i$,

$$0 = \partial_k P = U^i_k F_i + U^i F_{ik} + \mu_k \xi + \mu \xi_k.$$

By Proposition 7.1, $\xi_k = aF_k$, and also $F_{ik} = (\Gamma^j_{ik} + C^j_{ik})F_j + g_{ik}\xi$ for C^j_{ik} the cubic form and Γ^j_{ik} the Levi-Civita connection with respect to the affine metric g_{ij}. Since we assume the cubic form is zero, we have $F_{ik} = \Gamma^j_{ik}F_j + g_{ik}\xi$. Thus

$$0 = U^i_k F_i + U^i(\Gamma^j_{ik}F_j + g_{ik}\xi) + \mu_k\xi + \mu a F_k,$$

and by splitting into the components on the basis $\{F_1, \ldots, F_n, \xi\}$, we find

$$U^j_k = -U^i\Gamma^j_{ik} - (1 + a\mu)\delta^j_k \quad \text{for } j, k = 1, \ldots, n, \tag{7.8}$$

$$\mu_k = -U^i g_{ik} \quad \text{for } k = 1, \ldots, n. \tag{7.9}$$

Now define $\Phi \colon \mathcal{L} \to \mathbb{R}$ by

$$\Phi(y) = g_{ij}U^iU^j - a\mu^2 - 2\mu = D^{\frac{1}{n+2}} s_{ij}U^iU^j - a\mu^2 - 2\mu.$$

Note $\Phi(y_0) = 0$ since by definition $U^i(y_0) = \mu(y_0) = 0$. So if we show $\Phi_k = 0$, then $\Phi(y) = 0$ for all y. By the definitions of Φ, U^i, μ, then we will have shown $y_0 \in \mathcal{F}_y$ and so $\mathcal{L} \subset \mathcal{F}_y$.

So in order to complete the proof of the theorem, we must check $\Phi_k = 0$.

So compute, using (7.8) and (7.9) above,

$$\begin{aligned}
\Phi_k &= \frac{1}{n+2} D^{\frac{1}{n+2}}(\ln D)_k s_{ij} U^i U^j + D^{\frac{1}{n+2}} s_{ijk} U^i U^j + 2D^{\frac{1}{n+2}} s_{ij} U^i_k U^j \\
&\quad - 2a\mu\mu_k - 2\mu_k \\
&= \frac{1}{n+2} D^{\frac{1}{n+2}}(\ln D)_k s_{ij} U^i U^j + D^{\frac{1}{n+2}} s_{ijk} U^i U^j + 2(a\mu+1) U^i g_{ik} \\
&\quad + 2D^{\frac{1}{n+2}} s_{ij} U^j [-U^l \Gamma^i_{lk} - (1+a\mu)\delta^i_k] \\
&= \frac{1}{n+2} D^{\frac{1}{n+2}}(\ln D)_k s_{ij} U^i U^j + D^{\frac{1}{n+2}} s_{ijk} U^i U^j - 2D^{\frac{1}{n+2}} s_{ij} U^j U^l \Gamma^i_{lk} \\
&= \frac{1}{n+2} D^{\frac{1}{n+2}}(\ln D)_k s_{ij} U^i U^j + D^{\frac{1}{n+2}} s_{ijk} U^i U^j - D^{\frac{1}{n+2}} s_{ij} \left[\frac{1}{n+2}(\ln D)_k \delta^i_l \right. \\
&\quad \left. + \frac{1}{n+2}(\ln D)_l \delta^i_k + s^{im} s_{lkm} - \frac{1}{n+2} s^{im} (\ln D)_m s_{lk} \right] \\
&= 0.
\end{aligned}$$

This completes the proof of Theorem 7.1. □

References

[1] B. Andrews. Contraction of convex hypersurfaces by their affine normal. *Journal of Differential Geometry*, 43(2):207–230, 1996.

[2] B. Andrews. Motion of hypersurfaces by Gauss curvature. *Pacific J. Math.*, 195(1):1–34, 2000.

[3] E. Calabi. Complete affine hyperspheres I. *Instituto Nazionale di Alta Matematica Symposia Mathematica*, 10:19–38, 1972.

[4] S. Y. Cheng and S. T. Yau. On the regularity of the solution of the n-dimensional Minkowski problem. *Comm. Pure Appl. Math.*, 29(5):495–516, 1976.

[5] S.-Y. Cheng and S.-T. Yau. On the regularity of the Monge-Ampère equation $\det((\partial^2 u/\partial x^i \partial x^j)) = F(x,u)$. *Communications on Pure and Applied Mathematics*, 30:41–68, 1977.

[6] S.-Y. Cheng and S.-T. Yau. On the real Monge-Ampère equation and affine flat structures. In *Proceedings of the 1980 Beijing Symposium on*

Differential Geometry and Differential Equations, Vol. 1,2,3 (Beijing, 1980), pages 339–370. Science Press, 1982.

[7] B. Chow. Deforming convex hypersurfaces by the nth root of the Gaussian curvature. *J. Differential Geom.*, 22(1):117–138, 1985.

[8] C. E. Gutiérrez and Q. Huang. A generalization of a theorem by Calabi to the parabolic Monge-Ampère equation. *Indiana Univ. Math. J.*, 47(4):1459–1480, 1998.

[9] R. S. Hamilton. Four-manifolds with positive curvature operator. *J. Differential Geom.*, 24(2):153–179, 1986.

[10] J. Loftin and M.-P. Tsui. Ancient solutions of the affine normal flow. to appear, Journal of Differential Geometry, math.DG/0602484.

[11] K. Nomizu and T. Sasaki. *Affine Differential Geometry: Geometry of Affine Immersions*. Cambridge University Press, 1994.

[12] R. T. Rockafellar. *Convex analysis*. Princeton Landmarks in Mathematics. Princeton University Press, Princeton, NJ, 1997. Reprint of the 1970 original, Princeton Paperbacks.

[13] K. Tso. Deforming a hypersurface by its Gauss-Kronecker curvature. *Comm. Pure Appl. Math.*, 38(6):867–882, 1985.