Solution to Problem Set #4

1. **(a)** (15 pts) Find parametric equations for the tangent line to the curve $r(t) = \langle t^3, 5t, t^4\rangle$ at the point $(-1, -5, 1)$.

(b) (15 pts) At what point on the curve $r(t) = \langle t^3, 5t, t^4\rangle$ is the normal plane (this is the plane that is perpendicular to the tangent line) parallel to the plane $12x + 5y + 16z = 3$?

Solution. (a) Solving $5t = -1$ (or $t^3 = -5$), we get $t = -1$. So we have $r(-1) = (-1, -5, 1)$. Taking the derivative of $r(t)$, we get $r'(t) = \langle 3t^2, 5, 4t^3\rangle$. Thus the tangent vector at $t = -1$ is $r'(-1) = \langle 3, 5, -4\rangle$. Therefore parametric equations for the tangent line is $x = -1+3t$, $y = -5 + 5t$ and $z = 1 - 4t$.

(b) The tangent vector at any time t is $r'(t) = \langle 3t^2, 5, 4t^3\rangle$. The normal vector of the normal plane is parallel to $r'(t) = \langle 3t^2, 5, 4t^3\rangle$. The normal vector of $12x+5y+16z = 3$ is $\langle 12, 5, 16\rangle$. So $\frac{12}{3t^2} = \frac{5}{5} = \frac{16}{4t^3}$. This implies that $3t^2 = 12$ and $4t^3 = 16$. So $t = \pm 2$ and $t = \pm\sqrt[3]{2}$. Thus we don't have a solution for this problem.

(Remark: The normal plane of this problem should have been $12x + 5y + 32z = 3$. Then we have $\frac{12}{3t^2} = \frac{5}{5} = \frac{32}{4t^3}$. So $3t^2 = 12$ and $4t^3 = 32$. So $t = \pm 2$ and $t = 2$. Hence $t = 2$ is a solution of $\frac{12}{3t^2} = \frac{5}{5} = \frac{32}{4t^3}$.

The points that we want to find is $r(2) = \langle 8, 10, 16\rangle$ and $r(-2) = \langle -8, -10, 16\rangle$.)

□

2. (25 pts, 10 for unit normal, 10 for unit tangent, 5 for curvature) Find the unit tangent T, unit normal N and unit binormal vectors B for the curve $r(t) = \langle\cos(2t), 2t, \sin(2t)\rangle$. Then calculate the curvature.

Solution. Given $r(t) = \langle\cos(2t), 2t, \sin(2t)\rangle$, we have $r'(t) = \langle -2\sin(2t), 2, 2\cos(2t)\rangle$ and $|r'(t)| = \sqrt{4\sin^2(2t) + 4 + 4\cos^2(2t)} = \sqrt{8}$. So the unit tangent vector is $T(t) = \frac{r'(t)}{|r'(t)|} = \frac{1}{\sqrt{8}}\langle -2\sin(2t), 2, 2\cos(2t)\rangle = \langle -\frac{\sin(2t)}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{\cos(2t)}{\sqrt{2}}\rangle$.

Now $T'(t) = \frac{1}{\sqrt{2}}\langle -2\cos(2t), 0, -2\sin(2t)\rangle$ and $|T'(t)| = \sqrt{2}$. So the unit normal vector is $N(t) = \frac{T'(t)}{|T'(t)|} = \langle -\cos(2t), 0, -\sin(2t)\rangle$.

The binormal vector is
$B(t) = T(t) \times N(t) = \langle -\cos(2t), 0, -\sin(2t)\rangle$

$$B(t) = T(t) \times N(t) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\frac{\sin(2t)}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{\cos(2t)}{\sqrt{2}} \\ -\cos(2t) & 0 & -\sin(2t) \end{vmatrix}$$
$$= \begin{vmatrix} \frac{1}{\sqrt{2}} & \frac{\cos(2t)}{\sqrt{2}} \\ 0 & -\sin(2t) \end{vmatrix} \vec{i} - \begin{vmatrix} -\frac{\sin(2t)}{\sqrt{2}} & \frac{\cos(2t)}{\sqrt{2}} \\ -\cos(2t) & -\sin(2t) \end{vmatrix} \vec{j} + \begin{vmatrix} -\frac{\sin(2t)}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\cos(2t) & 0 \end{vmatrix} \vec{k}$$
$$= \langle -\frac{\sin(2t)}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, \frac{\cos(2t)}{\sqrt{2}}\rangle.$$

The curvature $k(t) = \frac{|T'(t)|}{|r'(t)|} = \frac{\sqrt{2}}{\sqrt{8}} = \frac{1}{2}$.

□

3. (15 pts) Find the arc-length of the curve $r(t) = \langle t^2, \ln(t), 2t\rangle$ when $1 \leq t \leq 2$.

Solution. Given $r(t) = \langle t^2, \ln(t), 2t\rangle$, we have $r'(t) = \langle 2t, \frac{1}{t}, 2\rangle$ and $|r'(t)| = \sqrt{4t^2 + \frac{1}{t^2} + 4} == \sqrt{(2t + \frac{1}{t})^2} = 2t + \frac{1}{t}$. Hence the arc-length of the curve $r(t) = \langle t^2, \ln(t), 2t\rangle$ between $1 \leq t \leq 2$ is $\int_1^2 |r'(t)|dt = \int_1^2 (2t + \frac{1}{t})dt = t^2 + \ln(t)|_1^2 = 4 + \ln(2) - (1 + \ln(1)) = 3 + \ln(2)$.

□

4. (30 pts, 10 for each) Find the domain of the following functions and sketch the level curves of the following functions for the listed k values.

4.(a) $f(x, y) = \frac{x^2-y^2}{x^2+y^2}$. $k = 0,\ 1,\ 2,\ 3.$

Solution. The domain of f is $\{(x, y)|x^2 + y^2 \neq 0\} = \{(x, y)|(x.y) \neq (0, 0)\}$.

The level curve for $k = 0$ is determined by $f(x, y) = 0$, i.e. $\frac{x^2-y^2}{x^2+y^2} = 0$. This is the same as $x^2 - y^2 = 0$, so $x = y$ or $x = -y$. But $(0, 0)$ is not in the domain. Therefore the level curve for $k = 0$ looks like the following graph.

The level curve for $k = 1$ is determined by $f(x, y) = 1$, i.e. $\frac{x^2-y^2}{x^2+y^2} = 1$. This is the same as $x^2 - y^2 = x^2 + y^2$, so $2y^2 = 0$ which is $y = 0$. But $(0, 0)$ is not in the domain. Therefore the level curve for $k = 1$ looks like the following graph.

The level curve for $k = 2$ is determined by $f(x, y) = 2$, i.e. $\frac{x^2-y^2}{x^2+y^2} = 2$. This is the same as $x^2 - y^2 = 2x^2 + 2y^2$, so $x^2 + 3y^2 = 0$ which is $(x, y) =$

$(0,0)$. But $(0,0)$ is not in the domain. Therefore the level curve for $k = 2$ is a empty set.

The level curve for $k = 3$ is determined by $f(x,y) = 3$, i.e. $\frac{x^2-y^2}{x^2+y^2} = 3$. This is the same as $x^2 - y^2 = 3x^2 + 3y^2$, so $2x^2 + 4y^2 = 0$ which is $(x,y) = (0,0)$. But $(0,0)$ is not in the domain. Therefore the level curve for $k = 3$ is a empty set.

□

4.(b) $g(x,y) = \frac{1}{1+x^2+y^2}$. $k = 0,\ 1,\ \frac{1}{2},\ \frac{1}{5}$.

Solution. The domain of g is $\{(x,y)|1 + x^2 + y^2 \neq 0\} = \{(x,y)|(x.y) \in R^2\}$.

The level curve for $k = 0$ is determined by $g(x,y) = 0$, i.e. $\frac{1}{1+x^2+y^2} = 0$ which has no solution. Therefore the level curve for $k = 0$ is a empty set.

The level curve for $k = 1$ is determined by $g(x,y) = 1$, i.e. $\frac{1}{1+x^2+y^2} = 1$ or $x^2 + y^2 = 0$. Thus $(x,y) = (0,0)$.

The level curve for $k = \frac{1}{2}$ is determined by $g(x,y) = \frac{1}{2}$, i.e. $\frac{1}{1+x^2+y^2} = \frac{1}{2}$ or $x^2 + y^2 = 1$.

The level curve for $k = \frac{1}{5}$ is determined by $g(x,y) = \frac{1}{5}$, i.e. $\frac{1}{1+x^2+y^2} = \frac{1}{5}$ or $x^2 + y^2 = 4$.

□

4.(c) $h(x, y) = \sqrt{x^2 - y^2}$. $k = 0,\ 1,\ 2,\ 3.$

Solution. The domain of h is $\{(x, y)|x^2 - y^2 \geq 0\} = \{(x, y)|x^2 \geq y^2\}$.

The level curve for $k = 0$ is determined by $h(x, y) = 0$, i.e. $\sqrt{x^2 - y^2} = 0$ or $x = y$ or $x = -y$.

The level curve for $k = 1$ is determined by $h(x, y) = 1$, i.e. $\sqrt{x^2 - y^2} = 1$ or $x^2 - y^2 = 1$.

The level curve for $k = 2$ is determined by $h(x, y) = 2$, i.e. $\sqrt{x^2 - y^2} = 2$ or $x^2 - y^2 = 4$.

The level curve for $k = 3$ is determined by $h(x, y) = 3$, i.e. $\sqrt{x^2 - y^2} = 3$ or $x^2 - y^2 = 3$.

□