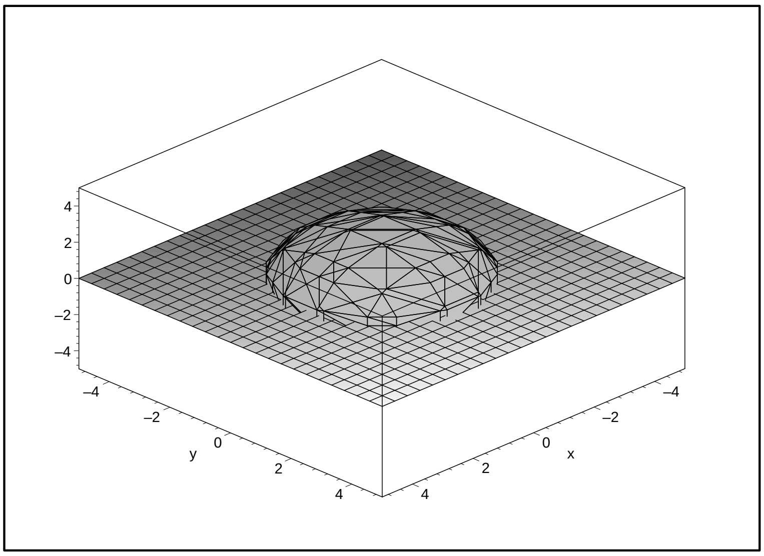
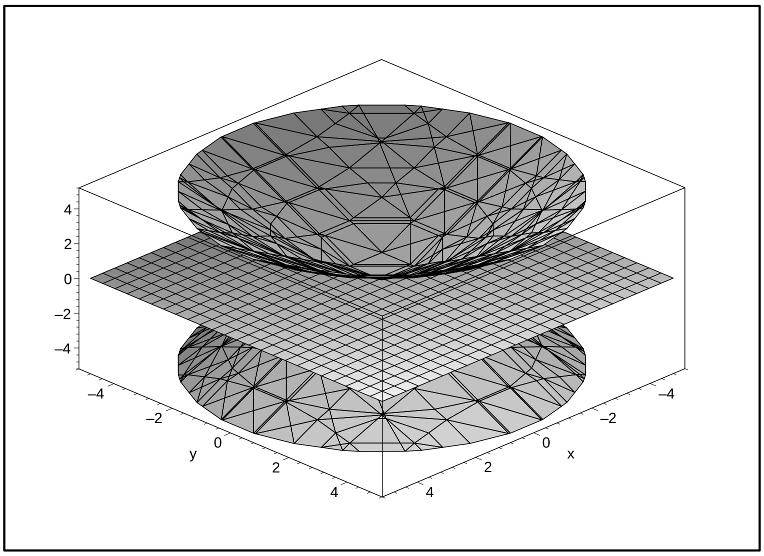
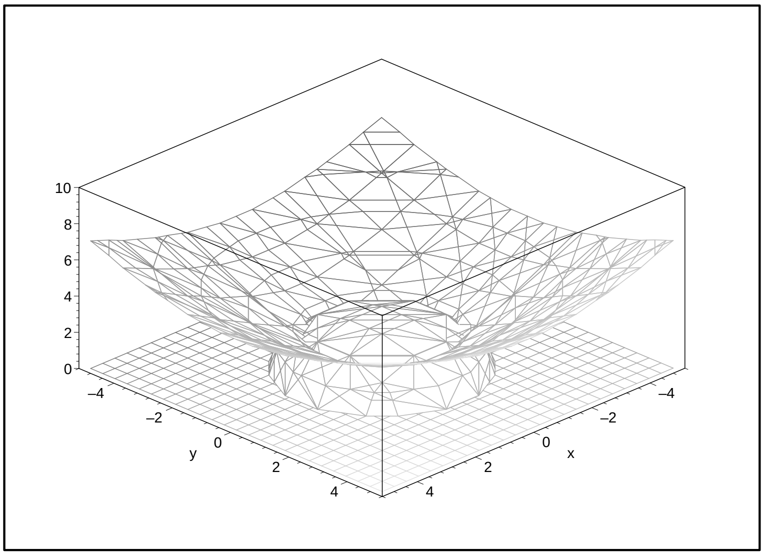
Solution to Problem Set #8

1. (20 pt) Find the volume of an ice cream cone bounded by the hemisphere
$z = \sqrt{8 - x^2 - y^2}$ and the cone $z = \sqrt{x^2 + y^2}$. The graphs above are the graphs of $z = \sqrt{8 - x^2 - y^2}$, $z = \sqrt{x^2 + y^2}$ and their intersection.

Solution.







The region is bounded above by the hemisphere $z = \sqrt{8 - x^2 - y^2}$ and below by the cone $z = \sqrt{x^2 + y^2}$. We have $\sqrt{x^2 + y^2} \leq z \leq \sqrt{8 - x^2 - y^2}$. Thus $x^2 + y^2 \leq z^2 \leq 8 - x^2 - y^2$ and $x^2 + y^2 \leq 4$

In polar coordinates, this region $x^2 + y^2 \leq 4$ is $R = \{(r, \theta) : 0 \leq r \leq 2, 0 \leq \theta \leq 2\pi\}$. Note that $\sqrt{8 - x^2 - y^2} = \sqrt{8 - r^2}$ and $\sqrt{x^2 + y^2} = \sqrt{r^2} = r$.

Hence, we can compute the volume of the ice cream cone by finding the volume under the graph of $\sqrt{8 - r^2}$ above the disk $R = \{(r, \theta) : 0 \leq r \leq 2, 0 \leq \theta \leq 2\pi\}$ and subtracting the volume under the graph of r above R. Therefore, we have

$$\begin{aligned}
\text{Volume} &= \int_0^{2\pi} \int_0^2 \left(\sqrt{8 - r^2}\right) r dr\, d\theta - \int_0^{2\pi} \int_0^2 \left(r\right) r dr\, d\theta \\
&= \int_0^{2\pi} \int_0^2 r\sqrt{8 - r^2} - r^2\, dr\, d\theta = \int_0^{2\pi} \left[-\tfrac{1}{3}(8 - r^2)^{3/2} - \tfrac{1}{3}r^3\right]_0^2 d\theta \\
&= \tfrac{1}{3} \int_0^{2\pi} \left(-4^{3/2} - 8 + 8^{3/2}\right) d\theta = \tfrac{1}{3}(16\sqrt{2} - 16) \int_0^{2\pi} d\theta \\
&= \tfrac{1}{3}(32\pi)(\sqrt{2} - 1)\,.
\end{aligned}$$

□

2. Evaluate the following integral by converting to polar coordinates.

(a) (10 pt)$\int_0^2 \int_0^{\sqrt{4-y^2}} (x^2 + y^2)^{\frac{3}{2}} dxdy$

(b) (10 pt) $\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sin(x^2 + y^2) dydx$

Solution. (a) The region of integration is $\{(x,y)0 \le x \le \sqrt{4-y^2}, 0 \le y \le y2\}$. The is the region in first quadrant. In polar coordinates, it is $R = \{(r,\theta) : 0 \le r \le 2, 0 \le \theta \le \frac{\pi}{2}\}$. We also have $(x^2+y^2)^{\frac{3}{2}} = (r^2)^{\frac{3}{2}} = r^3$ and
$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sin(x^2+y^2)dydx = \int_0^{2\pi}\int_0^1 \sin(r^2)\cdot rdrd\theta$
$= \int_0^{2\pi} -\frac{\cos(r^2)}{2}|0^1 d\theta = \int_0^{2\pi}(-\frac{\cos(1)}{2}+\frac{1}{2})d\theta = (-\frac{\cos(1)}{2}+\frac{1}{2})\cdot 2\pi = -\cos(1)+1.$

(b) The region of integration is $\{(x,y)| -\sqrt{1-x^2} \le y \le \sqrt{1-x^2}, -1 \le x \le 1\}$. Note that $-\sqrt{1-x^2} = y$ and $\sqrt{1-x^2} = y$ imply $x^2+y^2 = 1$. Since $-1 \le x \le 1$, we know that R is the disk inside the unit circle.

In polar coordinates, it is $R = \{(r,\theta) : 0 \le r \le 1, 0 \le \theta \le 2\pi\}$. We also have $x^2+y^2 = r^2$ and
$\int_0^2\int_0^{\sqrt{4-y^2}}(x^2+y^2)^{\frac{3}{2}}dxdy = \int_0^{\frac{\pi}{2}}\int_0^2 r^3\cdot rdrd\theta$
$== \int_0^{\frac{\pi}{2}}\int_0^2 r^4 drd\theta = \int_0^{\frac{\pi}{2}} \frac{r^5}{5}|_0^2 d\theta = \int_0^{\frac{\pi}{2}} \frac{32}{5}d\theta = \frac{32}{5}\cdot\frac{\pi}{2} = \frac{16\pi}{5}.$ □

3. **(a)** (10 pt)For $a > 0$ find the volume under the graph of $z = e^{-(x^2+y^2)}$ above the disk $x^2+y^2 \le a^2$.
(b) (10 pt)What happens to the volume as $a \to \infty$.

Solution. In polar coordinates, the disk is described by the inequalities $0 \le r \le a$, $0 \le \theta \le 2\pi$ and the function is e^{-r^2}. Hence, the volume under the graph and above the disk is

$$\text{Volume} = \int_0^a\int_0^{2\pi} e^{-r^2}\, rdr\, d\theta = 2\pi\int_0^a re^{-r^2}\, dr = 2\pi\left[-\tfrac{1}{2}e^{-r^2}\right]_0^a = \pi\left(1-e^{-a^2}\right).$$

Since $\lim\limits_{a\to\infty}\pi\left(1-e^{-a^2}\right) = \pi$, the volume under the graph of $e^{-x^2-y^2}$ above the entire xy-plane is π. □

4. Consider a thin plate that occupies the region D bounded by the parabola $y = 1-x^2$, $x = 0$ and $y = 0$ in the first quadrant with density function $\rho(x,y) = x$.
(a) (10 pt) Find the mass of the thin plate.
(b) (10 pt)Find the center of mass of the thin plate.
(c) (10 pt)Find moments of inertia I_x, I_y and I_0.

Solution. (a) The graph $y = 1-x^2$ intersect with $y = 0$ at $1-x^2 = 0$, i.e. $x = \pm 1$. We also know that $y = 1-x^2 \ge 0$ when $-1 \le x \le 1$. The region of integration is $R = \{(x,y)|0 \le x \le 1, 0 \le y \le 1-x^2$.

The mass is $m = \int\int_R \rho(x,y)dA = \int_0^1\int_0^{1-x^2} xdydx = \int_0^1\int_0^{1-x^2} xy|_0^{1-x^2}dx = \int_0^1 x(1-x^2)dx = \int_0^1(x-x^3)dx = (\frac{x^2}{2}-\frac{x^4}{4})|_0^1 = \frac{1}{2}-\frac{1}{4} = \frac{1}{4}$.

(b) The center of mass $= (\frac{\int\int_R \rho(x,y)xdA}{m}, \frac{\int\int_R \rho(x,y)ydA}{m}$.

Now $\int\int_R \rho(x,y)x dA = \int_0^1\int_0^{1-x^2} x{\cdot}x dydx = \int_0^1\int_0^{1-x^2} x^2 dydx \int_0^1\int_0^{1-x^2} x^2y|_0^{1-x^2}dx = \int_0^1 x^2(1-x^2)dx = \int_0^1(x^2-x^4)dx = (\frac{x^3}{3}-\frac{x^5}{5})|_0^1 = \frac{1}{3}-\frac{1}{5} = \frac{2}{15}$.

$\int\int_R \rho(x,y)ydA = \int_0^1\int_0^{1-x^2} xydydx \int_0^1\int_0^{1-x^2} x\frac{y^2}{2}|_0^{1-x^2}dx = \int_0^1 x\frac{(1-x^2)^2}{2}dx = \int_0^1 x\frac{(1-x^2)^2}{2}dx = \int_0^1 x\frac{x^4-2x^2+1}{2}dx = \int_0^1 \frac{x^5-2x^3+x}{2}dx = (\frac{x^6}{12}-\frac{2x^4}{8}+\frac{x^2}{4})|_0^1 = \frac{1}{12}-\frac{2}{8}+\frac{1}{4} = \frac{1}{12}$. So the center of mass is $(\frac{\frac{2}{15}}{\frac{1}{4}}, \frac{\frac{1}{12}}{\frac{1}{4}}) = (\frac{8}{15}, \frac{1}{3})$.

(c) $I_x = \int\int_R \rho(x,y)y^2 dA = \int_0^1\int_0^{1-x^2} xy^2 dydx \int_0^1\int_0^{1-x^2} x\frac{y^3}{3}|_0^{1-x^2}dx = \int_0^1 x\frac{(1-x^2)^3}{3}dx$. Let $u = 1-x^2$. Then $du = -2xdx$, $xdx = -\frac{du}{2}$ and $\int x\frac{(1-x^2)^3}{3}dx = -\int \frac{(u)^3}{6}du = -\frac{u^4}{24} = -\frac{(1-x^2)^4}{24}$.

So $I_x = -\frac{(1-x^2)^4}{24}|_0^1 = \frac{1}{24}$.

$I_y = \int\int_R \rho(x,y)x^2 dA = \int_0^1\int_0^{1-x^2} xx^2 dydx \int_0^1\int_0^{1-x^2} x^3y|_0^{1-x^2}dx = \int_0^1 x^3(1-x^2)dx = \int_0^1(x^3-x^5)dx = \frac{x^4}{4}-\frac{x^6}{6}|_0^1 = \frac{1}{12}$.

Note that $I_0 = \int\int_R \rho(x,y)(x^2+y^2)dA = I_x + I_y = \frac{1}{20}+\frac{1}{24} = \frac{11}{120}$.

$\square$