HW 6 Solution

(1)

$$\text{(7.2 Problem 26)}\ \int_0^{\frac{\pi}{6}} e^x \sin(x)dx$$

Solution: First, we find $\int e^x \sin(x)dx$. I will do a more general integral $\int e^{ax}\sin(bx)dx$.

This is a typical "integration by parts" example. We start with $u = e^{ax}$ and $dv = \sin(bx)dx$.

Then we have $du = ae^{ax}dx$ (use chain rule here) and $v = \int \sin(bx)dx = -\frac{\cos(bx)}{b}$ (note that $(\cos(bx))' = -\sin(bx)\cdot b$ and $(-\frac{\cos(bx)}{b})' = \sin(bx)$.

Using integration by parts, we have

$$\int \underbrace{e^{ax}}_{u}\underbrace{\sin(bx)dx}_{dv} = \underbrace{e^{ax}}_{u}\cdot\underbrace{(-\frac{\cos(bx)}{b})}_{v} - \int \underbrace{(-\frac{\cos(bx)}{b})}_{v}\cdot\underbrace{ae^{ax}dx}_{du}$$

$= -\frac{e^{ax}\cos(bx)}{b} + \frac{a}{b}\int e^{ax}\cos(bx)dx$

Now we have $\int e^{ax}\sin(bx)dx = -\frac{e^{ax}\cos(bx)}{b} + \frac{a}{b}\int e^{ax}\cos(bx)dx$. We do not get the answer now. We need to find $\int e^{ax}\cos(bx)dx$ using IBP again.

Let $u =$ and $dv = \cos(bx)dx$. Then we have $du = ae^{ax}dx$ (use chain rule here) and $v = \int \cos(bx)dx = \frac{\sin(bx)}{b}$ (note that $(\sin(bx))' = \cos(bx)\cdot b$ and $(\frac{\sin(bx)}{b})' = \cos(bx)$.

Using integration by parts, we have

$$\int \underbrace{e^{ax}}_{u}\underbrace{\cos(bx)dx}_{dv} = \underbrace{e^{ax}}_{u}\cdot\underbrace{\frac{\sin(bx)}{b}}_{v} - \int \underbrace{\frac{\sin(bx)}{b}}_{v}\cdot\underbrace{ae^{ax}dx}_{du}$$

$= \frac{e^{ax}\sin(bx)}{b} - \frac{a}{b}\int e^{ax}\sin(bx)dx$

Now we have $\int e^{ax}\cos(bx)dx = \frac{e^{ax}\sin(bx)}{b} - \frac{a}{b}\int e^{ax}\sin(bx)dx$.

Now we combine $\int e^{ax}\sin(bx)dx = -\frac{e^{ax}\cos(bx)}{b} + \frac{a}{b}\int e^{ax}\cos(bx)dx$ and $\int e^{ax}\cos(bx)dx = \frac{e^{ax}\sin(bx)}{b} - \frac{a}{b}\int e^{ax}\sin(bx)dx$ to get

$$\begin{aligned}\int e^{ax}\sin(bx)dx &= -\frac{e^{ax}\cos(bx)}{b} + \frac{a}{b}\Big(\frac{e^{ax}\sin(bx)}{b} - \frac{a}{b}\int e^{ax}\sin(bx)dx\Big)\\ &= -\frac{e^{ax}\cos(bx)}{b} + \frac{a}{b}\cdot\frac{e^{ax}\sin(bx)}{b} - \frac{a}{b}\cdot\frac{a}{b}\int e^{ax}\sin(bx)dx\\ &= -\frac{e^{ax}\cos(bx)}{b} + \frac{ae^{ax}\sin(bx)}{b^2} - \frac{a^2}{b^2}\int e^{ax}\sin(bx)dx\end{aligned} \tag{0.0.1}$$

Thus

$$\int e^{ax}\sin(bx)dx + \frac{a^2}{b^2}\int e^{ax}\sin(bx)dx = \frac{-be^{ax}\cos(bx) + ae^{ax}\sin(bx)}{b^2} + c,$$

$(1+\frac{a^2}{b^2})(\int e^{ax}\sin(bx)dx) = \frac{-be^{ax}\cos(bx)+ae^{ax}\sin(bx)}{b^2}+c$,
$\frac{b^2+a^2}{b^2}(\int e^{ax}\sin(bx)dx) = \frac{-be^{ax}\cos(bx)+ae^{ax}\sin(bx)}{b^2}+c$ and
$\int e^{ax}\sin(bx)dx = \frac{b^2}{a^2+b^2}\frac{-be^{ax}\cos(bx)+ae^{ax}\sin(bx)}{b^2}+C = \frac{-be^{ax}\cos(bx)+ae^{ax}\sin(bx)}{a^2+b^2}+C$.

So $\int e^x\sin(x)dx = \frac{-e^x\cos(x)+e^x\sin(x)}{2}+C$ and $\int_0^{\frac{\pi}{6}} e^x\sin(x)dx = \frac{-e^x\cos(x)+e^x\sin(x)}{2}\Big|_0^{\frac{\pi}{6}}$
$= \frac{-e^{\frac{\pi}{6}}\cos(\frac{\pi}{6})+e^{\frac{\pi}{6}}\sin(\frac{\pi}{6})}{2} - \left(\frac{-e^0\cos(0)+e^0\sin(0)}{2}\right) = \frac{-e^{\frac{\pi}{6}}\frac{\sqrt{3}}{2}+e^{\frac{\pi}{6}}\frac{1}{2}}{2} - \left(\frac{-1}{2}\right) = \frac{-e^{\frac{\pi}{6}}\sqrt{3}}{4}+\frac{e^{\frac{\pi}{6}}}{4}+\frac{1}{2}$

If you do the problem $\int_0^{\frac{\pi}{6}} e^x\cos(x)dx$, you should get $\int e^x\cos(x)dx = \frac{e^x\cos(x)+e^x\sin(x)}{2}+C$. Then $\int_0^{\frac{\pi}{6}} e^x\cos(x)dx = \frac{e^x\cos(x)+e^x\sin(x)}{2}\Big|_0^{\frac{\pi}{6}}$
$= \frac{e^{\frac{\pi}{6}}\cos(\frac{\pi}{6})+e^{\frac{\pi}{6}}\sin(\frac{\pi}{6})}{2} - \left(\frac{e^0\cos(0)+e^0\sin(0)}{2}\right) = \frac{e^{\frac{\pi}{6}}\frac{\sqrt{3}}{2}+e^{\frac{\pi}{6}}\frac{1}{2}}{2} - \left(\frac{1}{2}\right) = \frac{e^{\frac{\pi}{6}}\sqrt{3}}{4}+\frac{e^{\frac{\pi}{6}}}{4}-\frac{1}{2}$

(2)

$$\text{(7.2 Problem 40)} \int \sin(\sqrt{x})dx$$

Hint:make an substitution first and then use integration by parts) Solution: Let $w=\sqrt{x}$. Then $dw = \frac{dx}{2\sqrt{x}}$, $2\sqrt{x}dw = dx$ and $dx = 2wdw$. Thus $\int\sin(\sqrt{x})dx = \int\sin(w)\cdot 2wdw = 2\int w\sin(w)dw$.
Now we find $\int w\sin(w)dw$ using integration by parts. Let $u=w$ and $dv=\sin(w)dw$. Then $du=dw$ and $v=\int\sin(w)dw = -\cos(w)$. Thus $\int w\sin(w)dw = w(-\cos(w)) - \int(-\cos(w))dw = -w\cos(w)+\int\cos(w)dw = -w\cos(w)+\sin(w)+C$. So $\int\sin(\sqrt{x})dx = 2(-w\cos(w)+\sin(w))+C = -2\sqrt{x}\cos(\sqrt{x})+2\sin(\sqrt{x})+C$

(3)

$$\int \arcsin(2x)dx$$

Hint: Try $u=\arcsin(2x)$ and $dv=dx$. Recall that $\frac{d}{dx}(\arcsin(ax)) = \frac{a}{\sqrt{1-a^2x^2}}$.
Solution: Let $u=\arcsin(2x)$ and $dv=dx$. Then $du = \frac{2}{\sqrt{1-4x^2}}dx$ and $v=\int dx = x$. Using integration by parts, we have $\int\arcsin(2x)dx = \arcsin(2x)x - \int x\cdot\frac{2}{\sqrt{1-4x^2}}dx = \arcsin(2x)x - \int\frac{2x}{\sqrt{1-4x^2}}dx$. We can use the substitution $u=1-4x^2$, $du=-8xdx$ and $-\frac{du}{8}=xdx$ to find
$\int\frac{2x}{\sqrt{1-4x^2}}dx = \int\frac{2}{\sqrt{u}}\cdot(-\frac{du}{8}) = -\int\frac{1}{4\sqrt{u}}du = -\frac{\sqrt{u}}{2}+C = -\frac{\sqrt{1-4x^2}}{2}+C$. Thus $\int\arcsin(2x)dx = x\arcsin(2x)+\frac{\sqrt{1-4x^2}}{2}+C$.

(4)

$$\text{(7.3 Problem 36)} \int \frac{x^4+3}{x^2-4x+3}dx$$

Hint: Do long division first. Then do partial fraction.
Solution: We can do a long division (see the below graph)

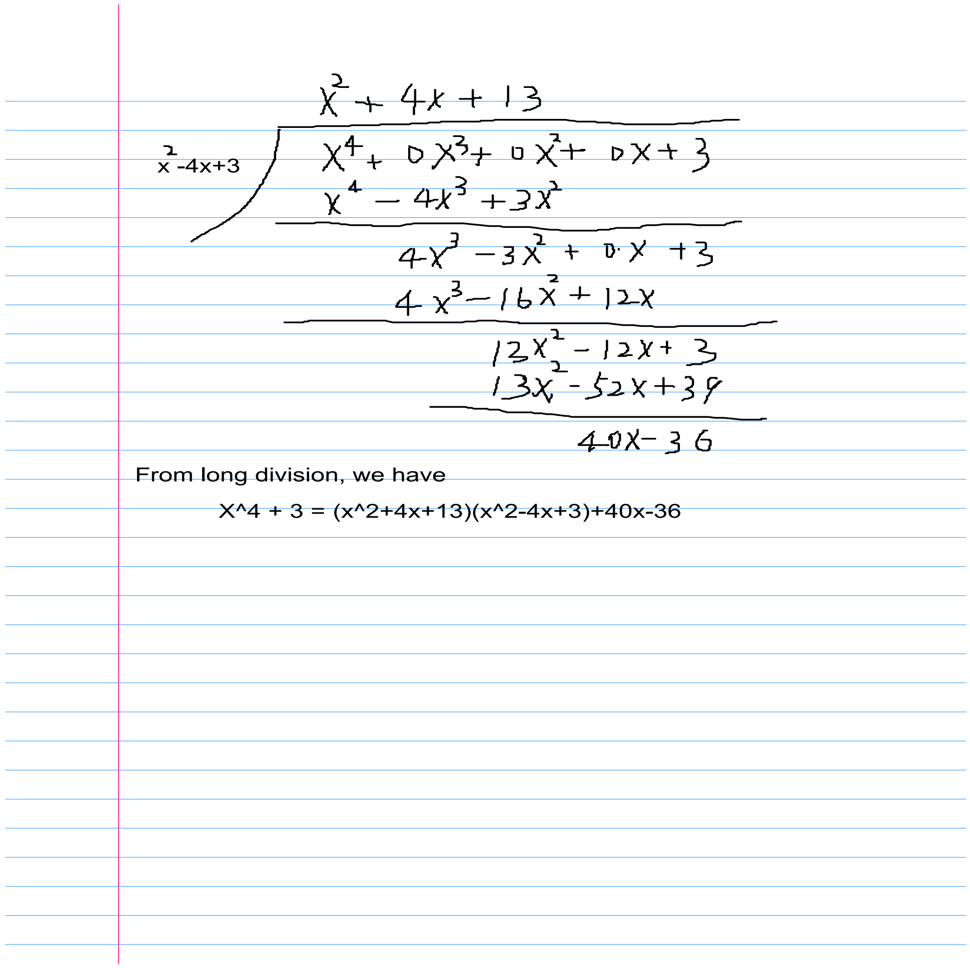

FIGURE 1. long division

to get $x^4 + 3 = (x^2 + 4x + 13) \cdot (x^2 - 4x + 3) + 40x - 36$. This gives

(0.0.2)
$$\frac{x^4+3}{x^2-4x+3} = \frac{(x^2+4x+13)\cdot(x^2-4x+3)+40x-36}{x^2-4x+3} = x^2+4x+13+\frac{40x-36}{x^2-4x+3}.$$

We can factor $x^2 - 4x + 3 = (x-1)(x-3)$. By partial fraction, we can find A and B such that

(0.0.3)
$$\frac{40x-36}{x^2-4x+3} = \frac{40x-36}{(x-1)(x-3)} = \frac{A}{x-1} + \frac{B}{x-3}.$$

Multiplying $(x-1)(x-3)$ to previous equation, we get $(x-1)(x-3)(\frac{40x-36}{(x-1)(x-3)}) = (x-1)(x-3)(\frac{A}{x-1} + \frac{B}{x-3})$ and $40x - 36 = A(x-3) + B(x-1)$. Expanding the the right hand side of

previous equation, we get $40x+36 = Ax - 3A + Bx - B$ and $40x+36 = (A+B)x - 3A - B$. Comparing the coefficient of x and the constant, we get
$A+B = 40$ and $-3A-B = -36$. From $A+B = 40$, we get $B = 40-A$. From $-3A - B = -36$ and $B = 40 - A$, we get $-3A - (40 - A) = -36$, $-2A = 4$ and $A = -2$. From $B = 40 - A$ and $A = -2$, we get $B = 40 - (-2) = 42$.

From equation (0.0.3), we have

$$\frac{40x - 36}{x^2 - 4x + 3} = -2 \cdot \frac{1}{x-1} + 42 \cdot \frac{1}{x-3}. \tag{0.0.4}$$

From equation (0.0.2) and equation (0.0.4), we have

$$\frac{x^4+3}{x^2-4x+3} = x^2 + 4x + 13 + \frac{40x-36}{x^2-4x+3} = x^2 + 4x + 13 - 2 \cdot \frac{1}{x-1} + 42 \cdot \frac{1}{x-3}.$$

Thus $\int \frac{x^4+3}{x^2-4x+3} dx = \int (x^2 + 4x + 13) dx - 2 \int \frac{1}{x-1} dx + 42 \int \frac{1}{x-3} dx$
$= \frac{x^3}{3} + 2x^2 + 13x - 2\ln|x-1| + 42 \ln|x-4| + C.$

5

$$\text{(7.3 Problem 18)} \int \frac{4x^2 - x - 1}{(x+1)^2(x-3)} dx$$

Hint: Try partial fraction $\frac{4x^2-x-1}{(x+1)^2(x-3)} = \frac{A}{x-3} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$ Solution: First, we use partial fraction to find A, B and C such that
$\frac{4x^2-x-1}{(x+1)^2(x-3)} = \frac{A}{x-3} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$.
Multiplying $(x-3)(x+1)^2$ to previous equation, we get
$(x-3)(x+1)^2(\frac{4x^2-x-1}{(x-3)(x+1)^2}) = (x-3)(x+1)^2(\frac{A}{x-3} + \frac{B}{x+1} + \frac{C}{(x+1)^2})$ and
$4x^2 - x - 1 = A(x+1)^2 + B(x-3)(x+1) + C(x-3)$. We expand the right hand side of the equation to get
$4x^2 - x - 1 = A(x^2 + 2x + 1) + B(x^2 - 2x - 3) + C(x-3)$,
$4x^2 - x - 1 = Ax^2 + 2Ax + A + Bx^2 - 2Bx - 3B + Cx - 3C$. Now we collect the terms on the right to get
$4x^2 - x - 1 = (A+B)x^2 + (2A - 2B + C)x + A - 3B - 3C$.
Comparing the coefficient of x^2, x and the constant, we get
$A + B = 4$, $2A - 2B + C = -1$ and $A - 3B - 3C = -1$.
From $2A - 2B + C = -1$, we get $C = -1 - 2A + 2B$. Plugging $C = -1 - 2A + 2B$ to $A - 3B - 3C = -1$, we get $A - 3B - 3(-1 - 2A + 2B) = -1$, $A - 3B + 3 + 6A - 6B = -1$ and $7A - 9B = -4$. From $A + B = 4$, we get $B = 4-A$. Plugging $B = 4-A$ into $7A-9B = -4$, we get $7A-9(4-A) = -4$, $16A - 36 = -4$, $16A = 32$ and $A = 2$. Using $B = 4 - A$ and $A = 2$, we get $B = 4 - 2 = 2$. Using $C = -1 - 2A + 2B$, $A = 2$ and $B = 2$, we get $C = -1 - 2 \cdot 2 + 2 \cdot 2 = 1 - 4 + 4 = -1$.
Thus we have $A = 2$, $B = 2$, $C = -1$ and $\frac{4x^2-x-1}{(x-3)(x+1)^2} = \frac{2}{x-3} + \frac{2}{x+1} - \frac{1}{(x+1)^2}$.
Thus $\int \frac{4x^2-x-1}{(x-3)(x+1)^2} dx = 2 \int \frac{1}{x-3} dx + 2 \int \frac{1}{x+1} dx - \int \frac{1}{(x+1)^2} dx$

$= 2\ln|x-3| + 2\ln|x+1| + \frac{1}{x+1} + C$. Here we have used $\int \frac{1}{ax+b}dx = \frac{\ln|ax+b|}{a}$ and $\int \frac{1}{(x+1)^2}dx = \int \frac{1}{u^2}du$ (with $u = x+1$ and $du = dx$) $= -\frac{1}{u} + C = -\frac{1}{x+1} + C$