MATH 1760 note on Oct. 2

Last time, we integrated $\int \frac{4x^2+x+4}{(x-4)(x+2)^2}dx$.
To integrate $\int \frac{4x^2+x+4}{(x-4)(x+2)^2}dx$, we tried do the partial fraction to find A, B and C such that
$\frac{4x^2+x+4}{(x-4)(x+2)^2} = \frac{A}{x-4} + \frac{B}{x+2} + \frac{C}{(x+2)^2}$

$$\frac{4x^2+x+4}{(x-4)\underbrace{(x+2)^2}_{\text{this creates two terms in partial fraction}}} = \frac{A}{x-4} + \underbrace{\frac{B}{x+2} + \frac{C}{(x+2)^2}}_{\text{these two terms comes from the factor } (x+2)^2}$$

Let me explain why we have two terms instead of one term. Look at the example $\frac{4x-5}{x^2}$. We can simplify this as

$$\frac{4x+5}{x^2} = \frac{4x}{x^2} + \frac{5}{x^2} = \frac{4}{x} + \frac{5}{x^2}.$$

So we can see the term x^2 in the bottom creates two terms in the partial fractions.

In general, the factor $(x-a)^k$ in the bottom will create k terms of the form $\frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \cdots + \frac{A_k}{(x-a)^k}$.

For example, the partial fraction of $\frac{x^3-x^2+x-4}{x^2(x-1)(x-2)^3}$ will be

$$\frac{x^3-x^2+x-4}{x^2(x-1)(x-2)^3} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} + \frac{D}{x-2} + \frac{E}{(x-2)^2} + \frac{F}{(x-2)^3}.$$

Next, we will look at the case where the bottom has the irreducible quadratic factor of the form $a(x-b)^2+c$ where $a>0$ and $c>0$ (like $2(x-3)^2+4$). First, we recall that a quadratic polynomial ax^2+bx+c can be factored as the product of linear factor if $b^2-4ac \geq 0$. If $b^2-4ac<0$ then ax^2+bx+c can not be factored and we say ax^2+bx+c is an irreducible quadratic factor.

Example 1 $\int \frac{3x-5}{x^2+1}dx$ Solution: It is obvious that x^2+1 can not be factored. We have $\frac{3x-5}{x^2+1} = \frac{3x}{x^2+1} - \frac{5}{x^2+1}$. So $\int \frac{3x-5}{x^2+1}dx = \int \frac{3x}{x^2+1}dx - \int \frac{5}{x^2+1}dx$. We have $\int \frac{5}{x^2+1}dx = 5\int \frac{1}{x^2+1}dx = 5\arctan(x)+C$. To integrate $\int \frac{3x}{x^2+1}dx$, we use the substitution $u = x^2+1$ and $du = 2xdx$ and $xdx = \frac{1}{2}du$. Thus $\int \frac{3x}{x^2+1}dx = 3\int \frac{1}{x^2+1}xdx = 3\int \frac{1}{u}\frac{1}{2}du = \frac{3}{2}\int \frac{1}{u}du = \frac{3}{2}\ln|u|+c = \frac{3}{2}\ln|x^2+1|$.

Thus we have $\int \frac{3x-5}{x^2+1}dx = \frac{3}{2}\ln|x^2+1| - 5\arctan(x)+C$.

Remark: Using similar idea, we have $\int \frac{ax+b}{x^2+1}dx = \frac{a}{2}\ln|x^2+1| + b\arctan(x)+C$.

Example 2 $\int \frac{x^2+x+1}{(x^2+1)^2}dx$ Solution: The irreducible quadratic factor $(x^2+1)^2$ in the bottom will create two terms of the form $\frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2}$. We try the partial fraction $\frac{x^2+x+1}{(x^2+1)^2} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2}$. See example 7 on p349 for the detail.

Remark: In general, the irreducible quadratic factor $(a^2(x-b)^2+c^2)^k$ in the bottom will create k terms of the form $\frac{A_1x+B_1}{a^2(x-b)^2+c^2} + \frac{A_2x+B_2}{(a^2(x-b)^2+c^2)^2} + \cdots + \frac{A_kx+B_k}{(a^2(x-b)^2+c^2)^k}$. For example, the partial fraction for $\frac{x^3-x^2+x-4}{x^2(x-1)(x-2)^3(x^2+1)^3}$ will be

$$\frac{x^3-x^2+x-4}{x^2(x-1)(x-2)^3(x^2+1)^3}$$
$$= \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} + \frac{D}{x-2} + \frac{E}{(x-2)^2} + \frac{F}{(x-2)^3} + \frac{GX+H}{x^2+1} + \frac{Ix+J}{(x^2+1)^2} + \frac{KX+L}{(x^2+1)^3}.$$

Example 3 $\int \frac{2x-7}{x^2-4x+8}dx$

Solution: The quadratic x^2-4x+8 is irreducible since $(-4)^2-4\cdot 8 = 16-32 = -16 < 0$. We can complete the square $x^2-4x+8 = (x^2-4x+4)+4 = (x-2)^2+2^2$. So $\frac{2x-7}{x^2-4x+8} = \frac{2x-7}{(x-2)^2+2^2}$. To integrate $\int \frac{2x-7}{(x-2)^2+2^2}dx$. we try the substitution $x-2 = 2u$. Then $dx = 2du$ and $x = 2u+2$. So $\int \frac{2x-7}{x^2-4x+8}dx = \int \frac{2(2u+2)-7}{2^2u^2+2^2}2du = \int \frac{4u+4-7}{4u^2+4}2du = \int \frac{4u-3}{2(u^2+1)}du = \int \frac{2u}{u^2+1}du - \frac{3}{2}\int \frac{1}{u^2+1}du = \ln|u^2+1| - \frac{3}{2}\arctan(u) + C = \ln|(\frac{x-2}{2})^2+1| - \frac{3}{2}\arctan(\frac{x-2}{2}) + C$. Here we use $x-2 = 2u$ and $u = \frac{x-2}{2}$.

Remark: To integrate $\int \frac{Dx+E}{a^2(x-b)^2+c^2}dx$, one tries the substitution $a(x-b) = cu$ to get $a^2(x-b)^2 = c^2u^2$. We have $cdu = adx$, $dx = \frac{c}{a}du$ and also $x-b = \frac{cu}{a}$, $x = b+\frac{cu}{a}$ and $u = \frac{a(x-b)}{c}$. Then $\int \frac{Dx+E}{a^2(x-b)^2+c^2}dx = \int \frac{D(b+\frac{cu}{a})+E}{c^2u^2+c^2}\frac{c}{a}du = \frac{1}{ac}\int \frac{\frac{cD}{a}u+Db+E}{u^2+1}du$.