Solution to Quiz 6

- (50 pts)

$$\int e^{3x}\sin(2x)dx$$

We start with $u = e^{3x}$ and $dv = \sin(2x)dx$.

Then we have $du = 3e^{3x}dx$ (use chain rule here) and $v = \int \sin(2x)dx = -\frac{\cos(2x)}{2}$ (note that $(\cos(2x))' = -\sin(2x)\cdot 2$ and $(-\frac{\cos(2x)}{2})' = \sin(2x)$.

Using integration by parts, we have

$\int \underbrace{e^{3x}}_{u}\underbrace{\sin(2x)dx}_{dv} = \underbrace{e^{3x}}_{u}\cdot\underbrace{(-\frac{\cos(2x)}{2})}_{v} - \int\underbrace{(-\frac{\cos(2x)}{2})}_{v}\cdot\underbrace{3e^{3x}dx}_{du}$

$= -\frac{e^{3x}\cos(2x)}{2} + \frac{3}{2}\int e^{3x}\cos(2x)dx$

Now we have $\int e^{3x}\sin(2x)dx = -\frac{e^{3x}\cos(2x)}{2} + \frac{3}{2}\int e^{3x}\cos(2x)dx$. We do not get the answer now. We need to find $\int e^{3x}\cos(2x)dx$ using IBP again.

Let $u = e^{3x}$ and $dv = \cos(2x)dx$. Then we have $du = 3e^{3x}dx$ (use chain rule here) and $v = \int\cos(2x)dx = \frac{\sin(2x)}{2}$ (note that $(\sin(2x))' = \cos(2x)\cdot 2$ and $(\frac{\sin(2x)}{2})' = \cos(2x)$.

Using integration by parts, we have

$\int \underbrace{e^{3x}}_{u}\underbrace{\cos(2x)dx}_{dv} = \underbrace{e^{3x}}_{u}\cdot\underbrace{\frac{\sin(2x)}{2}}_{v} - \int\underbrace{\frac{\sin(2x)}{2}}_{v}\cdot\underbrace{3e^{3x}dx}_{du}$

$= \frac{e^{3x}\sin(2x)}{2} - \frac{3}{2}\int e^{3x}\sin(2x)dx$

Now we have $\int e^{3x}\cos(2x)dx = \frac{e^{3x}\sin(2x)}{2} - \frac{3}{2}\int e^{3x}\sin(2x)dx$.

Now we combine $\int e^{3x}\sin(2x)dx = -\frac{e^{3x}\cos(2x)}{2} + \frac{3}{2}\int e^{3x}\cos(2x)dx$ and $\int e^{3x}\cos(2x)dx = \frac{e^{3x}\sin(2x)}{2} - \frac{3}{2}\int e^{3x}\sin(2x)dx$ to get

$$\begin{aligned}\int e^{3x}\sin(2x)dx &= -\frac{e^{3x}\cos(2x)}{2} + \frac{3}{2}\Big(\frac{e^{3x}\sin(2x)}{2} - \frac{3}{2}\int e^{3x}\sin(2x)dx\Big)\\ &= -\frac{e^{3x}\cos(2x)}{2} + \frac{3}{2}\cdot\frac{e^{3x}\sin(2x)}{2} - \frac{3}{2}\cdot\frac{3}{2}\int e^{3x}\sin(2x)dx \\ &= -\frac{e^{3x}\cos(2x)}{2} + \frac{3e^{3x}\sin(2x)}{4} - \frac{9}{4}\int e^{3x}\sin(2x)dx\end{aligned} \tag{0.0.1}$$

Thus

$$\int e^{3x}\sin(2x)dx + \frac{9}{4}\int e^{3x}\sin(2x)dx = \frac{-2e^{3x}\cos(2x) + 3e^{3x}\sin(2x)}{4} + c,$$

$(1 + \frac{9}{4})(\int e^{3x}\sin(2x)dx) = \frac{-2e^{3x}\cos(2x)+3e^{3x}\sin(2x)}{4} + c$,

$\frac{13}{4}(\int e^{3x}\sin(2x)dx) = \frac{-2e^{3x}\cos(2x)+3e^{3x}\sin(2x)}{4} + c$ and

$\int e^{3x}\sin(2x)dx = \frac{4}{13}\frac{-2e^{3x}\cos(2x)+3e^{3x}\sin(2x)}{4} + C = \frac{-2e^{3x}\cos(2x)+3e^{3x}\sin(2x)}{13} + C.$

- (50 pts)
$$\int \frac{x^4 - 3x^3 - x^2 + 10x - 10}{x^2 - 3x + 2} dx$$
Solution: From long division

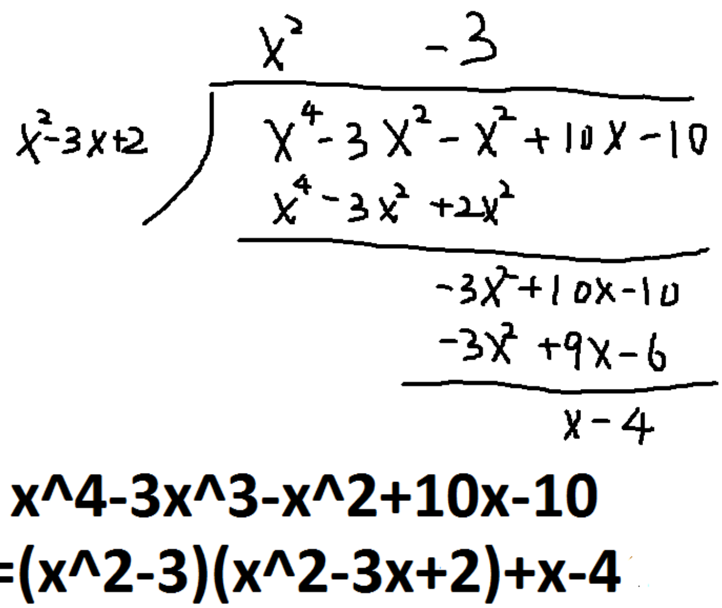

FIGURE 1. Graph for problem 6

, we have $x^4 - 3x^3 - x^2 + 10x - 10 = (x^2 - 3)(x^2 - 3x + 2) + x - 4$. So $\frac{x^4-3x^3-x^2+10x-10}{x^2-3x+2} = \frac{(x^2-3)(x^2-3x+2)+x-4}{x^2-3x+2} = x^2 - 3 + \frac{x-4}{x^2-3x+2}$ For $\frac{x-4}{x^2-3x+2}$, we can try the partial fraction $\frac{x-4}{x^2-3x+2} = \frac{x-4}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2}$. Multiplying $(x-1)(x-2)$, we have $x - 4 = A(x-1) + B(x-2)$. Plugging $x = 2$ and $x = 1$, we get $A = -2$ and $B = 3$. Thus $\frac{x-4}{x^2-3x+2} = -\frac{2}{x-1} + \frac{3}{x-2}$ and $\frac{x^4-3x^3-x^2+10x-10}{x^2-3x+2} = x^2 - 3 - \frac{2}{x-1} + \frac{3}{x-2}$. Hence $\int \frac{x^4-3x^3-x^2+10x-10}{x^2-3x+2} dx = \int \left(x^2 - 3 - \frac{2}{x-1} + \frac{3}{x-2}\right) dx = \frac{x^3}{3} - 3x - 2\ln|x-1| + 3\ln|x-2| + C$.

- (10 pts)

$$\text{Bonus problem} \int \arcsin(3x)dx$$

Solution: Let $u = \arcsin(3x)$ and $dv = dx$. Recall that $\frac{d}{dx}(\arcsin(ax)) = \frac{a}{\sqrt{1-a^2x^2}}$.
Then $du = \frac{3}{\sqrt{1-9x^2}}dx$ and $v = \int dx = x$. Using integration by parts, we have $\int \arcsin(3x)dx = \arcsin(3x)x - \int x \cdot \frac{3}{\sqrt{1-9x^2}}dx = \arcsin(3x)x - \int \frac{3x}{\sqrt{1-9x^2}}dx$.
We can use the substitution $u = 1 - 9x^2$, $du = -18xdx$ and $-\frac{du}{18} = xdx$ to find
$\int \frac{3x}{\sqrt{1-9x^2}}dx = \int \frac{3}{\sqrt{u}} \cdot (-\frac{du}{18}) = -\int \frac{1}{6\sqrt{u}}du = -\frac{\sqrt{u}}{3} + C = -\frac{\sqrt{1-9x^2}}{3} + C$. Thus $\int \arcsin(3x)dx = x\arcsin(3x) + \frac{\sqrt{1-9x^2}}{3} + C$.