Solution to Quiz 7

- (50 pts) $\int \frac{4x^3-8x^2+2x-3}{x^2(x^2+1)}dx$

  Solution: From the partial fraction $\frac{4x^3-8x^2+2x-3}{x^2(x^2+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+1}$, we get $4x^3-8x^2+2x-3 = Ax(x^2+1)+B(x^2+1)+(Cx+D)x^2$, $x^3-2x^2-2x+3 = Ax^3+Ax+Bx^2+B+Cx^3+Dx^2$ and $4x^3-8x^2+2x-3 = (A+C)x^3+(B+D)x^2+Ax+B$. Thus $A+C=4$, $B+D=-8$, $A=2$ and $B=-3$ Hence$A=2$, $B=-3$, $C=4-A=2$ and $D=-8-B=-5$. Thus $\frac{4x^3-8x^2+2x-3}{x^2(x^2+1)} = \frac{2}{x} - \frac{3}{x^2} + \frac{2x-5}{x^2+1}$ and $\int \frac{4x^3-8x^2+2x-3}{x^2(x^2+1)}dx = 2\ln|x| + \frac{3}{x} + \ln|x^2+1| - 5\arctan(x) + C$. We have used $\int \frac{2x-5}{x^2+1}dx = \int \frac{2x}{x^2+1}dx - \frac{5}{x^2+1}dx = \ln|x^2+1| - 5\arctan(x) + C$.

- (50 pts) Determine whether each integral is convergent or divergent. If the integral is convergent, compute its value.

  1.

  $$\int_1^\infty \frac{1}{x^{\frac{1}{3}}}dx$$

  Solution: $\int_1^\infty \frac{1}{x^{\frac{1}{3}}}dx = \lim_{b\to\infty}\int_1^b \frac{1}{x^{\frac{1}{3}}}dx = \lim_{b\to\infty}\frac{3}{2}x^{\frac{2}{3}}\Big|_1^b = \lim_{b\to\infty}\frac{3}{2}b^{\frac{2}{3}} - \frac{3}{2} = \infty$. So $\int_1^\infty \frac{1}{x^{\frac{1}{3}}}dx$ diverges.

  2.

  $$\int_e^\infty \frac{1}{x(\ln x)^3}dx$$

  l We use $u=\ln x$ and $du = \frac{1}{x}dx$ to integrate $\int \frac{1}{x(\ln x)^3}dx = \int \frac{1}{u^3}du = -\frac{1}{2}u^{-2} + C = -\frac{1}{2(\ln x)^2} + C$ and $\int_e^\infty \frac{1}{x(\ln x)^3}dx = \lim_{b\to\infty} -\frac{1}{2(\ln x)^2}\Big|_e^b = \lim_{b\to\infty} -\frac{1}{2(\ln b)^2} + \frac{1}{2} = \frac{1}{2}$. We have used $\ln e = 1$.

- (Bonus problem 20 pts)

  $$\int \frac{6x-5}{x^2+4x+13}dx$$

  Solution: Completing the square, we get $x^2+4x+13 = x^2+4x+4+9 = (x+2)^2+3^2$. Let $x+2=3u$. Then $dx = 3du$ and $x = 3u-2$. $\int \frac{6x-5}{x^2+4x+13}dx = \int \frac{6x-5}{(x+2)^2+3^3}dx = \int \frac{6\cdot(3u-2)-5}{3^2u+3^2}3du = \int \frac{18u-17}{9(u^2+1)}3du = \int \frac{18u-17}{3(u^2+1)}du = \int \frac{6u}{u^2+1}du - \frac{17}{3}\int \frac{1}{u^2+1}du = 3\ln|u^2+1| - \frac{17}{3}\arctan(u) + C = 3\ln|(\frac{x+2}{3})^2+1| - \frac{17}{3}\arctan(\frac{x+2}{3}) + C$. In the last step, we have used $u = \frac{x+2}{3}$.