Solution to Review Problem for Midterm #2

Midterm II: Monday, October 28. in class Topics: 7.1-7.4, 8.1-8.2 and 12.1

No calculator is allowed in the exam. You should know how to solve these problems without a calculator.

In the following, I will use the follow formula frequently.
$\int e^{ax}dx = \frac{e^a x}{a} + C$, $\int \frac{1}{ax+b}dx = \frac{\ln|ax+b|}{a} + C$,
$\int \frac{x}{ax^2+b}dx = \frac{1}{2a}\cdot \ln|ax^2+b| + C$, $\int \sec x dx = \ln|\sec x + \tan x| + C$,
$\int \csc x dx = -\ln|\csc x + \cot x| + C$, $\int \tan x dx = \ln|\sec x| + C$,
and $\int \cot x dx = \ln|\sin x| + C$.

1. Evaluate the following indefinite integrals:

1 $\int \frac{x+1}{(x^2+2x+10)^4}dx$
Solution: Let $u = x^2+2x+10$. Then $du = (2x+2)dx$ and $\frac{du}{2} = (x+1)dx$. Thus $\int \frac{x+1}{(x^2+2x+10)^4}dx = \int \frac{1}{u^4}\frac{du}{2} = \frac{1}{2}\int u^{-4}du = \frac{1}{2}\cdot(-\frac{1}{3})u^{-3} + C = -\frac{1}{6u^3} + C = -\frac{1}{6(x^2+2x+10)^3} + C$.

2 $\int xe^{-3x}dx$ Solution: We use integration by parts. Let $u = x$ and $dv = e^{-3x}dx$. Then $du = dx$ and $v = \int e^{-3x}dx = \frac{-e^{-3x}}{3}$. So $\int xe^{-3x}dx = x(\frac{-e^{-3x}}{3}) - \int(\frac{-e^{-3x}}{3})dx = -\frac{xe^{-3x}}{3} + \frac{1}{3}\int e^{-3x}dx = -\frac{xe^{-3x}}{3} + \frac{1}{3}\cdot(-\frac{e^{-3x}}{3}) + C = -\frac{xe^{-3x}}{3} - \frac{e^{-3x}}{9} + C$.

3 $\int x\sin(3x)$ Solution: Let $u = x$ and $dv = \sin(3x)dx$. Then $du = dx$ and $v = \int \sin(3x)dx = -\frac{\cos(3x)}{3}$. So $\int x\sin(3x)dx = x\cdot(-\frac{\cos(3x)}{3}) - \int(-\frac{\cos(3x)}{3})dx = -\frac{x\cos(3x)}{3} + \frac{1}{3}\int\cos(3x)dx = -\frac{x\cos(3x)}{3} + \frac{1}{3}\cdot\frac{\sin(3x)}{3} + C = -\frac{x\cos(3x)}{3} + \frac{\sin(3x)}{9} + C$.

4 $\int_0^3 x\sqrt{x+1}dx$ Solution: Let $u = \sqrt{x+1}$. Then $du = \frac{1}{2\sqrt{x+1}}dx$, $2\sqrt{x+1}du = dx$ i.e. $2udu = dx$. From $u = \sqrt{x+1}$, we also have $u^2 = x+1$. So $x = u^2 - 1$. Thus $\int x\sqrt{x+1}dx = \int(u^2-1)u\cdot 2udu = \int 2u^4 - 2u^2du = \frac{2}{5}u^5 - \frac{2}{3}u^3 + C = \frac{2}{5}(\sqrt{x+1})^5 - \frac{2}{3}(\sqrt{x+1})^3 + C$.
So $\int_0^3 x\sqrt{x+1}dx = \frac{2}{5}(\sqrt{x+1})^5 - \frac{2}{3}(\sqrt{x+1})^3\Big|_0^3 = \frac{2}{5}(\sqrt{3+1})^5 - \frac{2}{3}(\sqrt{3+1})^3 - \left(\frac{2}{5}(\sqrt{0+1})^5 - \frac{2}{3}(\sqrt{0+1})^3\right) = \frac{2}{5}(2)^5 - \frac{2}{3}(2)^3 - \left(\frac{2}{5} - \frac{2}{3}\right) = \frac{64}{5} - \frac{16}{3} - \frac{2}{5} + \frac{2}{3} = \frac{62}{5} - \frac{14}{3} = \frac{186-70}{15} = \frac{116}{15}$.

5 $\int \frac{dx}{1+\sqrt{x}}$ Solution: Let $u = \sqrt{x}+1$. Then $du = \frac{1}{2\sqrt{x}}dx$., i.e. $2\sqrt{x}du = dx$ and $2(u-1)du = dx$ where we have used $\sqrt{x} = u - 1$.
Hence $\int \frac{dx}{1+\sqrt{x}}dx = \int \frac{2(u-1)}{u}du = \int 2 - \frac{2}{u}du = 2u - 2\ln|u| + C$
$= 2(\sqrt{x}+1) - 2\ln|1+\sqrt{x}| + C$.

6 $\int \frac{1}{\sqrt{x}(1+\sqrt{x})}dx$ Solution: Let $u = \sqrt{x}+1$. Then $du = \frac{1}{2\sqrt{x}}dx$ and $2du = \frac{1}{\sqrt{x}}dx$. Hence $\int \frac{1}{\sqrt{x}(1+\sqrt{x})}dx = \int \frac{1}{(1+\sqrt{x})}\cdot\frac{1}{\sqrt{x}}dx = \int \frac{1}{u}\cdot 2du = \int \frac{2}{u}du = 2\ln|u| + C = 2\ln|1+\sqrt{x}| + C$.

7 $\int \sin x\sqrt{\cos x}dx$ Solution:Let $u = \cos x$. Then $du = -\sin x dx$ and $\sin(x)dx = -du$. $\int \sin x\sqrt{\cos x}dx = -\int \sqrt{u}du = -\frac{2}{3}u^{\frac{3}{2}} + C = -\frac{2}{3}(\cos x)^{\frac{3}{2}} + C$.

8 $\int \cos^5 x dx$ Solution: $\int \cos^5 x dx = \int(\cos^2 x)^2 \cos x dx = \int(1-\sin^2 x)^2 \cos x dx$ where we have used $\cos^2 x = 1 - \sin^2 x$.
Let $u = \sin x$. Then $du = \cos x dx$. Then $\int \cos^5 x dx = \int(1-\sin^2 x)^2 \cos x dx = \int(1-u^2)^2 du = \int(1-2u^2+u^4)du$
$= u - \frac{2}{3}u^3 + \frac{1}{5}u^5 + C = \sin x - \frac{2}{3}(\sin x)^3 + \frac{1}{5}(\sin x)^5 + C$

9 $\int_0^{\frac{\pi}{3}} \sin x \sec^2 x dx$ Solution: Note that $\int \sin x \sec^2 x dx = \int \frac{\sin x}{\cos^2 x}dx$. Let $u = \cos x$. Then $du = -\sin x dx$ and $-du = \sin x dx$. Then $\int \sin x \sec^2 x dx = \int \frac{1}{\cos^2 x} \cdot \sin x dx = -\int u^{-2}du = u^{-1} + C = \frac{1}{\cos x} + C$.
Hence $\int_0^{\frac{\pi}{3}} \sin x \sec^2 x dx = \frac{1}{\cos x}\Big|_0^{\frac{\pi}{3}} = \frac{1}{\cos(\frac{\pi}{3})} - \frac{1}{\cos(0)} = 2 - 1 = 1$. We have used $\cos(0) = 1$ and $\cos(\frac{\pi}{3}) = \frac{1}{2}$.

10 $\int x\tan^{-1}(x)dx$ Solution: Let $u = \arctan(x)$ and $dv = xdx$. Then $du = \frac{1}{x^2+1}$ and $v = \int xdx = \frac{x^2}{2}$. Thus $\int x\arctan(x)dx = \arctan(x) \cdot \frac{x^2}{2} - \int \frac{x^2}{2} \cdot \frac{1}{x^2+1}dx = \frac{x^2\arctan(x)}{2} - \frac{1}{2}\int \frac{x^2}{x^2+1}dx = \frac{x^2\arctan(x)}{2} - \frac{1}{2}\int \frac{(x^2+1)-1}{x^2+1}dx = \frac{x^2\arctan(x)}{2} - \frac{1}{2}\int(1-\frac{1}{x^2+1})dx = \frac{x^2\arctan(x)}{2} - \frac{1}{2}(x - \arctan(x)) + C$.

11 $\int \frac{x^3}{(1+x^2)^5}dx$ Solution: Let $u = 1 + x^2$. Then $du = 2xdx$ and $xdx = \frac{du}{2}$. Also we have $x^2 = u-1$ Thus $\int \frac{x^3}{(1+x^2)^5}dx = \int \frac{x^2}{(1+x^2)^5} \cdot xdx = \int \frac{u-1}{u^5} \cdot \frac{du}{2} = \frac{1}{2}\int \frac{u-1}{u^5}du = \frac{1}{2}\int(u^{-4} - u^{-5})du = \frac{1}{2} \cdot (-\frac{1}{3}u^{-3} + \frac{1}{4}u^{-4}) + C = -\frac{1}{6}u^{-3} + \frac{1}{8}u^{-4} + C = -\frac{1}{6}(1+x^2)^{-3} + \frac{1}{8}(1+x^2)^{-4} + C$

12 $\int \sec^4 x \tan^3 x dx$ Solution: Let $u = \tan(x)$. Then $du = \sec^2(x)dx$. Note that $sec^2(x) = 1 + \tan^2(x)$. So $\int \sec^4 x \tan^3 x dx = \int \sec^2 x \cdot \tan^3 x \cdot \sec^2 x dx = \int(1+\tan^2(x)) \cdot \tan^3 x \cdot \sec^2 x dx == \int(1+u^2) \cdot u^3 x du = \int(u^3 + u^5)du = \frac{u^4}{4} + \frac{u^6}{6} + C = \frac{tan^4(x)}{4} + \frac{tan^6(x)}{6} + C$.

13 $\int e^{ax}\sin(bx)dx$ This is a typical "integration by parts" example. We start with $u = e^{ax}$ and $dv = \sin(bx)dx$.
Then we have $du = ae^{ax}dx$ (use chain rule here) and $v = \int \sin(bx)dx = -\frac{\cos(bx)}{b}$ (note that $(\cos(bx))' = -\sin(bx) \cdot b$ and $(-\frac{\cos(bx)}{b})' = \sin(bx)$.
Using integration by parts, we have

$$\int \underbrace{e^{ax}}_{u}\underbrace{\sin(bx)dx}_{dv} = \underbrace{e^{ax}}_{u} \cdot \underbrace{(-\frac{\cos(bx)}{b})}_{v} - \int \underbrace{(-\frac{\cos(bx)}{b})}_{v} \cdot \underbrace{ae^{ax}dx}_{du}$$

$= -\frac{e^{ax}\cos(bx)}{b} + \frac{a}{b}\int e^{ax}\cos(bx)dx$

Now we have $\int e^{ax}\sin(bx)dx = -\frac{e^{ax}\cos(bx)}{b} + \frac{a}{b}\int e^{ax}\cos(bx)dx$. We do not get the answer now. We need to find $\int e^{ax}\cos(bx)dx$ using IBP again.

Let $u = e^{ax}$ and $dv = \cos(bx)dx$. Then we have $du = ae^{ax}dx$ (use chain rule here) and $v = \int \cos(bx)dx = \frac{\sin(bx)}{b}$ (note that $(\sin(bx))' = \cos(bx) \cdot b$ and $(\frac{\sin(bx)}{b})' = \cos(bx)$.

Using integration by parts, we have

$\int \underbrace{e^{ax}}_{u} \underbrace{\cos(bx)dx}_{dv} = \underbrace{e^{ax}}_{u} \cdot \underbrace{\frac{\sin(bx)}{b}}_{v} - \int \underbrace{\frac{\sin(bx)}{b}}_{v} \cdot \underbrace{ae^{ax}dx}_{du}$

$= \frac{e^{ax}\sin(bx)}{b} - \frac{a}{b}\int e^{ax}\sin(bx)dx$

Now we have $\int e^{ax}\cos(bx)dx = \frac{e^{ax}\sin(bx)}{b} - \frac{a}{b}\int e^{ax}\sin(bx)dx$.

Now we combine $\int e^{ax}\sin(bx)dx = -\frac{e^{ax}\cos(bx)}{b} + \frac{a}{b}\int e^{ax}\cos(bx)dx$ and $\int e^{ax}\cos(bx)dx = \frac{e^{ax}\sin(bx)}{b} - \frac{a}{b}\int e^{ax}\sin(bx)dx$ to get

$$\begin{aligned}
\int e^{ax}\sin(bx)dx &= -\frac{e^{ax}\cos(bx)}{b} + \frac{a}{b}\Big(\frac{e^{ax}\sin(bx)}{b} - \frac{a}{b}\int e^{ax}\sin(bx)dx\Big) \\
&= -\frac{e^{ax}\cos(bx)}{b} + \frac{a}{b}\cdot\frac{e^{ax}\sin(bx)}{b} - \frac{a}{b}\cdot\frac{a}{b}\int e^{ax}\sin(bx)dx \\
&= -\frac{e^{ax}\cos(bx)}{b} + \frac{ae^{ax}\sin(bx)}{b^2} - \frac{a^2}{b^2}\int e^{ax}\sin(bx)dx
\end{aligned} \tag{0.0.1}$$

Thus

$$\int e^{ax}\sin(bx)dx + \frac{a^2}{b^2}\int e^{ax}\sin(bx)dx = \frac{-be^{ax}\cos(bx) + ae^{ax}\sin(bx)}{b^2} + c,$$

$(1 + \frac{a^2}{b^2})(\int e^{ax}\sin(bx)dx) = \frac{-be^{ax}\cos(bx)+ae^{ax}\sin(bx)}{b^2} + c,$

$\frac{b^2+a^2}{b^2}(\int e^{ax}\sin(bx)dx) = \frac{-be^{ax}\cos(bx)+ae^{ax}\sin(bx)}{b^2} + c$ and

$\int e^{ax}\sin(bx)dx = \frac{b^2}{a^2+b^2}\frac{-be^{ax}\cos(bx)+ae^{ax}\sin(bx)}{b^2} + C = \frac{-be^{ax}\cos(bx)+ae^{ax}\sin(bx)}{a^2+b^2} + C$.

14 $\int e^{ax}\cos(bx)dx$ Solution:Let $u = e^{ax}$ and $dv = \cos(bx)dx$. Then we have $du = ae^{ax}dx$ (use chain rule here) and $v = \int \cos(bx)dx = \frac{\sin(bx)}{b}$ (note that $(\sin(bx))' = \cos(bx) \cdot b$ and $(\frac{\sin(bx)}{b})' = \cos(bx)$.

Using integration by parts, we have

$\int \underbrace{e^{ax}}_{u} \underbrace{\cos(bx)dx}_{dv} = \underbrace{e^{ax}}_{u} \cdot \underbrace{\frac{\sin(bx)}{b}}_{v} - \int \underbrace{\frac{\sin(bx)}{b}}_{v} \cdot \underbrace{ae^{ax}dx}_{du}$

$= \frac{e^{ax}\sin(bx)}{b} - \frac{a}{b}\int e^{ax}\sin(bx)dx$.

Now we have $\int e^{ax}\cos(bx)dx = \frac{e^{ax}\sin(bx)}{b} - \frac{a}{b}\int e^{ax}\sin(bx)dx$.

To integrate $\int e^{ax}\sin(bx)dx$, we try $u = e^{ax}$ and $dv = \sin(bx)dx$. Then we have $du = ae^{ax}dx$ (use chain rule here) and $v = \int \sin(bx)dx = -\frac{\cos(bx)}{b}$ (note that $(\cos(bx))' = -\sin(bx) \cdot b$ and $(-\frac{\cos(bx)}{b})' = \sin(bx)$.

Using integration by parts, we have

$\int \underbrace{e^{ax}}_{u} \underbrace{\sin(bx)dx}_{dv} = \underbrace{e^{ax}}_{u} \cdot \underbrace{(-\frac{\cos(bx)}{b})}_{v} - \int \underbrace{(-\frac{\cos(bx)}{b})}_{v} \cdot \underbrace{ae^{ax}dx}_{du}$

$= -\frac{e^{ax}\cos(bx)}{b} + \frac{a}{b}\int e^{ax}\cos(bx)dx$

Now we combine $\int e^{ax}\cos(bx)dx = \frac{e^{ax}\sin(bx)}{b} - \frac{a}{b}\int e^{ax}\sin(bx)dx$ and $\int e^{ax}\sin(bx)dx = -\frac{e^{ax}\cos(bx)}{b} + \frac{a}{b}\int e^{ax}\cos(bx)dx$.
to get

$$\begin{aligned}\int e^{ax}\cos(bx)dx &= \frac{e^{ax}\sin(bx)}{b} - \frac{a}{b}\Big(-\frac{e^{ax}\cos(bx)}{b} + \frac{a}{b}\int e^{ax}\cos(bx)dx\Big)\\ &= \frac{e^{ax}\sin(bx)}{b} - \frac{a}{b}\cdot(-\frac{e^{ax}\cos(bx)}{b}) - \frac{a}{b}\cdot\frac{a}{b}\int e^{ax}\cos(bx)dx\\ &= \frac{e^{ax}\sin(bx)}{b} + \frac{ae^{ax}\cos(bx)}{b^2} - \frac{a^2}{b^2}\int e^{ax}\cos(bx)dx\end{aligned} \tag{0.0.2}$$

Thus

$$\int e^{ax}\cos(bx)dx + \frac{a^2}{b^2}\int e^{ax}\cos(bx)dx = \frac{e^{ax}\sin(bx)}{b} + \frac{ae^{ax}\cos(bx)}{b^2} + c,$$

$(1+\frac{a^2}{b^2})(\int e^{ax}\cos(bx)dx) = \frac{e^{ax}\sin(bx)}{b} + \frac{ae^{ax}\cos(bx)}{b^2} + c$,
$\frac{b^2+a^2}{b^2}(\int e^{ax}\cos(bx)dx) = \frac{e^{ax}\sin(bx)}{b} + \frac{ae^{ax}\cos(bx)}{b^2} + c$ and
$\int e^{ax}\cos(bx)dx = \frac{b^2}{a^2+b^2}\Big(\frac{e^{ax}\sin(bx)}{b} + \frac{ae^{ax}\cos(bx)}{b^2}\Big) + C$
$= \frac{be^{ax}\sin(bx)+ae^{ax}\cos(bx)}{a^2+b^2} + C$.

15 $\int \frac{dx}{e^x+1}$ Solution:Let $u = e^x+1$. Then $du = e^x dx = (u-1)dx$. So $dx = \frac{du}{u-1}$. $\int \frac{dx}{e^x+1} = \int \frac{1}{u(u-1)}du$. By partial fractions, $\frac{1}{u(u-1)} = \frac{1}{u-1} - \frac{1}{u}$. $\int \frac{dx}{e^x+1} = \ln|u-1| - \ln|u| + C = \ln|e^x| - \ln|e^x+1| + C = x - \ln|e^x+1| + C$.

16 $\int \frac{\ln x}{x^2}dx$ Solution:Let $u = \ln(x)$ and $dv = \frac{1}{x^2}dx$. Then $du = \frac{1}{x}dx$ and $v = \int \frac{1}{x^2}dx = -\frac{1}{x}$. So $\int \frac{\ln x}{x^2}dx = \int \ln(x)\cdot\frac{1}{x^2}dx = \ln(x)\cdot(-\frac{1}{x}) - \int(-\frac{1}{x})\cdot\frac{1}{x}dx = -\frac{\ln(x)}{x} + \int\frac{1}{x^2}dx = -\frac{\ln(x)}{x} - \frac{1}{x} + C$.

17 $\int x^2\ln x dx$ Solution:Let $u = \ln(x)$ and $dv = x^2dx$. Then $du = \frac{1}{x}dx$ and $v = \int x^2dx = \frac{x^3}{3}$. So $\int x^2\ln xdx = \int \ln(x)\cdot x^2dx = \ln(x)\cdot\frac{x^3}{3} - \int\frac{x^3}{3}\cdot\frac{1}{x}dx = \frac{x^3\ln(x)}{3} - \int\frac{x^2}{3}dx = \frac{x^3\ln(x)}{3} - \frac{x^3}{9} + C$.

18 $\int \frac{\ln x}{x}dx$ Solution:Let $u = \ln(x)$ Then $du = \frac{1}{x}dx$ and $\int\frac{\ln x}{x}dx = \int udu = \frac{u^2}{2} + C = \frac{(\ln(x))^2}{2} + C$.

19 $\int \sqrt{x}\sin(\sqrt{x})dx$ Solution: Let $u = \sqrt{x}$. Then $du = \frac{1}{2\sqrt{x}}dx$, $2\sqrt{x}du = dx$ and $2udu = dx$. $\int\sqrt{x}\sin(\sqrt{x})dx = \int u\sin u(2udu) = 2\int u^2\sin(u)du$.
By integration by parts twice, we have
$\int u^2\sin udu = -u^2\cos u + 2u\sin u + 2\cos u + C$.

Thus $\int \sqrt{x}\sin(\sqrt{x})dx$
$= -(\sqrt{x})^2\cos(\sqrt{x}) + 2(\sqrt{x})\sin(\sqrt{x}) + 2\cos(\sqrt{x}) + C.$

20 $\int \frac{\sin(\sqrt{x})}{\sqrt{x}}dx$ Solution:$\int \frac{\sin(\sqrt{x})}{\sqrt{x}}dx = -2\cos(\sqrt{x}) + C.$
(By substitution, $u = \sqrt{x}$, $du = \frac{1}{2\sqrt{x}}dx$.)

21 $\int \frac{\sin^{-1}(\sqrt{x})}{\sqrt{x}}dx$ Solution:By substitution, $u = \sqrt{x}$.
$\int \frac{\sin^{-1}(\sqrt{x})}{\sqrt{x}}dx = 2\int \sin^{-1} u du + C$
By integration by parts, we have
$\int \sin^{-1} u du = u\sin^{-1} u + \sqrt{1-u^2} + C..$
Thus $\int \frac{\sin^{-1}(\sqrt{x})}{\sqrt{x}}dx = 2\sqrt{x}\sin^{-1}(\sqrt{x}) + \sqrt{1-x} + C..$

22 $\int x\sec^2(x)dx$ Solution:Let $u = x$ and $dv = \sec^2(x)dx$. Then $du = dx$ and $v = \tan x$.
$\int x\sec^2(x)dx = x\tan x - \int \tan x dx = x\tan x - \ln|\sec x| + C.$

23 $\int \arcsin(2x)dx$ Solution: $u = \arcsin(2x)$ and $dv = dx$. Recall that $\frac{d}{dx}(\arcsin(ax)) = \frac{a}{\sqrt{1-a^2x^2}}$.
Then $du = \frac{2}{\sqrt{1-4x^2}}dx$ and $v = \int dx = x$. Using integration by parts, we have $\int \arcsin(2x)dx = \arcsin(2x)x - \int x \cdot \frac{2}{\sqrt{1-4x^2}}dx = \arcsin(2x)x - \int \frac{2x}{\sqrt{1-4x^2}}dx$. We can use the substitution $u = 1-9x^2$, $du = -8xdx$ and $-\frac{du}{8} = xdx$ to find
$\int \frac{2x}{\sqrt{1-4x^2}}dx = \int \frac{2}{\sqrt{u}} \cdot (-\frac{du}{8}) = -\int \frac{1}{4\sqrt{u}}du = -\frac{\sqrt{u}}{2} + C = -\frac{\sqrt{1-4x^2}}{2} + C$. Thus $\int \arcsin(3x)dx = x\arcsin(2x) + \frac{\sqrt{1-4x^2}}{2} + C.$

24 $\int \tan x \ln(\cos x)dx$ Solution:Let $u = \cos x$. Then $du = -\sin x dx$. $\int \tan x\ln(\cos x)dx = \int \frac{\sin x}{\cos x}\ln(\cos x)dx = -\int \frac{\ln u}{u}du = -\ln|\ln u| + C$ (By making a substitution $w = \ln u$.) $= -\ln|\ln\cos x| + C$.

25 $\int \ln(x^2+1)dx$ Solution:Let $u = \ln(x^2+1)$ and $dv = dx$. Then $du = \frac{2x}{x^2+1}$ and $v = x$. By integration by parts, $\int \ln(x^2+1)dx = x\ln(x^2+1) - \int \frac{2x^2}{x^2+1}$. By long division, we have $2x^2 = 2(x^2+1) - 2$
$\frac{2x^2}{x^2+1} = \frac{2(x^2+1)-2}{x^2+1} = 2 - \frac{2}{x^2+1}$.
So $\int \frac{2x^2}{x^2+1}dx = \int(2 - \frac{2}{x^2+1})dx = 2x - 2\arctan(x) + C.$
Thus $\int \ln(x^2+1)dx = x\ln(x^2+1) - 2x + 22\arctan(x) + C.$

26 $\int x^3\sqrt{1+x^2}dx$ Solution:Let $u = 1+x^2$. Then $du = 2xdx$ and $xdx = \frac{du}{2}$. Note that $x^2 = u-1$. Then $\int x^2\sqrt{1+x^2}xdx = \int(u-1)\sqrt{u}\frac{du}{2} = \frac{1}{2}\int(u^{\frac{3}{2}} - u^{\frac{1}{2}}) = \frac{1}{2}(\frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}}) + C = \frac{1}{5}u^{\frac{5}{2}} - \frac{1}{3}u^{\frac{3}{2}} + C = \frac{1}{5}(1+x^2)^{\frac{5}{2}} - \frac{1}{3}(1+x^2)^{\frac{3}{2}} + C.$

27 $\int \frac{x^2+10x+12}{x^3+8x^2+12x}dx$ Solution: $\frac{x^2+10x+12}{x^3+8x^2+16x}\frac{x^2+10x+12}{x(x^2+8x+16)} = \frac{x^2+10x+12}{x(x+2)(x+6)} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{x+6}$. Multiply $x(x+2)(x+6)$ to both sides, we have $x^2+10x+12 = A(x+2)(x+6) + Bx(x+6) + Cx(x+2)$.

Plug in $x = 0$, we have $12 = 12A$ and $A = 1$. Plug in $x = -2$, we have $4 - 20 + 12 = -8B$, $8B = -4$ and $B = -\frac{1}{2}$ Plug in $x = -6$, we have $36 - 60 + 12 = 24C$, $24C = -12$ and $C = -\frac{1}{2}$
So $\frac{x^2+10x+12}{x^3+8x^2+16x} = \frac{1}{x} - \frac{1}{2(x+2)} - \frac{1}{2(x+6)}$.
Thus $\int \frac{x^2+10x+12}{x^3+8x^2+16x}dx = \ln|x| - \frac{1}{2}\ln|x+2| - \frac{1}{2}\ln|x+6| + C$.

28 $\int \frac{e^{4t}}{(e^{2t}-1)^3}dt$ Solution: Let $u = e^{2t} - 1$. Then $du = 2e^{2t}dt$ and $e^{2t} = u + 1$. $\int \frac{e^{4t}}{(e^{2t}-1)^3}dt = \int \frac{e^{2t}}{(e^{2t}-1)^3}e^{2t}dt = \int \frac{u+1}{u^3}du = \int \frac{1}{u^2}du + \int \frac{1}{u^3}du = -\frac{1}{u} - \frac{1}{2u^2} + C = -\frac{1}{e^{2t}-1} - \frac{1}{2}\frac{1}{(e^{2t}-1)^2} + C$

29 $\int \frac{x^2}{x^4-1}dx$ Solution: $\frac{x^2}{x^4-1} = \frac{x^2}{(x^2-1)(x^2+1)} = \frac{x^2}{(x-1)(x+1)(x^2+1)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{Cx+D}{x^2+1}$.
One should get $A = \frac{1}{4}$, $B = -\frac{1}{4}$, $C = 0$ and $D = \frac{1}{2}$.
$\frac{x^2}{x^4-1} = \frac{1}{4}\cdot\frac{1}{x-1} - \frac{1}{4}\cdot\frac{1}{x+1} + \frac{1}{2}\cdot\frac{1}{x^2+1}$.
Thus $\int \frac{x^2}{x^4-1}dx = \frac{1}{4}\ln|x-1| - \frac{1}{4}\ln|x+1| + \frac{1}{2}\tan^{-1}x + C$.

30 $\int \frac{x^2}{(x+2)^{10}}dx$ Solution: Let $u = x + 2$. Then $du = dx$ and $x = u - 2$.
$\int \frac{x^2}{(x+2)^{10}}dx = \int \frac{(u-2)^2}{u^{10}}du = \int \frac{u^2-4u+4}{u^{10}}du = \int u^{-8} - 4u^{-9} + 4u^{-10}du$
$= -\frac{1}{7}u^{-7} + \frac{4}{8}u^{-8} - \frac{4}{9}u^{-9} + C = -\frac{1}{7(x+2)^7} + \frac{1}{2(x+2)^8} - \frac{4}{9(x+2)^9} + C$.

31 $\int \frac{2x-6}{x^2+4x+13}dx$ Solution: There is a typo in the original problem. This problem should be $\int \frac{2x-6}{x^2+4x+13}dx$.
Completing the square, we get $x^2 + 4x + 13 = x^2 + 4x + 4 + 9 = (x+2)^2 + 3^2$. Let $x + 2 = 3u$. Then $dx = 3du$ and $x = 3u - 2$. $\int \frac{2x-6}{x^2+4x+13}dx = \int \frac{2x-6}{(x+2)^2+3^3}dx = \int \frac{2\cdot(3u-2)-6}{3^2u+3^2}3du = \int \frac{6u-10}{9(u^2+1)}3du = \int \frac{6u-10}{3(u^2+1)}du = \int \frac{2u}{u^2+1}du - \frac{10}{3}\int \frac{1}{u^2+1}du = \ln|u^2+1| - \frac{10}{3}\arctan(u) + C = \ln|(\frac{x+2}{3})^2 + 1| - \frac{10}{3}\arctan(\frac{x+2}{3}) + C$. In the last step, we have used $u = \frac{x+2}{3}$.

32 $\int \frac{x^3-1}{x^3+x}dx$ Solution: From the long division, we have $x^3 - 1 = (x^3 + x) - x - 1$. $\int \frac{x^3-1}{x^3+x}dx = \int \frac{x^3+x-x-1}{x^3+x} = \int(1 - \frac{x+1}{x^3+x})dx$
$\frac{x+1}{x^3+x} = \frac{x+1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$.
Multiply $x(x^2+1)$ to both sides,
we have $x + 1 = A(x^2+1) + (Bx+C)x$.
Plug in $x = 0$, we have $A = 1$.
By comparing the coefficient, we have $B = -1$ and $C = 1$.
Thus $\frac{x+1}{x^3+x} = \frac{1}{x} + \frac{(-1x+1)}{x^2+1} = \frac{1}{x} - \frac{x}{x^2+1} + \frac{1}{x^2+1}$.
So $\int \frac{x+1}{x^3+x}dx = \ln|x| - \frac{1}{2}\ln|x^2+1| + \arctan x + C$.
Therefore $\int \frac{x^3-1}{x^3+x}dx = x - \ln|x| + \frac{1}{2}\ln|x^2+1| - \arctan x + C$.

33 $\int \frac{x+1}{x^3-x^2}dx$ Solution: $\frac{x+1}{x^3-x^2} = \frac{x+1}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1}$.

Multiply $x^2(x-1)$ to both sides, we have
$x+1 = Ax(x-1)+B(x-1)+Cx^2$.
Plug in $x=0$, we have $B=-1$.
Plug in $x=1$, we have $C=2$.
Thus $x+1 = Ax(x-1)-1(x-1)+2x^2 = Ax^2-Ax-x+1+2x^2 = (A+2)x^2-(A+1)x+1$.
By comparing the coefficient, we have $A=-2$.
So $\frac{x+1}{x^3-x^2} = -\frac{2}{x}-\frac{1}{x^2}+\frac{2}{x-1}$.
Thus $\int \frac{x+1}{x^3-x^2}dx = -2\ln|x|+\frac{1}{x}+2\ln|x+1|+C$

2. Determine whether each integral is convergent or divergent. If the integral is convergent, compute its value.

(a) $\int_1^\infty \frac{1}{x^{\frac{1}{3}}}dx$ Solution: $\int_1^\infty \frac{1}{x^{\frac{1}{3}}}dx = \lim_{b\to\infty}\int_1^b \frac{1}{x^{\frac{1}{3}}}dx = \lim_{b\to\infty}\int_1^b x^{-\frac{1}{3}}dx = \lim_{b\to\infty} \frac{3}{2}x^{\frac{2}{3}}\Big|_1^b == \lim_{b\to\infty}\frac{3}{2}b^{\frac{2}{3}}-\frac{3}{2} = \infty$. So $\int_1^\infty \frac{1}{x^{\frac{1}{3}}}dx$ diverges. Here we have used the fact that $\lim_{b\to\infty} b^p = \infty$ if $p>0$.

(b) $\int_1^\infty \frac{1}{x^{\frac{5}{4}}}dx$ Solution: $\int_1^\infty \frac{1}{x^{\frac{5}{4}}}dx = \lim_{b\to\infty}\int_1^b \frac{1}{x^{\frac{5}{4}}}dx = \lim_{b\to\infty}\int_1^b x^{-\frac{5}{4}}dx = \lim_{b\to\infty} -4x^{-\frac{1}{4}}\Big|_1^b = \lim_{b\to\infty} -4b^{-\frac{1}{4}}+4 = 4$. So $\int_1^\infty \frac{1}{x^{\frac{5}{4}}}dx$ converges. Here we have used the fact that $\lim_{b\to\infty} b^q = 0$ if $q<0$.

(c) $\int_0^\infty \frac{x^2}{x^3+1}dx$ Solution: Note that $\int \frac{x^2}{x^3+1}dx = \frac{1}{3}\ln|x^3+1|+C$ by substituting $u=x^3+1$ and $\frac{du}{3} = x^2dx$. Thus $\int_1^\infty \frac{x^2}{x^3+1}dx = \lim_{b\to\infty}\int_1^b \frac{x^2}{x^3+1}dx = \lim_{b\to\infty}\ln|x^3+1|\Big|_1^b = \lim_{b\to\infty}\ln|b^3+1|-\ln 1 = \infty$. So $\int_0^\infty \frac{x^2}{x^3+1}dx$ diverges. We have used $\lim_{b\to\infty} b^3+1=\infty$ and $\lim_{x\to\infty}\ln x = \infty$.

(d) $\int_e^\infty \frac{1}{x(\ln x)^3}dx$ Solution: We have used $u=\ln x$ and $du = \frac{1}{x}dx$ to integrate $\int \frac{1}{x(\ln x)^3}dx = \int \frac{1}{u^3}du = -\frac{1}{2}u^{-2}+C = -\frac{1}{2}\frac{1}{(\ln x)^2}+C$ and $\int_e^\infty \frac{1}{x(\ln x)^3}dx = \lim_{b\to\infty} -\frac{1}{2}\frac{1}{(\ln x)^2}|_e^b = \lim_{b\to\infty} -\frac{1}{2}\frac{1}{(\ln b)^2}+\frac{1}{2}\frac{1}{(\ln e)^2} = \frac{1}{2}$. We have used $\ln e = 1$.

(e) $\int_{-\infty}^\infty x^3dx$ Solution: Since $\int_0^\infty x^3dx = \lim_{b\to\infty}\int_0^b x^3dx = \lim_{b\to\infty}\frac{b^4}{4} = \infty$, so $\int_{-\infty}^\infty x^3dx$ diverges.

(f) $\int_{-\infty}^\infty x^2e^{-x^3}dx$ Solution: We first integrate $\int x^2e^{-x^3}dx$. Let $u=-x^3$. Then $du = -3x^2dx$ and $x^2dx = -\frac{du}{3}$. Hence $\int x^2e^{-x^3}dx = \int e^u\cdot(-\frac{du}{3}) = -\frac{e^u}{3}+C = -\frac{e^{-x^3}}{3}+C$. Thus $\int_0^\infty x^2e^{-x^3}dx = \lim_{b\to\infty}\int_0^b x^2e^{-x^3}dx = \lim_{b\to\infty} -\frac{e^{-x^3}}{3}\Big|_0^b = \lim_{b\to\infty} -\frac{e^{-b^3}}{3}+\frac{1}{3} = \frac{1}{3}$. Now we look at $\int_{-\infty}^0 x^2e^{-x^3}dx$. We have $\int_{-\infty}^0 x^2e^{-x^3}dx = \lim_{b\to-\infty}\int_b^0 x^2e^{-x^3}dx = \lim_{b\to-\infty} -\frac{e^{-x^3}}{3}\Big|_b^0 = \lim_{b\to-\infty} -\frac{1}{3}+\frac{e^{-b^3}}{3} = \infty$. Here have have used $\lim_{b\to-\infty} -b^3 = \infty$ and $lim_{x\to\infty}e^x = \infty$. Thus $\int x^2e^{-x^3}dx$ diverges. Note that $\int_{-\infty}^\infty x^2e^{-x^3}dx$

converges if both $\int_{-\infty}^{0} x^2e^{-x^3}dx$ and $\int_{0}^{\infty} x^2e^{-x^3}dx$ converge. Note that $\int_{-\infty}^{\infty} x^2e^{-x^3}dx$ diverges if either $\int_{-\infty}^{0} x^2e^{-x^3}dx$ or $\int_{0}^{\infty} x^2e^{-x^3}dx$ diverges.

3. Solve the differential equation (a) $\frac{dy}{dx} = y^2 - 4y + 3$ with $y(0) = 3$ (b)$\frac{dy}{dx} = y^2 - 4y + 3$ with $y(0) = 2$

Solution: (a) Note that $y^2 - 4y + 3 = (y-1)(y-3)$ So $y = 1$ and $y = 3$ are equilibrium of the differential equation $\frac{dy}{dx} = y^2 - 4y + 3$ So if $y(0) = 3$ then $y(x) = 3$ for all x.

(b)Note that $\frac{dy}{dx} = y^2 - 4y + 3 = (y-1)(y-3)$. Separating the variable, we have $\int \frac{1}{(y-1)(y-3)}dy = \int dx$. $\frac{1}{(y-1)(y-3)} = \frac{A}{y-1} + \frac{B}{y-3}$. Multiplying $(y-1)(y-3)$ to both sides, we have $1 = A(y-3) + B(y-1)$. Plugging $y = 1$ and $y = 3$, we have $A = -\frac{1}{2}$ and $B = \frac{1}{2}$. Thus $\frac{1}{(y-1)(y-3)} = -\frac{1}{2} \cdot \frac{1}{y-1} + \frac{1}{2} \cdot \frac{1}{y-3}$ and $\int \frac{1}{(y-1)(y-3)}dy = \int(-\frac{1}{2} \cdot \frac{1}{y-1} + \frac{1}{2} \cdot \frac{1}{y-3})dy = -\frac{1}{2}\ln|y-1| + \frac{1}{2}\ln|y-3| = \frac{1}{2}\ln|\frac{y-3}{y-1}| + c$. Thus $\int \frac{1}{(y-1)(y-3)}dy = \int dx$ can be integrated to get $\frac{1}{2}\ln|\frac{y-3}{y-1}| = x + c$, $\ln|\frac{y-3}{y-1}| = 2x + c_1$ (here $c_1 = 2c$) $\frac{y-3}{y-1} = e^{2x+c_1} = e^{c_1} \cdot e^{2x} = Ce^{2x}$ where $C = e_1^c$. From $\frac{y-3}{y-1} = Ce^{2x}$, we have $y - 3 = Ce^{2x}(y-1)$, $y - 3 = Ce^{2x}y - Ce^{2x}$, $y - Ce^{2x}y = 3 - Ce^{2x}$, $y(1 - Ce^{2x}) = 3 - Ce^{2x}$and $y = \frac{3-Ce^{2x}}{1-Ce^{2x}}$. Using the initial condition $y(0) = 2$, we have $\frac{3-Ce^0}{1-Ce^0} = 2$, $\frac{3-C}{1-C} = 2$, $3 - C = 2 - 2C$, $-C + 2C = 2 - 3$ and $C = -1$. Thus $y = \frac{3+e^{2x}}{1+e^{2x}}$.

Remark: Note that if you try the initial condition $y(0) = 3$, we have $\frac{3-C}{1-C} = 3$, $3 - C = 3 - 3C$, $2C = 0$ and $C = 0$. Thus $y = 3$

4. Suppose that $\frac{dy}{dt} = -y^2(y-3)(y-5)$

(a) Determine the equilibria of this differential equation. Solution: Solving $y^2(y-3)(y-5) = 0$, we have we have $y = 0$, $y = 3$ or $y = 5$. So the equilibria of this differential equation are $y = 0$, $y = 3$ or $y = 5$.

(b) Graph $\frac{dy}{dt}$ as a function of y, and use your graph to discuss the stability of the equilibria. Solution: We plug in the value $y = -1 < 0$, $0 < y = 1 < 3$ and $3 < y = 4 < 5$ and $5 < y = 6$ to $-y^2(y-3)(y-5)$.

y	-1	1	4
$-y^2(y-3)(y-5)$	$-(-1)^2(-1-3)(-1-5)$	$-1^2(1-3)(1-5)$	$-4^2(4-3)(4-5)$
sign	-	-	+

y	6
$-y^2(y-3)(y-5)$	$-6^2(6-3)(6-5)$
sign	-

From the graph below, we know that $y = 0$ is unstable, $y = 3$ is unstable, $y = 5$ is stable.

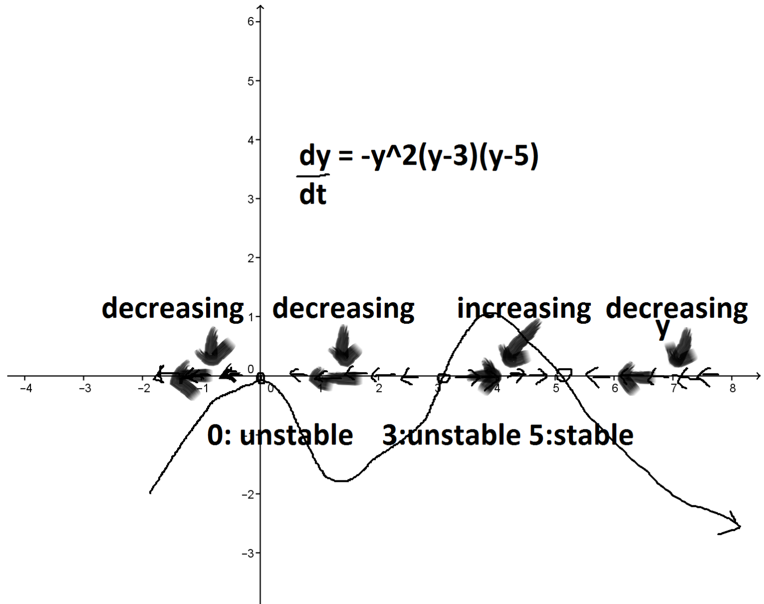

FIGURE 1. Graph for problem 4

(c) What can you say about the solution $\lim_{t\to\infty} y(t)$ if $y(0) = 1$ or $y(0) = 4$? Solution: If $y(0) = 1$, we know that y is decreasing to 0 and $\lim_{t\to\infty} y(t) = 0$. If $y(0) = 4$, we know that y is increasing to 5 and $\lim_{t\to\infty} y(t) = 5$.

5. Suppose that $\frac{dy}{dx} = g(y)$ and the graph of $\frac{dy}{dt}$ as a function of y is given by the figure above

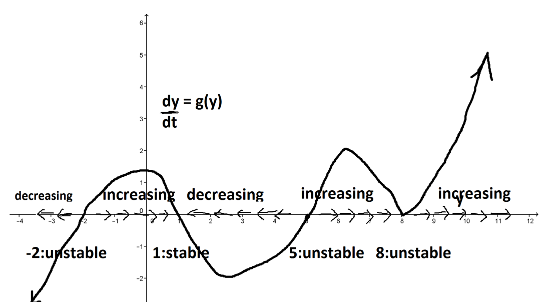

FIGURE 2. Graph for problem 5

(a) Determine the equilibria of this differential equation. Solution: The equilibrium of this differential equation are $y = -2$, $y = 1$, $y = 5$ and $y = 8$

(b) Use the graph to discuss the stability of the equilibria. Solution: From the graph on next page, we know that $y = -2$ is unstable, $y = 1$ is stable, $y = 5$ is unstable and $y = 8$ is unstable.

(c) What can you say about $\lim_{t\to\infty} y(t)$ if $y(0) = 3$ or $y(0) = 6$? Solution: If $y(0) = 3$, we know that y is decreasing to 1 and $lim_{t\to\infty}y(t) = 1$. If $y(0) = 6$, we know that y is increasing to 8 and $lim_{t\to\infty}y(t) = 8$.

6. A standard deck contains 52 different cards. In how many ways can you select 7 cards from the deck? The order of the card is not important. So the answer is $C(52, 7) = \frac{52!}{7!(52-7)!} = \frac{52!}{7!45!} = \frac{52\cdot51\cdot50\cdot49\cdot48\cdot47\cdot46}{7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1}$

7. Suppose you want to plant a flower bed with 3 different plants. You can choose from among 5 plants How many different choices do you have? The order of the flower is not important. So the answer is $C(5, 3) = \frac{5!}{3!(5-3)!} = \frac{5!}{3!2!} = \frac{5\cdot4\cdot3}{3\cdot2\cdot1} = 10$.

8. A committee of 2 people must be chosen from a group of 4. The committee consists of a president, a vice president. How many committees can be selected? Solution:There are 4 ways to choose the president and 3 ways to choose the vice president. So the answer is $4 \cdot 3 = 12$. (The order is important because the role of the committee member is different. The answer is $P(4, 2) = \frac{4!}{2!} = 4 \cdot 3 = 12$.)

9. An amino acid is encoded by triplet nucleotides (three nucleotides). How many different amino acids are possible if there are 4 different nucleotides that can be chosen for a triple? Solution: There are 4 possible choices for each nucleotides. The answer is $4 \cdot 4 \cdot 4 = 64$.

10. You have just enough time to play 3 different songs out of 5 from your favorite CD. In how many ways can you program your CD player to play the 3 songs? Solution: The order of the song is important. So the answer is $P(5, 3) = \frac{5!}{(5-3)!} = \frac{5!}{2!} = 5 \cdot 4 \cdot 3 = 60$.

11. Suppose that you want to investigate the effects of leaf damage on the performance of drought-stressed plants. You plan to use 5 levels of leaf damage and 3 different watering protocol, you plan to to have 4 replicates. What is the total number of replicates? Solution: The answer is $5 \cdot 3 \cdot 4 = 60$.

12. Ten children are divided up into three groups, of 2, 3 and 3 children, respectively. In how many ways can this be done if the order within each group is not important? Solution: The order of the children is not important. So the answer is $\frac{10!}{2!3!5!}$.