

SUMMING SOME ZETA SERIES

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Problem 1². Let $p > 2$ be a real number. Find the sum

$$\sum_{n=1}^{\infty} \left(\zeta(p) - \frac{1}{1^p} - \frac{1}{2^p} - \cdots - \frac{1}{n^p} \right),$$

where ζ denotes the **Riemann-Zeta** function¹.

Problem 2³. Let $p > 3$ be a real number. Find the sum

$$\sum_{n=1}^{\infty} n \left(\zeta(p) - \frac{1}{1^p} - \frac{1}{2^p} - \cdots - \frac{1}{n^p} \right),$$

where ζ denotes the **Riemann-Zeta** function.

Generalization 1. Let $p > 2$ be a real number and let $j \geq 0$ be a nonnegative integer. Find the sum

$$\sum_{n=1}^{\infty} \left(\zeta(p) - \frac{1}{1^p} - \frac{1}{2^p} - \cdots - \frac{1}{(n+j)^p} \right).$$

Generalization 2. Let $p > 3$ be a real number and let $j \geq 0$ be a nonnegative integer. Find the sum

$$\sum_{n=1}^{\infty} n \left(\zeta(p) - \frac{1}{1^p} - \frac{1}{2^p} - \cdots - \frac{1}{(n+j)^p} \right).$$

Generalization 3. Let $p > k + 2$ be a real number, $k \in \mathbb{N}$, and let $j \geq 0$ be a nonnegative integer. Find the sum

$$\sum_{n=1}^{\infty} n^k \left(\zeta(p) - \frac{1}{1^p} - \frac{1}{2^p} - \cdots - \frac{1}{(n+j)^p} \right).$$

Some alternating Zeta series.

Problem 4. Let $p > 1$ be a real number. Find the sum

$$\sum_{n=1}^{\infty} (-1)^{n-1} \left(\zeta(p) - \frac{1}{1^p} - \frac{1}{2^p} - \cdots - \frac{1}{n^p} \right),$$

where ζ denotes the **Riemann-Zeta** function.

¹The Riemann-Zeta function is defined by $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$, for $\Re(s) > 1$.

²Submitted to Math. Reflections.

³To appear in Elemente der Mathematik.

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Problem 5. Let $p > 2$ be a real number. Find the sum

$$\sum_{n=1}^{\infty} (-1)^{n-1} n \left(\zeta(p) - \frac{1}{1^p} - \frac{1}{2^p} - \cdots - \frac{1}{n^p} \right),$$

where ζ denotes the **Riemann-Zeta** function.

Problem 6. Let $p > 1$ be a natural number. Find the sum

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^p} \left(1 - \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{(-1)^{n-1}}{n^p} \right).$$

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