ADJOINTS OF LINEAR FRACTIONAL COMPOSITION OPERATORS ON WEIGHTED HARDY SPACES

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ABSTRACT. It is well known that on the Hardy space $H^2(\mathbb{D})$ or weighted Bergman space $A^2_\alpha(\mathbb{D})$ over the unit disk, the adjoint of a linear fractional composition operator equals the product of a composition operator and two Toeplitz operators. On $S^2(\mathbb{D})$, the space of analytic functions on the disk whose first derivatives belong to $H^2(\mathbb{D})$, Heller showed that a similar formula holds modulo the ideal of compact operators. In this paper we investigate what the situation is like on other weighted Hardy spaces.

1. INTRODUCTION

Let $\mathbb{D}$ denote the open unit disk in the complex plane. Let $\varphi : \mathbb{D} \to \mathbb{D}$ be an analytic map. The composition operator C_φ is defined by $C_\varphi f = f \circ \varphi$, where f is an analytic function on $\mathbb{D}$. Composition operators have been studied extensively on Hilbert spaces of analytic functions such as the Hardy space H^2, the weighted Bergman spaces A^2_α $(\alpha > -1)$ and the Dirichlet space $\mathcal{D}$, just to name a few. The reader is referred to the excellent books [5] and [11] for more details. Of particular interest was finding the formula for the adjoint C^*_φ on these spaces. Cowen [3] found the formula for C^*_φ on H^2 for the case φ is a linear fractional self-map of $\mathbb{D}$ (we shall call such C_φ a linear fractional composition operator). Cowen showed that if $\varphi(z) = (az + b)/(cz + d)$ is a linear fractional mapping of $\mathbb{D}$ into itself then

$$C^*_\varphi = M_g C_\sigma M^*_h, \tag{1.1}$$

where $\sigma(z) = (\bar{a}z - \bar{c})/(-\bar{b}z + \bar{d})$ is the Kreĭn adjoint of φ and M_g and M_h are multiplication operators with symbols $g(z) = (-\bar{b}z + \bar{d})^{-1}$ and $h(z) = cz + d$. Cowen's formula was later extended by Hurst [9] to weighted Bergman spaces A^2_α with $\alpha > -1$. Such formulas initiated more studies of the adjoint of linear fractional composition operators on different spaces of analytic functions and on H^2 for general rational symbols. See [6, 4, 10, 7, 1] and the references therein.

Recently, Heller [8] investigated the adjoint of C_φ acting on the space $S^2(\mathbb{D})$, which consists of analytic functions on $\mathbb{D}$ whose first derivative belongs to H^2. Let $\mathcal{K}$ denote the ideal of compact operators on $S^2(\mathbb{D})$. Heller obtained the following results.

Theorem A. *Let $\varphi(z) = az/(cz + d)$ be a holomorphic self-map of the disk and consider C_φ acting on $S^2(\mathbb{D})$. Then*

$$C^*_\varphi = M^*_G C_\sigma \ \textit{mod}\ \mathcal{K},$$

where $G(z) = (-c/a)z + 1$ and $\sigma(z) = (\bar{a}/\bar{d})z - \bar{c}/\bar{d}$ is the Kreĭn adjoint of φ.

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Theorem B. *Let $\varphi(z) = \lambda(z+u)/(1+\bar{u}z)$, $|\lambda| = 1$, $|u| < 1$, be an automorphism of the disk and consider C_φ acting on $S^2(\mathbb{D})$. Then*

$$C_\varphi^* = M_G^* C_{\varphi^{-1}} M_{1/H} \ mod\ \mathcal{K},$$

where $G(z) = -\overline{\lambda u}\, z + 1$ and $H(z) = \bar{u}z + 1$.

For a general linear fractional self-map φ, a formula for C_φ^* modulo the compact operators can be obtained by combining the above two results. Certain simplification of the above formulas was also presented in [8]. It is curious to us that Heller's formulas are not of the same form as Cowen's formula (1.1): the order of the multiplication operators is different. The purpose of the paper is to investigate the adjoints of linear fractional composition operators in a more general setting. We then explain how to recover Heller's formulas from our results.

All of the spaces mentioned above belong to the class of weighted Hardy spaces $H^2(\beta)$, where $\beta = \{\beta(n)\}_{n\geq 1}^{\infty}$ is a sequence of positive numbers. These spaces are Hilbert spaces of analytic functions on the unit disk in which the monomials $\{z^n : n \geq 0\}$ form an orthogonal basis with $\|z^n\| = \beta(n)$. We shall show that it is possible to obtain Cowen's formula modulo compact operators not only on $S^2(\mathbb{D})$ but also on a wide subclass of weighted Hardy spaces $H^2(\beta)$. Our strategy involves the family of weighted Bergman spaces A^2_α ($\alpha \in \mathbb{R}$) studied by Zhao and Zhu [12]. We use the exact formulas for the reproducing kernels of A^2_α to obtain Cowen's type formula for C_φ^* on these spaces first. We then extend our formulas to $H^2(\beta)$ for appropriate weight sequences whose term β_n behaves asymptotically as $\|z^n\|_\alpha$.

2. Adjoint formulas on A^2_α

In this section we study the adjoint of composition operators acting on weighted Bergman spaces A^2_α for $a \in \mathbb{R}$. The standard weighted Bergman spaces are defined for measures $dA_\alpha(z) = (1-|z|^2)^\alpha dA(z)$ with $\alpha > -1$. Zhao and Zhu [12] extended this definition to the case where α is any real number. For any $\alpha \in \mathbb{R}$, the space A^2_α consists of holomorphic functions f on $\mathbb{D}$ with the property that there exists an integer $k \geq 0$ with $\alpha + 2k > -1$ such that $(1-|z|^2)^k f^{(k)}(z)$ belongs to $L^2(\mathbb{D}, dA_\alpha)$, or equivalently, $f^{(k)}$ belongs to $A^2_{\alpha+2k}$. It is well know that this definition is consistent with the traditional definition for $\alpha > -1$. The reader is referred to [12] for a detailed study of A^2_α. Note that any function that is analytic on an open neighborhood of the closed unit disk belongs to A^2_α for all α.

In [12, Section 11], it was shown that each A^2_α is a reproducing kernel Hilbert space. When equipped with an appropriate inner product, the kernel of A^2_α can be computed explicitly. Depending on the value of α, we obtain three types of kernels. For each type, we show that the operator

$$C_\varphi^* - M_g C_\sigma M_h^*$$

is either zero or has finite rank, where g and h are certain analytic functions associated with φ.

For $\alpha + 2 > 0$, the kernel is

$$K_\alpha(z, w) = \frac{1}{(1 - z\bar{w})^{\alpha+2}}, \tag{2.1}$$

and

$$\|z^m\|_\alpha = \sqrt{\frac{m!\,\Gamma(\alpha+2)}{\Gamma(m+\alpha+2)}}, \quad m = 0, 1, 2 \ldots,$$

which behaves asymptotically as $m^{-(\alpha+1)/2}$ by Stirling's formula.

When $\alpha+2$ is negative and non-integer such that $-N < \alpha+2 < -N+1$ for some positive integer N, the kernel takes the form

$$K_\alpha(z,w) = \frac{(-1)^N}{(1-z\bar{w})^{\alpha+2}} + Q(z\bar{w}), \tag{2.2}$$

where Q is an analytic polynomial of degree N. In this case, for $m > N$,

$$\|z^m\|_\alpha = \sqrt{(-1)^N \frac{m!\,\Gamma(\alpha+2)}{\Gamma(m+\alpha+2)}},$$

which also behaves asymptotically as $m^{-(\alpha+1)/2}$.

In the case $\alpha+2 = -N$, where N is a non-negative integer, the kernel has the form

$$K_\alpha(z,w) = \big(\bar{w}z - 1\big)^N \log\big(\frac{1}{1-\bar{w}z}\big) + Q(\bar{w}z), \tag{2.3}$$

where Q is an analytic polynomial of degree N. For $m > N$, we have

$$\|z^m\|_\alpha = \sqrt{\frac{1}{A_m}},$$

where A_m is the coefficient of z^m in the Taylor expansion

$$(z-1)^N \log\frac{1}{1-z} = \sum_{k=0}^{\infty} A_k z^k.$$

The argument in the paragraph preceding [12, Theorem 44] shows that $\|z^m\|_\alpha$ behaves asymptotically as $m^{(N+1)/2} = m^{-(\alpha+1)/2}$ as well.

Remark 2.1. For any real number α, we see that $\|z^m\|_\alpha$ behaves asymptotically as $m^{-(\alpha+1)/2}$ when $m \to \infty$.

For the Hardy and weighted Bergman spaces (which may be identified as A^2_α with $\alpha \geq -1$), it is well known that their multiplier spaces are exactly the same as $H^\infty(\mathbb{D})$ and any composition operator induced by a holomorphic self-map of $\mathbb{D}$ is bounded. However, such results do not hold for other values of α. On the other hand, it turns out, as we shall show below, that all multiplication and composition operators discussed in this paper are bounded on all A^2_α.

For two positive quantities A and B, we write $A \lesssim B$ if there exists a constant $c > 0$ independent of the variables under consideration such that $A \leq cB$. We write $A \approx B$ if $A \lesssim B$ and $B \lesssim A$.

Let $m \geq 0$ be an integer. Recall [12, Theorem 13] that for any real number α, a function f belongs to A^2_α if and only if the mth derivative $f^{(m)}$ belongs to $A^2_{\alpha+2m}$ and

$$\|f\|_\alpha \approx \|f^{(m)}\|_{\alpha+2m}. \tag{2.4}$$

Also, if $\alpha_1 < \alpha_2$ then

$$\|\cdot\|_{\alpha_2} \lesssim \|\cdot\|_{\alpha_1}. \tag{2.5}$$

Lemma 2.2. *Let α be a real number and m be a positive integer such that $\alpha+2m > -1$. There exists a positive constant C such that if u is a function holomorphic on an open neighborhood of the closed unit disk, then M_u is a bounded operator on A^2_α and*

$$\|M_u\| \le C \max\{\|u^{(j)}\|_{L^\infty(\mathbb{D})} : 0 \le j \le m\}. \tag{2.6}$$

Proof. To simplify the notation, we put

$$\|u\|_{m,\infty} = \max\{\|u^{(j)}\|_{L^\infty(\mathbb{D})} : 0 \le j \le m\}.$$

For any $f \in A^2_\alpha$, using (2.4), we compute

$$\begin{aligned}\|uf\|_\alpha \approx \|(uf)^{(m)}\|_{\alpha+2m} &= \Big\|\sum_{j=0}^{m}\binom{m}{j}u^{(m-j)}f^{(j)}\Big\|_{\alpha+2m}\\ &\le \sum_{j=0}^{m}\binom{m}{j}\|u^{(m-j)}\|_{L^\infty(\mathbb{D})}\|f^{(j)}\|_{\alpha+2m}\\ &\le \|u\|_{m,\infty}\sum_{j=0}^{m}\binom{m}{j}\|f^{(j)}\|_{\alpha+2m}.\end{aligned}$$

Moreover, for any $0 \le j \le m$, by (2.4) and (2.5), we have

$$\|f^{(j)}\|_{\alpha+2m} \approx \|f\|_{\alpha+2m-2j} \lesssim \|f\|_\alpha.$$

Consequently,

$$\|uf\|_\alpha \lesssim \|u\|_{m,\infty}\|f\|_\alpha \sum_{j=0}^{m}\binom{m}{j} = 2^m\|u\|_{m,\infty}\|f\|_\alpha.$$

This implies (2.6) with a constant C independent of u. □

Lemma 2.3. *Let φ be a holomorphic self-map of $\mathbb{D}$ such that φ extends to a holomorphic function on an open neighborhood of the closed unit disk. Then C_φ is a bounded operator on A^2_α for any real number α.*

Proof. Fix any real number $\gamma > -1$. We shall prove that C_φ is bounded on $A^2_{-2k+\gamma}$ for all integers $k \ge 0$ by induction on k. This immediately yields the conclusion of the lemma.

Since $\gamma > -1$, A^2_γ is the weighted Bergman space with weight $(1-|z|^2)^\gamma$. It is well known that C_φ is bounded on A^2_γ, which proves our claim for the case $k = 0$. Now assume that C_φ is bounded on $A^2_{-2k+\gamma}$ for some integer $k \ge 0$. We would like to show that C_φ is bounded on $A^2_{-2k-2+\gamma}$. Since C_φ is a closed operator, it suffices to show that for any h in $A^2_{-2k-2+\gamma}$, the composition $h\circ\varphi$ belongs to $A^2_{-2k-2+\gamma}$ as well. This, in turn, is equivalent to the requirement that $(h\circ\varphi)'$ belongs to $A^2_{-2k+\gamma}$. We have $(h\circ\varphi)' = (h'\circ\varphi)\cdot\varphi'$. Since h is in $A^2_{-2k-2+\gamma}$, the derivative h' belongs to $A^2_{-2k+\gamma}$. By the induction hypothesis, $h'\circ\varphi = C_\varphi h'$ belongs to $A^2_{-2k+\gamma}$ as well. On the other hand, by our assumption about φ, Lemma 2.2 shows that multiplication by φ' is a bounded operator on $A^2_{-2k+\gamma}$. Consequently, $(h'\circ\varphi)\cdot\varphi'$ is an element of $A^2_{-2k+\gamma}$, which is what we wish to show. □

As in Heller's work, our adjoint formula for C_φ holds modulo finite rank or compact operators. We first recall a description of finite rank operators on Hilbert spaces, see, for example, [2, Exercise II.4.8].

Let $\mathcal{H}$ be a Hilbert space. For non-zero vectors $u, v \in \mathcal{H}$, we use $u \otimes v$ to denote the rank one operator $(u \otimes v)(h) = \langle h, v\rangle u$ for $h \in \mathcal{H}$.

Lemma 2.4. *A bounded linear operator $A : \mathcal{H} \to \mathcal{H}$ has rank at most m if and only if there exist $f_1, \dots, f_m$ and $g_1, \dots, g_m$ belonging to $\mathcal{H}$ such that*

$$A = f_1 \otimes g_1 + \cdots + f_m \otimes g_m.$$

When $\mathcal{H}$ is a reproducing kernel Hilbert space of analytic function, Lemma 2.4 takes a different form which will be useful for us. This result is probably well known but we provide a proof for the reason of completeness.

Lemma 2.5. *Let $\mathcal{H}$ be a Hilbert space of analytic functions on the unit disk with reproducing kernel K. Let $\mathcal{X}$ be the set of functions on $\mathbb{D} \times \mathbb{D}$ of the form*

$$f_1(z)\overline{g_1(w)} + \cdots + f_m(z)\overline{g_m(w)}, \tag{2.7}$$

where $f_1, \dots, f_m$ and $g_1, \dots, g_m$ belong to $\mathcal{H}$ and m is a positive integer. Then a bounded linear operator $A : \mathcal{H} \longrightarrow \mathcal{H}$ has finite rank if and only if the function $(z, w) \mapsto \langle AK_w, K_z\rangle$ belongs to $\mathcal{X}$. Here $K_w(z) = K(z, w)$ for $z, w \in \mathbb{D}$.

Proof. By Lemma 2.4, a bounded linear operator $A : \mathcal{H} \longrightarrow \mathcal{H}$ has finite rank if and only if there exist a positive integer m and functions $f_1, \dots, f_m$ and $g_1, \dots, g_m$ belonging to $\mathcal{H}$ such that

$$A = f_1 \otimes g_1 + \cdots + f_m \otimes g_m.$$

For $1 \le i, j \le m$, we have

$$\langle (f_j \otimes g_j)K_w, K_z\rangle = \langle K_w, g_j\rangle\langle f_j, K_z\rangle = f_j(z)\overline{g_j(w)}.$$

The conclusion of the lemma now follows from the density of the linear span of $\{K_w : w \in \mathbb{D}\}$. □

Suppose $\varphi(z) = (az + b)/(cz + d)$ is a linear fractional self-map of the unit disk. Let $\sigma(z) = (\bar{a}z - \bar{c})/(-\bar{b}z + \bar{d})$ be the Kreĭn adjoint of φ. It is known that σ is also a self-map of $\mathbb{D}$. Let $\eta(z) = (cz + d)^{-1}$ and $\mu(z) = -\bar{b}z + \bar{d}$. Then η and μ are bounded analytic functions on a neighborhood of the closed unit disk and

$$1 - \overline{\varphi(w)}z = \mu(z)\big(1 - \bar{w}\sigma(z)\big)\overline{\eta(w)}.$$

Consequently, by choosing appropriate branches of the logarithms, we have

$$\log\big(1 - \overline{\varphi(w)}z\big) = \log(\mu(z)) + \log\big(1 - \bar{w}\sigma(z)\big) + \log(\overline{\eta(w)}). \tag{2.8}$$

Therefore, for any real number γ,

$$\Big(1 - \overline{\varphi(w)}z\Big)^{\gamma} = \mu(z)^{\gamma}\Big(1 - \bar{w}\sigma(z)\Big)^{\gamma}\overline{\eta(w)^{\gamma}} \tag{2.9}$$

for z, w in $\mathbb{D}$.

We are now in a position to discuss the adjoints of composition operators induced by linear fractional maps. In the following theorem, we consider the first two types of kernels.

Theorem 2.6. *Let α be a real number such that $\alpha + 2$ is not zero nor a negative integer. Let $\varphi(z) = (az + b)/(cz + d)$ be a linear fractional self-map of the unit disk and σ be its Kreĭn adjoint. Let $g(z) = (-\bar{b}z + \bar{d})^{-\alpha-2}$ and $h(z) = (cz + d)^{\alpha+2}$ for $z \in \mathbb{D}$. Then $C_\varphi^* - M_g C_\sigma M_h^*$ has finite rank on A^2_α. In the case $\alpha + 2 > 0$, we actually have the identity $C_\varphi^* = M_g C_\sigma M_h^*$.*

Remark 2.7. Lemmas 2.2 and 2.3 show that the operators C_φ, C_σ, M_g and M_h are all bounded on A^2_α.

Proof of Theorem 2.6. As we mentioned before, the case $\alpha > -1$ was considered by Hurst [9]. His proof works also for $-2 < \alpha \leq -1$ since the kernels have the same form. Here we only need to investigate the case $-N < \alpha + 2 < -N + 1$ for some positive integer N. To simplify the notation, let $\gamma = -(\alpha+2)$. We then rewrite the kernel as $K(z,w) = (-1)^N(1-\bar{w}z)^\gamma + Q(\bar{w}z)$ for $z, w \in \mathbb{D}$. Set $K_w(z) = K(z,w)$ for $z, w \in \mathbb{D}$. We shall make use of the following identities, which are well known:

$$M_h^* K_w = \overline{h(w)} K_w, \quad M_g^* K_z = \overline{g(z)} K_z, \quad C_\varphi^* K_w = K_{\varphi(w)}.$$

We now compute

$$\begin{aligned}
\langle (C_\varphi^* - M_g C_\sigma M_h^*) K_w, K_z \rangle &= K(z, \varphi(w)) - g(z) K(\sigma(z), w)\overline{h(w)} \\
&= (-1)^N (1 - \overline{\varphi(w)} z)^\gamma + Q(\overline{\varphi(w)} z) \\
&\quad - g(z)\Big((-1)^N (1 - \bar{w}\sigma(z))^\gamma + Q(\bar{w}\sigma(z))\Big)\overline{h(w)} \\
&= Q(\overline{\varphi(w)} z) - g(z) Q(\bar{w}\sigma(z))\overline{h(w)} \quad \text{(using (2.9))}.
\end{aligned}$$

Since g and h are analytic on a neighborhood of the closed unit disk and Q is a polynomial, the last function has the form (2.7). Consequently, Lemma 2.5 shows that $C_\varphi^* - M_g C_\sigma M_h^*$ has finite rank. □

The following theorem considers the third type of kernel.

Theorem 2.8. *Let α be a real number such that $\alpha+2$ is zero or a negative integer. Let $\varphi(z) = (az+b)/(cz+d)$ be a linear fractional self-map of the unit disk and σ be its Kreĭn adjoint. Let $g(z) = (-\bar{b}z + \bar{d})^{-\alpha-2}$ and $h(z) = (cz+d)^{\alpha+2}$ for $z \in \mathbb{D}$. Then $C_\varphi^* - M_g C_\sigma M_h^*$ has finite rank on A^2_α.*

Proof. Let $N = -\alpha - 2$. Then N is a nonnegative integer. Recall that the kernel in this case has the form

$$K(z,w) = \big(\bar{w}z - 1\big)^N \log\big(\frac{1}{1-\bar{w}z}\big) + Q(\bar{w}z)$$

for $z, w \in \mathbb{D}$, where Q is an analytic polynomial. We compute

$$\begin{aligned}
&\langle (C_\varphi^* - M_g C_\sigma M_h^*) K_w, K_z \rangle \\
&= K\big(z, \varphi(w)\big) - g(z) K\big(\sigma(z), w\big)\overline{h(w)} \\
&= -\big(\overline{\varphi(w)} z - 1\big)^N \log\big(1 - \overline{\varphi(w)} z\big) + g(z)(\bar{w}\sigma(z) - 1)^N \log(1 - \bar{w}\sigma(z))\overline{h(w)} \\
&\qquad + Q(\overline{\varphi(w)} z) - g(z) Q(\bar{w}\sigma(z))\overline{h(w)}.
\end{aligned}$$

Since $g(z)(\bar{w}\sigma(z) - 1)^N \overline{h(w)} = (\overline{\varphi(w)} z - 1)^N$, using (2.8), we simplify the first two terms in the last expression as

$$\begin{aligned}
&\big(\overline{\varphi(w)} z - 1\big)^N \Big(-\log(1 - \overline{\varphi(w)} z) + \log(1 - \bar{w}\sigma(z))\Big) \\
&\qquad = -\big(\overline{\varphi(w)} z - 1\big)^N \Big(\log(\mu(z)) + \log(\overline{\eta(w)})\Big),
\end{aligned}$$

where $\eta(z) = (cz+d)^{-1}$ and $\mu(z) = -\bar{b}z + \bar{d}$ for $z \in \mathbb{D}$. Consequently,

$$\begin{aligned}\langle (C_\varphi^* - M_g C_\sigma M_h^*)K_w, K_z\rangle = -\big(\overline{\varphi(w)}z - 1\big)^N \Big(\log(\mu(z)) + \log(\overline{\eta(w)})\Big) \\ + Q(\overline{\varphi(w)}z) - g(z)Q(\bar{w}\sigma(z))\overline{h(w)}.\end{aligned}$$

Since N is a nonnegative integer and Q is a polynomial, the expression on the right hand side is an element of the form (2.7). Lemma 2.5 shows that $C_\varphi^* - M_g C_\sigma M_h^*$ has finite rank. □

3. Adjoint formulas on $H^2(\beta)$

In this section we would like to generalize the results in Section 2 to certain weighted Hardy spaces $H^2(\beta)$. We begin with an auxiliary result. For $s = 1, 2$, consider a Hilbert space H_s of analytic functions on the unit disk such that

$$\langle z^j, z^\ell\rangle = \begin{cases} 0 & \text{if } j \neq \ell, \\ \beta_s^2(j) & \text{if } j = \ell. \end{cases}$$

Here, $\{\beta_s(n)\}_{n=0}^\infty$ is a sequence of positive real numbers with $\liminf_{n\to\infty} \beta_s(n)^{1/n} = 1$. Such restriction guarantees that elements of H_s are analytic functions on the unit disk, see for example, [5, Exercise 2.1.10]. Assume that

$$\lim_{n\to\infty} \frac{\beta_2(n)}{\beta_1(n)} = \alpha > 0. \tag{3.1}$$

It is clear that the norms on H_1 and H_2 are equivalent. We claim that there is a compact operator $K : H_2 \to H_2$ such that for all functions $f, g \in H_1$,

$$\alpha^2\langle f, g\rangle_1 = \langle f, g\rangle_2 + \langle Kf, g\rangle_2. \tag{3.2}$$

In fact, define the operator $K : H_2 \to H_2$ by

$$K(z^n) = \Big(\frac{\alpha^2\beta_1(n)^2}{\beta_2(n)^2} - 1\Big)z^n,$$

for $n = 0, 1, \ldots$ and extend by linearity and continuity to all H_2. We see that K is a self-adjoint diagonal operator with respect to the orthonormal basis of monomials. By (3.1), [2, Proposition II.4.6] shows that K is a compact operator on H_2, hence on H_1 as well. It is clear that (3.2) holds for $f(z) = z^j$ and $g(z) = z^\ell$ if $j \neq \ell$. If $j = \ell$, then we compute

$$\begin{aligned}\alpha^2\langle z^j, z^j\rangle_1 = \alpha^2\beta_1^2(j) &= \beta_2^2(j) + \Big(\frac{\alpha^2\beta_1(j)^2}{\beta_2(j)^2} - 1\Big)\beta_2^2(j) \\ &= \langle z^j, z^j\rangle_2 + \langle Kz^j, z^j\rangle_2.\end{aligned}$$

Linearity and boundedness of K then shows that (3.2) holds for all $f, g \in H_1$.

Proposition 3.1. *Let A be a bounded linear operator on H_1 (hence, A is also bounded on H_2). Let B_s be the adjoint of A on H_s for $s = 1, 2$. Then $B_2 - B_1$ is a compact operator on H_2 (hence, on H_1 as well).*

Proof. For $f, g \in H_2$, we have

$$\begin{aligned}\langle B_2(I+K)f, g\rangle_2 &= \langle (I+K)f, Ag\rangle_2 \quad \text{(since } B_2 \text{ is the adjoint of } A \text{ in } H_2)\\ &= \alpha^2\langle f, Ag\rangle_1 \quad \text{(by (3.2))}\\ &= \alpha^2\langle B_1 f, g\rangle_1 \quad \text{(since } B_1 \text{ is the adjoint of } A \text{ in } H_1)\\ &= \langle (I+K)B_1 f, g\rangle_2 \quad \text{(by (3.2))}.\end{aligned}$$

This implies $B_2(I+K) = (I+K)B_1$, which shows that $B_2 - B_1 = KB_1 - B_2K$. Since K is compact on H_2, we conclude that $B_2 - B_1$ is compact as well. □

We now state and prove our main result in this section.

Theorem 3.2. *Let t be a real number. Suppose $\beta = \{\beta(n)\}_{n=0}^{\infty}$ is a sequence of positive numbers such that*

$$\lim_{n\to\infty} \frac{\beta(n)}{n^t} = \ell, \tag{3.3}$$

where $0 < \ell < \infty$. Let $\varphi(z) = (az+b)/(cz+d)$ be a linear fractional self-map of the unit disk and σ be its Kreĭn adjoint. Let $g(z) = (-\bar{b}z + \bar{d})^{2t-1}$ and $h(z) = (cz+d)^{-2t+1}$. Then the difference $C_\varphi^* - M_g C_\sigma M_h^*$ is a compact operator on $H^2(\beta)$.

Proof. Let $\alpha = -2t-1$. Then $t = -(\alpha+1)/2$ and we have

$$\begin{aligned}\lim_{m\to\infty} \frac{\beta(m)}{\|z\|_\alpha} &= \Big(\lim_{m\to\infty} \frac{\beta(m)}{m^t}\Big)\Big(\lim_{m\to\infty} \frac{m^t}{\|z^m\|_\alpha}\Big)\\ &= \ell \lim_{m\to\infty} \frac{m^{-(\alpha+1)/2}}{\|z^m\|_\alpha}.\end{aligned}$$

The last limit is a finite positive number by Remark 2.1. This, in particular, says that the spaces A_α^2 and $H^2(\beta)$ are the same with equivalent norms. For any bounded operator T on these spaces, we write $T^{*,\alpha}$ for the adjoint of T as an operator on A_α^2 and $T^{*,\beta}$ for the adjoint of T as an operator on $H^2(\beta)$.

By Theorems 2.6 and 2.8, the difference $K = C_\varphi^{*,\alpha} - M_g C_\sigma M_h^{*,\alpha}$ is compact on A_α^2, hence on $H^2(\beta)$ as well.

On the other hand, applying Proposition 3.1 with $H_1 = A_\alpha^2$ and $H_2 = H^2(\beta)$, we have $C_\varphi^{*,\beta} = C_\varphi^{*,\alpha} + K_1$ and $M_h^{*,\beta} = M_h^{*,\alpha} + K_2$ for some compact operators K_1, K_2 on $H^2(\beta)$. Consequently,

$$\begin{aligned}C_\varphi^{*,\beta} - M_g C_\sigma M_h^{*,\beta} &= (C_\varphi^{*,\alpha} + K_1) - M_g C_\sigma (M_h^{*,\alpha} + K_2)\\ &= C_\varphi^{*,\alpha} - M_g C_\sigma M_h^{*,\alpha} + K_1 - M_g C_\sigma K_2\\ &= K + K_1 - M_g C_\sigma K_2,\end{aligned}$$

which is compact on $H^2(\beta)$. This completes the proof of the theorem. □

We now explain how one obtains Heller's results from our Theorem 3.2. Let φ be a holomorphic self-map of the unit disk. We shall consider two particular cases: the case $\varphi(0) = 0$ and the case φ is an automorphism.

Corollary 3.3. *Let $\beta = \{\beta(n)\}_{n=0}^{\infty}$ be a sequence of positive numbers satisfying the condition (3.3). Let $\varphi(z) = az/(cz+d)$ be a holomorphic self-map of the disk and consider C_φ acting on $H^2(\beta)$. Then we have*

$$C_\varphi^* = M_G^* C_\sigma \ \ mod\ \mathcal{K},$$

where $G(z) = \big(-(c/a)z+1\big)^{2t-1}$ *and* $\sigma(z) = (\bar a/\bar d)z - \bar c/\bar d$ *is the Kreĭn adjoint of* φ.

Proof. Theorem 3.2 shows that

$$C_\varphi^* = M_g C_\sigma M_h^* \mod \mathcal{K}, \tag{3.4}$$

where $g(z) = (\bar d)^{2t-1}$, $h(z) = (cz+d)^{-2t+1}$. Since g is a constant function, we may combine it with h and rewrite (3.4) as $C_\varphi^* = C_\sigma M_{h_1}^* \mod \mathcal{K}$, where $h_1(z) = \big(d/(cz+d)\big)^{2t-1}$. It then follows that $C_\sigma = C_\varphi^* M_{1/h_1}^* \mod \mathcal{K}$. Now, a direct calculation verifies that $h_1 = G\circ\varphi$. We then compute

$$C_\sigma = C_\varphi^* M_{1/G\circ\varphi}^* = \Big(M_{1/G\circ\varphi} C_\varphi\Big)^* = \Big(C_\varphi M_{1/G}\Big)^* = M_{1/G}^* C_\varphi^*.$$

Multiplying by M_G^* on the left gives $C_\varphi^* = M_G^* C_\sigma \mod \mathcal{K}$ as desired. □

Corollary 3.4. *Let* $\beta = \{\beta(n)\}_{n=0}^\infty$ *be a sequence of positive numbers satisfying the condition* (3.3). *Let* $\varphi(z) = \lambda(z+u)/(1+\bar u z)$, $|\lambda| = 1$, $|u| < 1$, *be an automorphism of the disk and consider* C_φ *acting on* $H^2(\beta)$. *Then*

$$C_\varphi^* = M_G^* C_{\varphi^{-1}} M_{1/H} \ \textit{mod}\ \mathcal{K},$$

where $G(z) = (-\overline{\lambda u}\, z+1)^{2t-1}$ *and* $H(z) = (\bar u z+1)^{2t-1}$.

Proof. It can be verified that $\sigma = \varphi^{-1}$. Theorem 3.2 gives

$$C_\varphi^* = M_g C_{\varphi^{-1}} M_h^* \mod \mathcal{K},$$

where $g(z) = (-\overline{\lambda u} z+1)^{2t-1}$ and $h(z) = (\bar u z+1)^{-2t+1}$. Taking adjoints gives

$$\begin{aligned} C_\varphi &= \Big(M_g C_{\varphi^{-1}} M_h^*\Big)^* \mod \mathcal{K} \\ &= M_h C_{\varphi^{-1}}^* M_g^* \mod \mathcal{K}, \end{aligned}$$

which implies

$$M_{1/h} C_\varphi M_{1/g}^* = C_{\varphi^{-1}}^* \mod \mathcal{K}.$$

Taking inverses then yields

$$C_\varphi^* = \big(C_{\varphi^{-1}}^*\big)^{-1} = \Big(M_{1/h} C_\varphi M_{1/g}^*\Big)^{-1} = M_g^* C_{\varphi^{-1}} M_h \mod \mathcal{K}.$$

Since $g = G$ and $h = 1/H$, the conclusion of the corollary follows. □

The space $S^2(\mathbb{D})$ can be identified as $H^2(\beta)$, where the weight sequence $\beta = \{\beta(n)\}_{n\ge 0}$ is given by $\beta(0) = 1$ and $\beta(n) = n$ for all $n \ge 1$. This sequence satisfies condition (3.3) with $t = 1$. Consequently, Theorem A follows from Corollary 3.3 and Theorem B follows from Corollary 3.4.

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References

[1] Paul S. Bourdon and Joel H. Shapiro, *Adjoints of rationally induced composition operators*, J. Funct. Anal. **255** (2008), no. 8, 1995–2012. MR 2462584 (2009m:47056)

[2] John B. Conway, *A course in functional analysis*, second ed., Graduate Texts in Mathematics, vol. 96, Springer-Verlag, New York, 1990. MR 1070713 (91e:46001)

[3] Carl C. Cowen, *Linear fractional composition operators on H^2*, Integral Equations Operator Theory **11** (1988), no. 2, 151–160. MR 928479 (89b:47044)

[4] Carl C. Cowen and Eva A. Gallardo-Gutiérrez, *A new class of operators and a description of adjoints of composition operators*, J. Funct. Anal. **238** (2006), no. 2, 447–462. MR 2253727 (2007e:47033)

[5] Carl C. Cowen and Barbara D. MacCluer, *Composition operators on spaces of analytic functions*, Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 1995. MR 1397026 (97i:47056)

[6] Eva A. Gallardo-Gutiérrez and Alfonso Montes-Rodríguez, *Adjoints of linear fractional composition operators on the Dirichlet space*, Math. Ann. **327** (2003), no. 1, 117–134. MR 2005124 (2004h:47036)

[7] Christopher Hammond, Jennifer Moorhouse, and Marian E. Robbins, *Adjoints of composition operators with rational symbol*, J. Math. Anal. Appl. **341** (2008), no. 1, 626–639. MR 2394110 (2009b:47035)

[8] Katherine Heller, *Adjoints of linear fractional composition operators on $S^2(\mathbb{D})$*, J. Math. Anal. Appl. **394** (2012), no. 2, 724–737. MR 2927493

[9] Paul R. Hurst, *Relating composition operators on different weighted Hardy spaces*, Arch. Math. (Basel) **68** (1997), no. 6, 503–513. MR 1444662 (98c:47040)

[10] María J. Martín and Dragan Vukotić, *Adjoints of composition operators on Hilbert spaces of analytic functions*, J. Funct. Anal. **238** (2006), no. 1, 298–312. MR 2253017 (2007e:47035)

[11] Joel H. Shapiro, *Composition operators and classical function theory*, Universitext: Tracts in Mathematics, Springer-Verlag, New York, 1993. MR 1237406 (94k:47049)

[12] Ruhan Zhao and Kehe Zhu, *Theory of Bergman spaces in the unit ball of $\mathbb{C}^n$*, Mém. Soc. Math. Fr. (N.S.) (2008), no. 115, vi+103 pp. MR 2537698 (2010g:32010)

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