ALGEBRAIC PROPERTIES AND THE FINITE RANK PROBLEM FOR TOEPLITZ OPERATORS ON THE SEGAL-BARGMANN SPACE

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ABSTRACT. We study three different problems in the area of Toeplitz operators on the Segal-Bargmann space in $\mathbb{C}^n$. Extending results obtained previously by the first author and Y.L. Lee, and by the second author, we first determine the commutant of a given Toeplitz operator with a radial symbol belonging to the class $\mathrm{Sym}_{>0}(\mathbb{C}^n)$ of symbols having certain growth at infinity. We then provide explicit examples of zero products of non-trivial Toeplitz operators. These examples show the essential difference between Toeplitz operators on the Segal-Bargmann space and on the Bergman space over the unit ball. Finally, we discuss the "finite rank problem". We show that there are no non-trivial rank one Toeplitz operators T_f for $f \in \mathrm{Sym}_{>0}(\mathbb{C}^n)$. In all these problems, the growth at infinity of the symbols plays a crucial role.

1. INTRODUCTION

The first part of the present paper is a continuation and extension of [4], where commuting Toeplitz operators on the Segal-Bargmann space $H^2(\mathbb{C}^n, d\mu)$ of Gaussian square integrable entire functions on $\mathbb{C}^n$ were analyzed. With the orthogonal projection P from the enveloping L^2-space onto $H^2(\mathbb{C}^n, d\mu)$ and a suitable complex-valued function f on $\mathbb{C}^n$, the Toeplitz operator T_f is defined as $T_f = PM_f|_{H^2(\mathbb{C}^n,d\mu)}$, where M_f denotes the operator of multiplication by f. We call f the symbol of T_f. When f is a radial function (i.e. $f(z) = f(|z|)$), it follows from the rotation-invariance of the Gaussian measure that the operator T_f is diagonalizable. In their previous work, the first author and Lee determined the set $\mathrm{Com}(T_f)$ of Toeplitz operators belonging to the commutant of T_f when $n = 1$. Recently in [3], the commutant $\mathrm{Com}(T_p)$ was also studied in the case of a non-radial monomial symbol $p(z) = z^\ell \bar{z}^k$ where ℓ, k are non-negative integers, or even for a more general quasi-homogeneous function on the complex plane. In [4] the higher dimensional situation was only considered for operators with polynomial symbols, which actually form an algebra. Due to this additional structure, they can be treated more easily than operators with general symbols.

Analogous results for Toeplitz operators acting on the Bergman space were proved earlier by Čučković and Rao in [6] for the unit disc $\mathbb{D}$ and subsequently in [13], where the second author extended the result in [6] to the Bergman space $A^2(\mathbb{B}_n)$ over the unit ball $\mathbb{B}_n$ in $\mathbb{C}^n$. In this paper, by combining the techniques in [4] and [13], we successfully describe $\mathrm{Com}(T_f)$ when f is a radial function belonging to the class $\mathrm{Sym}_{>0}(\mathbb{C}^n)$ of generally unbounded functions. The class $\mathrm{Sym}_{>0}(\mathbb{C}^n)$ consists of all functions g on $\mathbb{C}^n$ for which the function $z \mapsto g(z)\exp(-c|z|^2)$ is bounded for all $c > 0$. In particular, $\mathrm{Sym}_{>0}(\mathbb{C}^n)$ contains all functions which have at most polynomial growth at infinity. We thus recover the results obtained in [4].

On the level of symbols in $L^\infty(\mathbb{B}_n)$ and in $\mathrm{Sym}_{>0}(\mathbb{C}^n)$, it turns out that the characterization of $\mathrm{Com}(T_f)$ is the same in the Bergman space and Segal-Bargmann space. More precisely, as the following theorem shows, $\mathrm{Com}(T_f)$ consists of Toeplitz operators whose symbols are invariant under the torus action on $\mathbb{B}_n$ and $\mathbb{C}^n$, respectively.

Theorem A. *Let $f \in L^\infty(\mathbb{B}_n)$ be non-trivial and radial on $\mathbb{B}_n$ (respectively, $f \in \mathrm{Sym}_{>0}(\mathbb{C}^n)$ be non-trivial and radial on $\mathbb{C}^n$) and $g \in L^\infty(\mathbb{B}_n)$ (respectively, $g \in \mathrm{Sym}_{>0}(\mathbb{C}^n)$). Then $[T_f, T_g] = 0$ on $A^2(\mathbb{B}_n)$ (respectively, $H^2(\mathbb{C}^n, d\mu)$) if and only if $g(e^{i\theta}z) = g(z)$ for a.e. $\theta \in \mathbb{R}$ and a.e. z in $\mathbb{B}_n$ (respectively, $\mathbb{C}^n$).*

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We in fact study a more general problem. For given radial functions f_1 and f_2 in $\mathrm{Sym}_{>0}(\mathbb{C}^n)$, we determine solutions g of the operator equation $T_{f_1}T_g = T_gT_{f_2}$ (see Theorem 3.4). Theorem A is then derived from the case $f_1 = f_2$. It was pointed out in [4] that Theorem A fails for Toeplitz operators on the Segal-Bargmann space whose symbols have higher order of growth at infinity, even in the framework of bounded operators. This fact is related to the distribution of zeros of the Mellin transform of a radial function.

In the second part of the paper we study zero-products of Toeplitz operators on the Segal-Bargmann space. On the Bergman space over the unit ball, the following result was recently proved by the second author in [12]: assume that for a finite number of bounded functions $f_1, \ldots, f_N$ on $\mathbb{B}_n$ all of which, except possibly one, are radial, the product $T_{f_1} \cdots T_{f_N}$ of Toeplitz operators vanishes, then one of the symbols f_j must be zero. We will show in Section 3 that the same statement is true for Toeplitz operators on $H^2(\mathbb{C}^n, d\mu)$ having symbols in $\mathrm{Sym}_{>0}(\mathbb{C}^n)$ when $N = 2$. Surprisingly, we found a counterexample in the case $N = 3$, even for bounded symbols.

Theorem B. *There are non-zero bounded radial functions f_0, f_1, f_2 on $\mathbb{C}^n$ such that $T_{f_0}T_{f_1}T_{f_2} = 0$ on the Segal-Bargmann space $H^2(\mathbb{C}^n, d\mu)$.*

It is still open (also for operators on Bergman spaces) whether $T_fT_g = 0$ implies $f = 0$ or $g = 0$ when f and g are arbitrary bounded functions. On the other hand, there are counterexamples in the case where at least one of the symbols f or g does not belong to $\mathrm{Sym}_{>0}(\mathbb{C}^n)$ (see Proposition 4.5).

It is well known [2] that the quantization map $\mathrm{Sym}_{>0}(\mathbb{C}^n) \ni f \mapsto T_f$ is one-to-one. On the other hand, Grudsky and Vasilevski showed in [9] the existence of non-zero radial symbols f of high growth order at infinity with $T_f = 0$. We give explicit examples of such radial functions f in Section 4. In these examples we deal with operator symbols $f \notin L^2(\mathbb{C}^n, d\mu)$ and we need to employ a natural extension [10, 11] of the above notion of Toeplitz operators. More precisely, for a measurable symbol f, we define the operator $\widetilde{T}_f$ so that with their maximal domains of definition we have $T_f \subseteq \widetilde{T}_f$. We then construct non-trivial radial symbols $f, g, h \notin L^2(\mathbb{C}^n, d\mu)$ for which the following is true (see Propositions 4.5 and 4.6).

Theorem C. *The analytic polynomials $\mathbb{P}[z]$ are contained in the domains of $\widetilde{T}_f, \widetilde{T}_g$, and $\widetilde{T}_h$; $\mathbb{P}[z]$ forms an invariant subspace for $\widetilde{T}_g$ and $\widetilde{T}_h$. Furthermore, $\widetilde{T}_f = 0$, and $\widetilde{T}_g, \widetilde{T}_h \neq 0$ but $\widetilde{T}_g\widetilde{T}_h = \widetilde{T}_h\widetilde{T}_g = 0$.*

In the last part of the paper we consider the finite rank problem for Toeplitz operators on $H^2(\mathbb{C}^n, d\mu)$ and give some partial results towards the question: *If the Toeplitz operator T_f with $f \in \mathrm{Sym}_{>0}(\mathbb{C}^n)$ has finite rank, does it follow that $f = 0$?* For Toeplitz operators on the Bergman space over a domain $\Omega \subset \mathbb{C}^n$, this problem had been considered for a long time and positive answers were given in the cases where Ω is bounded or where $\Omega = \mathbb{C}^n$ and f has compact support [1, 5, 14, 16]. It fact, Alexandrov and Rozenblum [1] obtained the affirmative answer even when f is replaced by a compactly supported distribution.

For Toeplitz operators with symbols in $\mathrm{Sym}_{>0}(\mathbb{C}^n)$ acting on $H^2(\mathbb{C}^n, d\mu)$, the problem remains open. We will show that the problem can be reduced to the complex one dimensional case $n = 1$. Moreover, we show the non-existence of non-trivial rank one Toeplitz operators.

Theorem D. *Let f be in $\mathrm{Sym}_{>0}(\mathbb{C}^n)$ such that T_f has at most rank one on the space of analytic polynomials, then $f(z) = 0$ for a.e z.*

Unfortunately, it seems not easy to generalize our proof of Theorem D to the case of higher ranks. On the other hand, our approach leads to a necessary criterion for a bounded finite rank operator on $H^2(\mathbb{C}^n, d\mu)$ to be represented as a Toeplitz operator. This criterion seems hard to be fulfilled and can be used to exclude various finite rank operators as candidates for Toeplitz operators.

The paper is organized as follows. In Section 2 we introduce the class of Toeplitz operators with (generally unbounded) symbols belonging to $\mathrm{Sym}_{>0}(\mathbb{C}^n)$. These operators can be considered as acting on a scale of Banach spaces [2, 4]. In particular, finite products of such operators are well-defined with dense domains. We also recall some of the results in [4] which will be used extensively in our proofs. In Section 3 we discuss the commuting problem. Section 4 provides examples of zero-products of non-trivial Toeplitz operators whose domains contain all analytic polynomials. The discussion of the finite rank problem for Toeplitz operators is contained in Section 5. Finally, we mention some open problems that are related to our results.

2. PRELIMINARIES

For a fixed positive integer n, let μ be the normalized Gaussian measure on $\mathbb{C}^n$ defined by

$$d\mu(z) = \pi^{-n} e^{-|z|^2} dV(z). \tag{2.1}$$

Here for $z, w \in \mathbb{C}^n$, we write $z \cdot \overline{w} = z_1\overline{w}_1 + \cdots + z_n\overline{w}_n$ and $|z| = \sqrt{z \cdot \overline{z}}$. Also, dV denotes the usual Lebesgue measure on $\mathbb{C}^n \cong \mathbb{R}^{2n}$. The *Segal-Bargmann space* $H^2(\mathbb{C}^n, d\mu)$ consists of all μ-square integrable entire functions on $\mathbb{C}^n$. It is well known that $H^2(\mathbb{C}^n, d\mu)$ is a closed subspace of $L^2(\mathbb{C}^n, d\mu)$. Let $\mathbb{N}_0$ denote the set of all non-negative integers. For any $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$ and $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$, we write $\alpha! = \alpha_1! \cdots \alpha_n!$ and $z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$. It is standard that the set $\mathcal{B} = \{e_\alpha(z) = (\alpha!)^{-1/2} z^\alpha \mid \alpha \in \mathbb{N}_0^n\}$ forms an orthonormal basis for $H^2(\mathbb{C}^n, d\mu)$, usually referred to as the standard orthonormal basis. The space $H^2(\mathbb{C}^n, d\mu)$ is in fact a reproducing kernel Hilbert space with kernel function $K_z(w) = K(w, z) = e^{w\cdot\overline{z}}$ for $(w, z) \in \mathbb{C}^n \times \mathbb{C}^n$. For any $h \in H^2(\mathbb{C}^n, d\mu)$ and $z \in \mathbb{C}^n$, we have $h(z) = \langle h, K_z\rangle$. Here $\langle \cdot, \cdot\rangle$ is the usual inner product in $L^2(\mathbb{C}^n, d\mu)$.

Let P denote the orthogonal projection from $L^2(\mathbb{C}^n, d\mu)$ onto $H^2(\mathbb{C}^n, d\mu)$. For any measurable function f, the Toeplitz operator T_f is defined as the compression of the multiplication operator M_f on $H^2(\mathbb{C}^n, d\mu)$, that is, $T_f = PM_f|_{H^2(\mathbb{C}^n, d\mu)}$. The natural domain of T_f is the space of all functions $h \in H^2(\mathbb{C}^n, d\mu)$ for which fh belongs to $L^2(\mathbb{C}^n, d\mu)$. For such a function h and for any $z \in \mathbb{C}^n$, using the reproducing property of the kernel functions, we have

$$T_f h(z) = \langle T_f h, K_z\rangle = \langle P(fh), K_z\rangle = \langle fh, K_z\rangle = \int_{\mathbb{C}^n} f(w)h(w)e^{z\cdot\overline{w}} d\mu(w). \tag{2.2}$$

For any number $c > 0$, we define the space

$$\mathcal{D}_c = \Big\{ f : \mathbb{C}^n \to \mathbb{C} \text{ measurable} \;\Big|\; \exists\, d > 0 \text{ such that } |f(z)| \le de^{c|z|^2} \text{ a.e. } z \in \mathbb{C}^n \Big\}.$$

One can check easily that if f belongs to $\mathcal{D}_c$ for some $c < 1/2$, then T_f has a dense domain in $H^2(\mathbb{C}^n, d\mu)$. In fact, the domain of T_f contains all entire functions belonging to $\mathcal{D}_{c'}$ with $0 < c' < 1/2 - c$. In this paper we are interested in products of Toeplitz operators. Unfortunately, a product of the form $T_{f_1}T_{f_2}$ for $f_1 \in \mathcal{D}_{c_1}$ and $f_2 \in \mathcal{D}_{c_2}$ can only be defined with a dense domain if the values of c_1 and c_2 satisfy certain restrictions. Because of this, we will follow [2] by restricting our attention to the space of symbols

$$\mathrm{Sym}_{>0}(\mathbb{C}^n) = \bigcap_{c>0} D_c = \big\{ f : \mathbb{C}^n \to \mathbb{C} \;\big|\; z \mapsto f(z)e^{-c|z|^2} \text{ is bounded for all } c > 0 \big\}.$$

With the increasing sequence $(c_j)_{j\in\mathbb{N}_0}$ defined by $c_j = 1/2 - 1/(2j + 2)$ and $\mathcal{H}_j = \mathcal{D}_{c_j} \cap H^2(\mathbb{C}^n, d\mu)$, one obtains a scale of Banach spaces

$$\mathbb{C} \cong \mathcal{H}_0 \subset \mathcal{H}_1 \subset \cdots \subset \mathcal{H}_j \subset \mathcal{H}_{j+1} \cdots \subset \mathcal{H} := \bigcup_{j\in\mathbb{N}} \mathcal{H}_j \subset H^2(\mathbb{C}^n, d\mu). \tag{2.3}$$

The norm on $\mathcal{H}_j$ is given by the restriction of the norm $\|f\|_j = \|e^{-c_j|\cdot|^2} f(\cdot)\|_\infty$ from $\mathcal{D}_{c_j}$ to $\mathcal{H}_j$. It was shown in [2] that both P and the multiplication operator M_f with $f \in \mathrm{Sym}_{>0}(\mathbb{C}^n)$ are continuous operators from $(D_{c_j}, \|\cdot\|_j)$ to $(D_{c_{j+1}}, \|\cdot\|_{j+1})$ for all $j \in \mathbb{N}_0$. In particular, the Toeplitz operator T_f maps $\mathcal{H}$ into $\mathcal{H}$. This shows that we can form finite products $T_{g_1} \cdots T_{g_m} : \mathcal{H} \to \mathcal{H}$ with $g_j \in \mathrm{Sym}_{>0}(\mathbb{C}^n)$ and consider them as densely defined operators on $H^2(\mathbb{C}^n, d\mu)$.

If f is a radial function, that is, $f(z) = \tilde{f}(|z|)$ for some function $\tilde{f}$ defined on $(0, \infty)$, then by the rotation-invariance of $d\mu$, the Toeplitz operator T_f is diagonal with respect to the standard orthonormal basis of $H^2(\mathbb{C}^n, d\mu)$. Furthermore, the eigenvalues of T_f are determined by values of the Mellin transform of

$\tilde{f}(r)e^{-r^2}$. We now remind the reader of the Mellin transform and state related known results that we will need. For a complex-valued function h on $(0,\infty)$, the Mellin transform of h is given by

$$\mathcal{M}[h](\zeta) = \int_0^\infty h(r)r^{\zeta-1}dr,$$

for all complex numbers ζ for which the integral exists. Under certain restrictions on h, the function $\mathcal{M}[h]$ is analytic on a vertical strip in the complex plane.

For complex-valued functions $k_1, k_2 : \mathbb{R}^+ \to \mathbb{C}$, the *Mellin convolution* of k_1 and k_2 is defined by

$$(k_1 * k_2)(x) = \int_0^\infty k_1(y)k_2\left(\frac{x}{y}\right)\frac{dy}{y},$$

for all $x > 0$ for which the integral exists. For suitable functions k_1 and k_2, the convolution $k_1 * k_2$ is defined on $\mathbb{R}^+$ and the Mellin convolution theorem says that $\mathcal{M}[k_1 * k_2] = \mathcal{M}[k_1]\cdot\mathcal{M}[k_2]$ on a certain vertical strip in the complex plane.

We will make intensive use of the following results on the Mellin transform. We state them here and refer the interested reader to [4] for the proofs.

Define the space

$$\mathcal{A} = \Big\{ u : \mathbb{R}^+ \to \mathbb{C} \text{ measurable} \;\Big|\; \exists C, c > 0 \text{ and } \exists \rho, \eta \geq 0 \text{ such that } |u(x)| \leq \frac{c}{x^\rho} \text{ for all } x \in (0,1], \\ \text{and } |u(x)| \leq Cx^\eta \text{ for all } x \in [1,\infty) \Big\}.$$

Proposition 2.1 ([4, Propositions 4.8 and 4.9]). *For any functions $u, v \in \mathcal{A}$, we put $f_u(x) = u(x)e^{-x^2}$ and $f_v(x) = v(x)e^{-x^2}$. Then the Mellin convolution $(f_u * f_v)(x)$ exists for all $x > 0$ and there is a function $h_1 \in \mathcal{A}$ such that*

$$(f_u * f_v)(x) = h_1(x)e^{-x} \quad \textit{for all } x \in \mathbb{R}^+.$$

In the case $\operatorname{supp}(v) \subset [0,1]$, there is a function $h_2 \in \mathcal{A}$ such that $(f_u * f_v)(x) = h_2(x)e^{-x^2}$ for $x \in \mathbb{R}^+$.

Proposition 2.2 ([4, Proposition 4.11]). *Let $u \in \mathcal{A}$ and $a \in (0,2]$. For any fixed integer $m_0 \in \mathbb{N}$, if*

$$\mathcal{M}[u(t)e^{-t}](ak+1) = \int_0^\infty u(t)e^{-t}t^{ak}dt = 0$$

for all integers $k \geq m_0$, then $u \equiv 0$ a.e. on $\mathbb{R}^+$.

Proposition 2.3 below was proved in [4] for functions u in $\mathcal{A}$. Here we point out that it remains valid for symbols $u : \mathbb{R}^+ \to \mathbb{C}$ with $u \circ |\cdot| \in \mathrm{Sym}_{>0}(\mathbb{C}^n)$, as well. We sketch here a proof and refer the reader to [4] for more details.

Proposition 2.3 ([4, Proposition 4.16]). *Let u be defined on $\mathbb{R}^+$ such that the function $w \mapsto u(|w|)$ belongs to $\mathrm{Sym}_{>0}(\mathbb{C}^n)$ and that the function*

$$\psi(\zeta) = \frac{\mathcal{M}[u(t)e^{-t^2}](2\zeta+2)}{\Gamma(\zeta+1)}, \; \zeta \in \mathbb{C} \textit{ with } \operatorname{Re}(\zeta) > 0 \textit{ (here } \Gamma \textit{ is the usual Gamma function)},$$

extends to a periodic entire function with period $j \in \mathbb{N}$. Then u must be a constant function.

Proof. Let $c \in (0,1)$ be fixed and choose $d > 0$ such that $|u(t)| \leq de^{ct^2}$ for all $t > 0$. It then follows, for any $\zeta \in \mathbb{C}$ with $\operatorname{Re}(\zeta) > 0$, that

$$\begin{aligned} \Big|\mathcal{M}[u(t)e^{-t^2}](2\zeta+2)\Big| &\leq \int_0^\infty |u(t)|e^{-t^2}t^{2\operatorname{Re}(\zeta)+1}\,dt \\ &\leq d\int_0^\infty e^{-(1-c)t^2}t^{2\operatorname{Re}(\zeta)+1}\,dt = d\cdot\frac{\Gamma\big(\operatorname{Re}(\zeta)+1\big)}{2(1-c)^{\operatorname{Re}(\zeta)+1}}. \end{aligned}$$

And hence for such ζ, we have

$$|\psi(\zeta)| \le \frac{d}{2(1-c)^{\mathrm{Re}(\zeta)+1}} \frac{\Gamma\big(\mathrm{Re}(\zeta)+1\big)}{|\Gamma(\zeta+1)|}.$$

Now the arguments on [4, p.480] together with the fact that $(1-c)^{-\mathrm{Re}(\zeta)+1}$ is bounded when ζ varies on any vertical strip of the form $\alpha \le \mathrm{Re}(\zeta) \le \alpha + j$ show the existence of a constant $C > 0$ for which $|\psi(\zeta)| \le Ce^{\frac{\pi}{2}|\zeta|}$ for all $\zeta \in \mathbb{C}$. It now follows by the exact same arguments as on [4, p.481 – 483] that

$$\mathcal{M}[u(t)e^{-t^2}](2\zeta+2) = \Gamma(\zeta+1) \sum_{|4\ell| \le j} a_\ell e^{\frac{2\pi i \ell \zeta}{j}} \quad \text{for all } \zeta \in \mathbb{C} \text{ with } \mathrm{Re}(\zeta) > 0. \tag{2.4}$$

Moreover, as it was shown on [4, p.482], for each integer ℓ satisfying $0 \ne |4\ell| \le j$, there are $\lambda_\ell, b_\ell \in \mathbb{C}$ with $0 < \mathrm{Re}(\lambda_\ell) \le 1$ such that for all $\zeta \in \mathbb{C}$ with $-1 < \mathrm{Re}(\zeta) < 0$ one has

$$\mathcal{M}\left[2a_0 e^{-t^2} + \sum_{0 \ne |4\ell| \le j} b_\ell e^{(\lambda_\ell - 1)t^2}\right](2\zeta+2) = \Gamma(\zeta+1) \sum_{|4\ell| \le j} a_\ell e^{\frac{2\pi i \ell \zeta}{j}}. \tag{2.5}$$

Since the Mellin transform is one-to-one, we obtain from (2.4) and (2.5) that

$$u(t) = 2a_0 + \sum_{0 \ne |4\ell| \le j} b_\ell \, e^{\lambda_\ell t^2}.$$

Finally, by using the boundedness of $u(t)e^{-\varepsilon t^2}$ on $\mathbb{R}^+$ for all $\varepsilon > 0$ we see that $u(t) \equiv 2a_0$ is a constant function. □

3. Commuting and zero-product problems

Let $\mathbb{K} = \{\zeta \in \mathbb{C} \mid \mathrm{Re}(\zeta) > 0\}$ be the right half of the complex plane. For a function $f : \mathbb{C}^n \to \mathbb{C}$, using the Gaussian measure (2.1), we define

$$\mathcal{F}[f](z) = \int_{\mathbb{C}^n} f(w)|w_1|^{2z_1} \cdots |w_n|^{2z_n} d\mu(w),$$

for any $z = (z_1, \dots, z_n) \in \overline{\mathbb{K}}^n$ for which the integral is defined. It follows from the Dominated Convergence Theorem and Morera's Theorem that if f belongs to $\mathcal{D}_c$ for some $c < 1$, i.e., the function $w \mapsto f(w)\exp(-c|w|^2)$ is bounded, then $\mathcal{F}[f]$ is defined, continuous on $\overline{\mathbb{K}}^n$, and analytic on $\mathbb{K}^n$.

In the case $n = 1$ and f is a complex-valued function on $\mathbb{C}$, the function $\mathcal{F}[f]$ is related to the Mellin transform by the formula

$$\mathcal{F}[f](\zeta) = \int_0^\infty \hat{f}(r) r^{2\zeta+1} e^{-r^2} dr = \mathcal{M}\left[\hat{f}e^{-r^2}\right](2\zeta+2) \quad \text{for } \zeta \in \overline{\mathbb{K}}, \tag{3.1}$$

where $\hat{f}(r) = \frac{1}{\pi}\int_0^{2\pi} f(re^{i\theta})d\theta$.

For any function f on $\mathbb{C}^n$ and any positive number t, define $V_t f(w) = f(tw)\exp((1-t^2)|w|^2)$ for $w \in \mathbb{C}^n$. It follows from a change of variables that if $z \in \overline{\mathbb{K}}^n$ such that $\mathcal{F}[f](z)$ exists, then $\mathcal{F}[V_t f](z)$ also exists and we have

$$\mathcal{F}[V_t f](z) = t^{-2(z_1 + \cdots + z_n + n)} \mathcal{F}[f](z) \quad \text{for any } t > 0. \tag{3.2}$$

The benefit of working with $V_t f$ is that for sufficiently large t, $V_t f$ is bounded even if f has certain exponential growth at infinity. More precisely, if f belongs to $\mathcal{D}_c$ for some $c < 1$, then $(V_t f)(w) \to 0$ as $|w| \to \infty$ for all $t > (1-c)^{-1/2}$.

In analyzing the commuting and zero-product problems for Toeplitz operators, we encounter analytic functions that vanish on the lattice $\mathbb{N}^n$ in $\overline{\mathbb{K}}^n$. Under certain restrictions on the growth at infinity, such functions must be identically zero as the following proposition shows.

Proposition 3.1. *Put $\mathcal{G} = \{\mathcal{F}[f] : f \in \mathcal{D}_c$ for some $c < 1\}$. Let G be a function defined on $\overline{\mathbb{K}}^n$ of the form $G = u_1 v_1 p_1 + \cdots + u_m v_m p_m$, where $u_1, \ldots, u_m$, $v_1, \ldots, v_m$ belong to $\mathcal{G}$ and $p_1, \ldots, p_m$ are polynomials. If $G(z) = 0$ for all $z \in \mathbb{N}^n$, then $G(z) = 0$ for all $z \in \overline{\mathbb{K}}^n$.*

Proof. Suppose $G = u_1 v_1 p_1 + \cdots + u_m v_m p_m$, where $u_j = \mathcal{F}[f_j]$, $v_j = \mathcal{F}[g_j] \in \mathcal{G}$ and $p_1, \ldots, p_m$ are polynomials. For $t > 0$, (3.2) gives

$$G_t(z) := \mathcal{F}[V_t f_1](z)\mathcal{F}[V_t g_1](z) p_1(z) + \cdots + \mathcal{F}[V_t f_m](z)\mathcal{F}[V_t g_m](z) p_m(z) = t^{-4(z_1+\cdots+z_n+n)} G(z).$$

Therefore, for $z \in \overline{\mathbb{K}}^n$, $G(z) = 0$ if and only if $G_t(z) = 0$. On the other hand, for sufficiently large t, the functions $V_t f_j, V_t g_j$ $(1 \le j \le m)$ are all bounded. Replacing f_j by $V_t f_j$, g_j by $V_t g_j$, and G by G_t if necessary, we may assume that all functions f_j, g_j are bounded.

We consider first the case $n = 1$. To avoid possible confusion, let us use ζ in place of z to denote a single complex variable. Let d be a positive integer that is strictly larger than the degrees of all the polynomials $p_1, \ldots, p_m$. We will show that $G(\zeta)/(\zeta+1)\cdots(\zeta+d)$ can be written as the Mellin transform of a certain function. First, note that

$$\mathcal{M}[2r^{2j-2}\chi_{[0,1]}(r)](2\zeta+2) = \int_0^1 2r^{2\zeta+2j-1}\,dr = \frac{1}{\zeta+j},$$

for $\zeta \in \overline{\mathbb{K}}$ and $j \ge 1$. Now using partial fractions, we conclude that for any polynomial $p(\zeta)$ with $\deg(p) < d$, there is a polynomial $\check{p}$ such that $\mathcal{M}[\check{p}\chi_{[0,1]}](2\zeta+2) = p(\zeta)/(\zeta+1)\cdots(\zeta+d)$ for $\zeta \in \overline{\mathbb{K}}$.

Using (3.1) together with the Mellin convolution theorem, we obtain

$$\begin{aligned}
\frac{G(\zeta)}{(\zeta+1)\cdots(\zeta+d)} &= \sum_{j=1}^m u_j(\zeta) v_j(\zeta) \frac{p_j(\zeta)}{(\zeta+1)\cdots(\zeta+d)} \\
&= \sum_{j=1}^m \mathcal{F}[f_j](\zeta)\cdot\mathcal{F}[g_j](\zeta)\cdot\frac{p_j(\zeta)}{(\zeta+1)\cdots(\zeta+d)} \\
&= \sum_{j=1}^m \mathcal{M}\big[\hat{f}_j e^{-r^2}\big](2\zeta+2)\cdot\mathcal{M}\big[\hat{g}_j e^{-r^2}\big](2\zeta+2)\cdot\mathcal{M}[\check{p}_j\chi_{[0,1]}](2\zeta+2) \\
&= \sum_{j=1}^m \mathcal{M}\big[\big(\hat{f}_j e^{-r^2}\big) * \big(\hat{g}_j e^{-r^2} * \check{p}_j\chi_{[0,1]}\big)\big](2\zeta+2),
\end{aligned}$$

for some polynomials $\check{p}_1, \ldots, \check{p}_m$. From Proposition 2.1 and the fact that $\hat{f}_j, \hat{g}_j$ are bounded on $\mathbb{R}^+$, we know that there are functions $\hat{h}_j$ $(1 \le j \le m)$ in $\mathcal{A}$ such that

$$\big(\hat{f}_j e^{-r^2}\big) * \big(\hat{g}_j e^{-r^2} * \check{p}_j\chi_{[0,1]}\big) = \hat{h}_j(r) e^{-r}.$$

Put $H = \sum_{j=1}^m \hat{h}_j$. For $\zeta \in \overline{\mathbb{K}}$, we have

$$G(\zeta) = (\zeta+1)\cdots(\zeta+d)\mathcal{M}\big[He^{-r}\big](2\zeta+2) = (\zeta+1)\cdots(\zeta+d)\int_0^\infty H(r)e^{-r}r^{2\zeta+1}\,dr.$$

Assume now that $G(\zeta) = 0$ for all $\zeta \in \mathbb{N}$. Since H is in $\mathcal{A}$, it follows from Proposition 2.2 (with $a = 2$) that $H = 0$ a.e. on $\mathbb{R}^+$ and as a consequence, $G(\zeta) = 0$ for all $\zeta \in \overline{\mathbb{K}}$.

Now consider the case $n > 1$. We write $w = (w_1, w') \in \mathbb{C}^n$ where $w' = (w_2, \ldots, w_n) \in \mathbb{C}^{n-1}$. For any bounded function $f : \mathbb{C}^n \to \mathbb{C}$, we write

$$\begin{aligned}
(3.3)\quad \mathcal{F}[f](z) &= \frac{1}{\pi^n}\int_0^\infty \Big(\int_{\mathbb{C}^{n-1}}\int_0^{2\pi} f(re^{i\theta}, w')\,d\theta\,|w_2|^{2z_2}\cdots|w_n|^{2z_n}e^{-|w'|^2}dV(w')\Big) r^{2z_1+1}e^{-r^2}dr \\
&= \mathcal{M}\big[F(r, z_2, \ldots, z_n)e^{-r^2}\big](2z_1+2),
\end{aligned}$$

where the function $F(r, z_2, \ldots, z_n)$ is defined by

$$F(r, z_2, \ldots, z_n) = \frac{1}{\pi^n} \int_{\mathbb{C}^{n-1}} \int_0^{2\pi} f(re^{i\theta}, w')\, d\theta\, |w_2|^{2z_2} \cdots |w_n|^{2z_n} e^{-|w'|^2} dV(w'). \tag{3.4}$$

Therefore,

$$\begin{aligned} G(z) &= \sum_{j=1}^m u_j(z) v_j(z) p_j(z) = \sum_{j=1}^m \mathcal{F}[f_j](z) \cdot \mathcal{F}[g_j](z) \cdot p_j(z) \\ &= \sum_{j=1}^m \mathcal{M}\big[F_j(r, z_2, \ldots, z_n) e^{-r^2}\big](2z_1 + 2) \cdot \mathcal{M}\big[G_j(r, z_2, \ldots, z_n) e^{-r^2}\big](2z_1 + 2) \cdot p_j(z), \end{aligned}$$

where (f_j, F_j) and (g_j, G_j) are related by (3.4). Note that for all j, the functions $F_j(r, z_2, \ldots, z_n)$ and $G_j(r, z_2, \ldots, z_n)$ are bounded in the variable r. Since $G(z_1, z') = 0$ for all $(z_1, z') \in \mathbb{N}^n$, it follows from the one dimensional case that $G(z_1, z') = 0$ for all $(z_1, z') \in \overline{\mathbb{K}} \times \mathbb{N}^{n-1}$. We now make the transformation to polar coordinates as in (3.3) with respect to the variable w_2 and obtain, with appropriate bounded functions $\tilde{F}_j$ and $\tilde{G}_j$,

$$G(z) = \sum_{j=1}^m \mathcal{M}\big[\tilde{F}_j(z_1, r, z_3, \ldots, z_n) e^{-r^2}\big](2z_2 + 2) \cdot \mathcal{M}\big[\tilde{G}_j(z_1, r, z_3, \ldots, z_n) e^{-r^2}\big](2z_2 + 2) \cdot p_j(z).$$

Since for each fixed $(z_1, z_3, \ldots, z_n) \in \overline{\mathbb{K}} \times \mathbb{N}^{n-2}$, the function $z_2 \mapsto G(z_1, z_2, \ldots, z_n)$ vanishes for all $z_2 \in \mathbb{N}$, we conclude, as in the one dimensional case again, that $G(z) = 0$ for all $(z_1, z_2, \ldots, z_n)$ in $\overline{\mathbb{K}}^2 \times \mathbb{N}^{n-2}$. Continuing this argument, we obtain the assertion of the proposition. □

The following lemma characterizes functions on $\mathbb{C}^n$ that are invariant under certain actions of the unit circle $\mathbb{T}$. This characterization will be important for us in analyzing commuting Toeplitz operators.

Lemma 3.2. *Let g be in $L^2(\mathbb{C}^n, d\mu)$, l be in $\mathbb{Z}^n$ and s be an integer. Then the following are equivalent.*

(a) $\int_{\mathbb{C}^n} g(w) w^m \bar{w}^k\, d\mu(w) = 0$ *for all multi-indices $m, k \in \mathbb{N}_0^n$ such that $(m-k)l \neq s$. Here we write $(m-k)l = (m_1 - k_1)l_1 + \cdots + (m_n - k_n)l_n$.*

(b) $g(\gamma^{l_1} z_1, \ldots, \gamma^{l_n} z_n) = \bar{\gamma}^s g(z)$ *for a.e. $\gamma \in \mathbb{T}$ and a.e. $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$.*

Proof. Define the function

$$\tilde{g}(z) = \frac{1}{2\pi} \int_0^{2\pi} g(e^{il_1\theta} z_1, \ldots, e^{il_n\theta} z_n) e^{i\theta s} d\theta,$$

for any $z \in \mathbb{C}^n$ for which the integral is defined. Since g belongs to $L^2(\mathbb{C}^n, d\mu)$, $\tilde{g}$ also belongs to $L^2(\mathbb{C}^n, d\mu)$, and hence $\tilde{g}(z)$ is defined for a.e. z. For such z, the identity $\tilde{g}(\gamma^{l_1} z_1, \ldots, \gamma^{l_n} z_n) = \bar{\gamma}^s \tilde{g}(z)$ holds for all $\gamma \in \mathbb{T}$. We see that (b) is equivalent to $g(z) = \tilde{g}(z)$ a.e. z.

For m, k in $\mathbb{N}_0^n$, using Fubini's Theorem and the rotation-invariance of μ, we obtain

$$\begin{aligned} \int_{\mathbb{C}^n} \tilde{g}(w) w^m \bar{w}^k d\mu(w) &= \frac{1}{2\pi} \int_0^{2\pi} \Big(\int_{\mathbb{C}^n} g(e^{il_1\theta} w_1, \ldots, e^{il_n\theta} w_n) w^m \bar{w}^k d\mu(w) \Big) e^{i\theta s} d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} \Big(\int_{\mathbb{C}^n} g(w) w^m \bar{w}^k d\mu(w) \Big) e^{i\theta((-m+k)l + s)} d\theta \\ &= \begin{cases} 0 & \text{if } (m-k)l \neq s; \\ \int_{\mathbb{C}^n} g(w) w^m \bar{w}^k d\mu(w) & \text{if } (m-k)l = s. \end{cases} \end{aligned}$$

It follows that (a) is equivalent to $\int_{\mathbb{C}^n} \tilde{g}(w)w^m\bar{w}^k d\mu(w) = \int_{\mathbb{C}^n} g(w)w^m\bar{w}^k d\mu(w)$ for all m, k in $\mathbb{N}_0^n$. This, in view of the fact that $\tilde{g} - g$ belongs to $L^2(\mathbb{C}^n, d\mu)$ and the span of $\{w^m\bar{w}^k : m, k \in \mathbb{N}_0^n\}$ is dense in $L^2(\mathbb{C}^n, d\mu)$, is equivalent to $g(w) = \tilde{g}(w)$ for a.e. w in $\mathbb{C}^n$. The conclusion of the lemma now follows. □

If a function $g \in L^2(\mathbb{C}^n, d\mu)$ depends only on $|z_1|, \dots, |z_n|$, then it follows from the rotation-invariant property of μ that the Toeplitz operator T_g is diagonal with respect to the standard orthonormal basis $\mathcal{B} = \{e_\alpha(z) = (\alpha!)^{-1/2}z^\alpha : \alpha \in \mathbb{N}_0^n\}$ of $H^2(\mathbb{C}^n, d\mu)$. The following corollary to Lemma 3.2 shows that the converse also holds true.

Corollary 3.3. *Let $g \in L^2(\mathbb{C}^n, d\mu)$ such that T_g is defined on the space $\mathbb{P}[z]$ of analytic polynomials and it is diagonal with respect to the orthonormal basis $\mathcal{B}$. Then $g(z) = g(|z_1|, \dots, |z_n|)$ for a.e. $z \in \mathbb{C}^n$, that is, g is radial in each component.*

Proof. Since T_g is diagonal with respect to $\mathcal{B}$, we have $\langle T_g e_\alpha, e_\beta\rangle = 0$ for all $\alpha \neq \beta \in \mathbb{N}_0^n$. This implies that for any $l \in \mathbb{Z}^n$, we have $\int_{\mathbb{C}^n} g(w)w^\alpha\bar{w}^\beta \, d\mu(w) = 0$ whenever $(\alpha - \beta)l \neq 0$. Lemma 3.2 now shows that $g(\gamma^{l_1}z_1, \dots, \gamma^{l_n}z_n) = g(z)$ for a.e. $z \in \mathbb{C}^n$ and $\gamma \in \mathbb{T}$. Since l was arbitrary, the conclusion of the corollary follows. □

If $g \in L^2(\mathbb{C}^n, d\mu)$ is a radial function, that is, $g(z) = g(|z|)$ for a.e. $z \in \mathbb{C}$, then the eigenvalue of T_g corresponding to the eigenvector e_α depends only on $|\alpha| = \alpha_1 + \dots + \alpha_n$. In fact, by integration in polar coordinates, we obtain

$$\begin{aligned}\langle T_g e_\alpha, e_\alpha\rangle &= \frac{1}{\alpha!}\int_{\mathbb{C}^n} g(w)|w_1|^{2\alpha_1}\cdots|w_n|^{2\alpha_n} d\mu(w) = \frac{1}{\alpha!}\mathcal{F}[g](\alpha)\\ &= \frac{1}{\Gamma(|\alpha|+n)}\int_0^\infty 2g(r)r^{2|\alpha|+2n-1}e^{-r^2}dr = \frac{\mathcal{M}[2g(r)e^{-r^2}](2|\alpha|+2n)}{\Gamma(|\alpha|+n)}.\end{aligned}$$

For any $s \in \mathbb{C}$ with $\mathrm{Re}(s) > -n$, we define $\omega(g,s) = \mathcal{M}[2g(r)e^{-r^2}](2s+2n)/\Gamma(s+n)$. Then $\omega(g,s)$ is analytic on its domain and for $\alpha \in \mathbb{N}_0^n$, we have

$$\omega(g,|\alpha|) = \frac{\mathcal{F}[g](\alpha)}{\alpha!} \quad \text{and} \quad T_g e_\alpha = \omega(g,|\alpha|)e_\alpha. \tag{3.5}$$

Since $\omega(g-c,s) = \omega(g,s) - c$ for any complex number c, we see that $s \mapsto \omega(g,s)$ is a constant function if and only if g is a constant function on $\mathbb{C}^n$.

Let $f_1, f_2 \in L^2(\mathbb{C}^n, d\mu)$ be two radial functions. In order to study the equation $T_{f_1}T_g = T_gT_{f_2}$, we have to investigate when the eigenvalues of T_{f_1} and T_{f_2} coincide. It turns out (as in the proof below) that we need to consider the set

$$\mathcal{Z}(f_1,f_2) = \Big\{d \in \mathbb{Z} \;\Big|\; \omega(f_1,s) = \omega(f_2,d+s) \text{ for all } s \in \mathbb{K} \text{ with } \mathrm{Re}(s) \text{ sufficiently large}\Big\}. \tag{3.6}$$

Without any restriction on the growth at infinity of the functions f_1 and f_2, it may be difficult to describe $\mathcal{Z}(f_1,f_2)$. However, if we assume that both f_1 and f_2 belong to $\mathrm{Sym}_{>0}(\mathbb{C}^n)$, then $\mathcal{Z}(f_1,f_2)$ is extremely simple. Indeed, if both f_1 and f_2 are constant functions, then it is clear that either $\mathcal{Z}(f_1,f_2) = \emptyset$ or $\mathcal{Z}(f_1,f_2) = \mathbb{Z}$. The former corresponds to the case $f_1 \not\equiv f_2$ and the latter corresponds to the case $f_1 \equiv f_2$.

In the case one of the functions f_1, f_2 is non-constant, we claim that there are also two possibilities: $\mathcal{Z}(f_1,f_2) = \emptyset$ or $\mathcal{Z}(f_1,f_2) = \{d\}$ for some integer d. In fact, suppose $d_1 < d_2$ and both belong to $\mathcal{Z}(f_1,f_2)$. Then for $s \in \mathbb{K}$ with $\mathrm{Re}(s)$ large enough, we have

$$\omega(f_1, s+d_2-d_1) = \omega(f_2, s+d_2) = \omega(f_1,s).$$

Because the map $s \mapsto \omega(f_1,s)$ is analytic on the set $\{s \in \mathbb{C} : \mathrm{Re}(s) > -n\}$, we conclude that the identity $\omega(f_1, s-n+1+d_2-d_1) = \omega(f_1, s-n+1)$ holds for all $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 0$. This shows that the function $s \mapsto \omega(f_1, s-n+1)$ extends to a periodic function with period $d_2 - d_1 \in \mathbb{N}$. Since

$$\omega(f_1, s-n+1) = \frac{2\mathcal{M}[f_1(r)e^{-r^2}](2(s-n+1)+2n)}{\Gamma((s-n+1)+n)} = \frac{2\mathcal{M}[f_1(r)e^{-r^2}](2s+2)}{\Gamma(s+1)},$$

Proposition 2.3 implies that f_1 is a constant function. This then shows that the function $s \mapsto \omega(f_2, s)$, which is the same as $\omega(f_1, s-d_1)$, is constant. Hence f_2 is also a constant function. This contradicts the assumption that one of the functions f_1, f_2 is non-constant. Therefore, the set $\mathcal{Z}(f_1, f_2)$ contains at most one element.

If it happens that $f_1 = f_2 = f$, a non-constant function, then since 0 clearly belongs to $\mathcal{Z}(f, f)$, we have $\mathcal{Z}(f, f) = \{0\}$.

If $f_1 = 0$ and f_2 is not the zero function, then since $\omega(f_1, s) \equiv 0$ and $\omega(f_2, s)$ is not identically zero, we see that $\mathcal{Z}(f_1, f_2) = \emptyset$.

We are now ready for the main result in this section.

Theorem 3.4. *Let $f_1, f_2 \in \mathrm{Sym}_{>0}(\mathbb{C}^n)$ be two radial functions, at least one of which is non-constant. Then exactly one of the following two cases occurs.*

(a) $\mathcal{Z}(f_1, f_2) = \emptyset$ and for any function $g \in \mathrm{Sym}_{>0}(\mathbb{C}^n)$, $T_{f_1}T_g = T_gT_{f_2}$ on the space of analytic polynomials if and only if $g(z) = 0$ a.e. $z \in \mathbb{C}^n$.

(b) $\mathcal{Z}(f_1, f_2) = \{d\}$ and for any function $g \in \mathrm{Sym}_{>0}(\mathbb{C}^n)$, $T_{f_1}T_g = T_gT_{f_2}$ on the space of analytic polynomials if and only if $g(\gamma z) = \bar{\gamma}^d g(z)$ for a.e. $\gamma \in \mathbb{T}$ and a.e. $z \in \mathbb{C}^n$.

Proof. Let g be a function in $\mathrm{Sym}_{>0}(\mathbb{C}^n)$. Then as we discussed in Section 2, the products $T_{f_1}T_g$ and $T_gT_{f_2}$ are defined on the space of analytic polynomials. For any multi-indices α and β, since $T_{\bar{f}_1}e_\beta = \omega(\bar{f}_1, |\beta|)e_\beta$ and $T_{f_2}e_\alpha = \omega(f_2, |\alpha|)e_\alpha$, we obtain

$$\langle T_{f_1}T_g e_\alpha, e_\beta\rangle = \langle P(ge_\alpha), \bar{f}_1 e_\beta\rangle = \langle ge_\alpha, T_{\bar{f}_1}e_\beta\rangle = \langle ge_\alpha, \omega(\bar{f}_1, |\beta|)e_\beta\rangle = \omega(f_1, |\beta|)\langle ge_\alpha, e_\beta\rangle,$$

and $\langle T_gT_{f_2}e_\alpha, e_\beta\rangle = \omega(f_2, |\alpha|)\langle ge_\alpha, e_\beta\rangle$. This shows that $T_{f_1}T_g = T_gT_{f_2}$ on analytic polynomials if and only if for all α and β,

$$0 = \Big(\omega(f_1, |\beta|) - \omega(f_2, |\alpha|)\Big)\langle ge_\alpha, e_\beta\rangle = \frac{1}{\sqrt{\alpha!}\sqrt{\beta!}}\Big(\omega(f_1, |\beta|) - \omega(f_2, |\alpha|)\Big)\int_{\mathbb{C}^n} g(w)w^\alpha \bar{w}^\beta d\mu(w).$$

Let m and k be two fixed multi-indices in $\mathbb{N}_0^n$. With $\alpha = m + l$ and $\beta = k + l$ for $l \in \mathbb{N}_0^n$, by (3.5), we obtain

$$\begin{aligned} G(l) &:= \Big(\frac{\mathcal{F}[f_1](k+l)}{(k+l)!} - \frac{\mathcal{F}[f_2](m+l)}{(m+l)!}\Big)\int_{\mathbb{C}^n} g(w)w^m\bar{w}^k|w_1|^{2l_1}\cdots|w_n|^{2l_n}d\mu(w) \\ &= \Big(\omega(f_1, |k|+|l|) - \omega(f_2, |m|+|l|)\Big)\int_{\mathbb{C}^n} g(w)w^{m+l}\bar{w}^{k+l}d\mu(w) = 0. \end{aligned} \tag{3.7}$$

Using the fact that $(m+k+l)!/(k+l)!$ and $(m+k+l)/(m+l)!$ are polynomials in l, one can verify that the function $(m+k+l)!\, G(l)$ satisfies the hypothesis of Proposition 3.1. Therefore, by Proposition 3.1, $G(l) = 0$ for all $l \in \mathbb{N}_0^n$ if and only if $G(l) = 0$ for all $l \in \overline{\mathbb{K}}^n$. This, by analyticity, is equivalent to either $\omega(f_1, |k|+|l|) = \omega(f_2, |m|+|l|)$ for all $l \in \overline{\mathbb{K}}^n$ or $\int_{\mathbb{C}^n} g(w)w^m\bar{w}^k|w^l|^2 d\mu(w) = 0$ for all $l \in \overline{\mathbb{K}}^n$. The former is equivalent to that $|m| - |k|$ belongs to $\mathcal{Z}(f_1, f_2)$. Since one of the functions f_1, f_2 is non-constant, by the discussion preceding the theorem, there are two possibilities.

(a) If $\mathcal{Z}(f_1, f_2) = \emptyset$, then (3.7) is equivalent to $\int_{\mathbb{C}^n} g(w)w^m\bar{w}^k|w^l|^2 d\mu(w) = 0$ for all $l \in \overline{\mathbb{K}}^n$, all $m, k \in \mathbb{N}_0^n$. This, in turn, is equivalent to $g(z) = 0$ for a.e. $z \in \mathbb{C}^n$.

(b) If $\mathcal{Z}(f_1, f_2) = d$, then (3.7) is equivalent to $\int_{\mathbb{C}^n} g(w)w^m\bar{w}^k|w^l|^2 d\mu(w) = 0$ for all $l \in \overline{\mathbb{K}}^n$ satisfying $|m| - |k| \neq d$, that is, $(m-k)\cdot(1, \ldots, 1) \neq d$. By Lemma 3.2, this is equivalent to $g(\gamma z) = \bar{\gamma}^d g(z)$ for a.e. $\gamma \in \mathbb{T}$ and a.e. $z \in \mathbb{C}^n$. The proof of the theorem is now completed. ◻

Using Theorem 3.4, we obtain results about the commuting and the zero-product problems for Toeplitz operators on $H^2(\mathbb{C}^n, d\mu)$.

Corollary 3.5. *Let $f \in \mathrm{Sym}_{>0}(\mathbb{C}^n)$ be a radial, non-constant function. Then for any $g \in \mathrm{Sym}_{>0}(\mathbb{C}^n)$, $[T_f, T_g] = 0$ on analytic polynomials $\mathbb{P}[z]$ if and only if $g(\gamma z) = g(z)$ for a.e. $z \in \mathbb{C}^n$.*

Proof. We see from the discussion preceding Theorem 3.4 that $\mathcal{Z}(f, f) = \{0\}$. The assertion of the corollary now follows from Theorem 3.4. □

Corollary 3.6. *Let f, g be in $\mathrm{Sym}_{>0}(\mathbb{C}^n)$ so that f is radial. If $T_f T_g = 0$ or $T_g T_f = 0$ on $\mathbb{P}[z]$, then f or g must be zero a.e. on $\mathbb{C}^n$.*

Proof. If f is a constant function, then $T_f T_g$ and $T_g T_f$ are constant multiples of T_g. The conclusion of the corollary follows.

Now assume that f is not a constant function. Put $h(z) = 0$ for all z. If $T_f T_g = 0$, then we have $T_f T_g = T_g T_h$. Since $\mathcal{Z}(f, h) = \emptyset$, Theorem 3.4 implies that g must be zero. The case $T_g T_f = 0$ is similar. □

On the Bergman space of the unit ball in $\mathbb{C}^n$, it was proved in [12] that if $f_1, \ldots, f_N$ are bounded functions all of which, except possibly one, are radial, then $T_{f_1} \cdots T_{f_N} = 0$ implies that one of the functions must be zero. Corollary 3.6 shows that this result holds on $H^2(\mathbb{C}^n, d\mu)$ for $N = 2$. Surprisingly, it fails when $N = 3$ as the next proposition shows.

In the proof of the proposition, we will make use of the following known integral formula (3.8). It can be found in [8, p.498] (formula 3.944(5)) without a proof (see also [4, Example 4.12] for details of the calculation).

$$\int_0^\infty r^{\zeta-1} \sin(ar) e^{-r} dr = (1 + a^2)^{-\zeta/2} \sin(\zeta \arctan a) \Gamma(\zeta), \tag{3.8}$$

where a is a positive number and ζ belongs to $\mathbb{K}$.

Proposition 3.7. *There exist bounded non-zero functions f_0, f_1, f_2 on $\mathbb{C}^n$ such that the operator $T_{f_0} T_{f_1} T_{f_2}$ is zero on $H^2(\mathbb{C}^n, d\mu)$.*

Proof. For $j = 0, 1, 2$, put $f_j(z) = |z|^{2j} e^{-|z|^2} \sin(2\sqrt{3}|z|^2)$ for $z \in \mathbb{C}^n$. Then T_{f_j} is diagonal with respect to the standard orthonormal basis. The eigenvalue corresponding to the eigenvector e_α is given by

$$\begin{aligned}
\omega(f_j, |\alpha|) &= \frac{1}{\alpha!} \int_{\mathbb{C}^n} f_j(z) |z^\alpha|^2 d\mu(z) \\
&= \frac{1}{\Gamma(|\alpha| + n)} \int_0^\infty 2\big(r^{2j} \sin(2\sqrt{3} r^2) e^{-r^2}\big) r^{2|\alpha|+2n-1} e^{-r^2} dr \quad \text{(integration in polar coordinates)} \\
&= \frac{2^{-j-|\alpha|-n}}{\Gamma(|\alpha| + n)} \int_0^\infty t^{j+|\alpha|+n-1} \sin(\sqrt{3} t) e^{-t} dt \quad \text{(change of variables } t = 2r^2) \\
&= \frac{2^{-j-|\alpha|-n}}{\Gamma(|\alpha| + n)} (1+3)^{-(j+|\alpha|+n)/2} \sin\big((j + |\alpha| + n) \arctan(\sqrt{3})\big) \Gamma(j + |\alpha| + n) \quad \text{(by (3.8))} \\
&= \frac{4^{-j-|\alpha|-n}}{\Gamma(|\alpha| + n)} \sin\Big(\frac{(j + |\alpha| + n)\pi}{3}\Big) \Gamma(j + |\alpha| + n).
\end{aligned}$$

It follows that the operator $T_{f_0} T_{f_1} T_{f_2}$ is a diagonal operator whose eigenvalue corresponding to the eigenvector e_α is a multiple of the product $\sin(\frac{(|\alpha|+n)\pi}{3}) \sin(\frac{(1+|\alpha|+n)\pi}{3}) \sin(\frac{(2+|\alpha|+n)\pi}{3})$, which is zero since one of the integers $|\alpha| + n, 1 + |\alpha| + n, 2 + |\alpha| + n$ is a multiple of 3. Therefore, $T_{f_0} T_{f_1} T_{f_2} = 0$. □

Note that the symbols f_j ($j = 0, 1, 2$) in Proposition 3.7 even vanish at exponential rate at infinity and hence the corresponding operators T_{f_j} are compact. Furthermore, Corollary 3.6 implies that the product

$T_{f_0}T_{f_1}$ cannot be represented as a Toeplitz operator T_g with $g \in \mathrm{Sym}_{>0}(\mathbb{C}^n)$. Coburn [7] provided examples of C^∞-functions φ for which the products $T_\varphi T_\varphi$ are not Toeplitz operators. However, the functions in his examples have exponential growth at infinity.

4. Extension of Toeplitz operators and the zero-product problem

It turns out that a statement analogous to Corollary 3.6 still holds when the symbols f and g have certain higher orders of growth at infinity. On the other hand, it fails to be true when the functions f and g grow too rapidly. Since we are dealing with symbols of high growth order at infinity, we need to describe an extension of the notion of Toeplitz operators for symbols not in $L^2(\mathbb{C}^n, d\mu)$. A detailed study of this extension and its relation with the usual Toeplitz operators can be found in [10].

Recall that the set $\{e_\alpha \mid \alpha \in \mathbb{N}_0^n\}$ is an orthonormal basis for $H^2(\mathbb{C}^n, d\mu)$ and that $\mathbb{P}[z]$ is the space of analytic polynomials, which is the linear span of $\{e_\alpha \mid \alpha \in \mathbb{N}_0^n\}$.

Definition 4.1. Let $f : \mathbb{C}^n \to \mathbb{C}$ be a measurable function. We define the domain $\mathcal{D}(\widetilde{T}_f)$ to be the space of all functions φ in $H^2(\mathbb{C}^n, d\mu)$ such that $f\varphi\overline{e}_\alpha$ is integrable with respect to $d\mu$ for all $\alpha \in \mathbb{N}_0^n$, and $\sum_\alpha \big|\int_{\mathbb{C}^n} f\varphi\overline{e}_\alpha\, d\mu\big|^2 < \infty$. We then define the operator $\widetilde{T} : \mathcal{D}(\widetilde{T}_f) \to H^2(\mathbb{C}^n, d\mu)$ by

$$\widetilde{T}_f\varphi = \sum_{\alpha\in\mathbb{N}_0^n} \Big(\int_{\mathbb{C}^n} f\varphi\overline{e}_\alpha\, d\mu\Big)e_\alpha \qquad \text{for } \varphi \in \mathcal{D}(\widetilde{T}_f). \tag{4.1}$$

Let f be as in Definition 4.1. Suppose φ is in the domain of the Toeplitz operator T_f defined in Section 2. Then $f\varphi$ belongs to $L^2(\mathbb{C}^n, d\mu)$, and since e_α also belongs to $L^2(\mathbb{C}^n, d\mu)$, $f\varphi\overline{e}_\alpha$ is integrable with respect to $d\mu$ for all $\alpha \in \mathbb{N}_0^n$. Furthermore,

$$T_f\varphi = \sum_\alpha \langle T_f\varphi, e_\alpha\rangle e_\alpha = \sum_\alpha \langle f\varphi, e_\alpha\rangle e_\alpha = \sum_\alpha \Big(\int_{\mathbb{C}^n} f\varphi\overline{e}_\alpha\, d\mu\Big)e_\alpha = \widetilde{T}_f\varphi.$$

This shows that $\widetilde{T}_f$ is an extension of T_f. In [10], Janas defined another extension Π_f of T_f and showed [10, Proposition 1.1] that $T_f \subseteq \Pi_f \subseteq \widetilde{T}_f$. For our purpose in this section we do not need this intermediate extension.

For a real number $t > 0$, define $(S_t h)(z) = t^{-n}h(t^{-1}z)$ for measurable functions h on $\mathbb{C}^n$. Then the restriction of S_t on $\mathbb{P}[z]$ is a diagonalizable operator which satisfies $S_t e_\alpha = t^{-n-|\alpha|}e_\alpha$ for $\alpha \in \mathbb{N}_0$. Recall that in Section 3 we defined $V_t f(z) = f(tz)\exp((1-t^2)|z|^2)$ for measurable functions f. It turns out that there is a simple relation between $\widetilde{T}_{V_t f}$ and $\widetilde{T}_f$ via S_t.

Lemma 4.2. *Let $t > 0$. For any $\varphi \in H^2(\mathbb{C}^n, d\mu)$ such that φ belongs to the domain of $\widetilde{T}_{V_t f}$ and $S_t\varphi$ belongs to the domain of $\widetilde{T}_f$, we have $\widetilde{T}_{V_t f}\varphi = S_t\widetilde{T}_f S_t\varphi$.*

Proof. For any multi-index α, a change of variables gives

$$\begin{aligned}\int_{\mathbb{C}^n} \varphi(V_t f)\overline{e}_\alpha\, d\mu &= \frac{1}{\pi^n}\int_{\mathbb{C}^n} \varphi(z)f(tz)\overline{e}_\alpha(z)\exp(-t^2|z|^2)dV(z)\\ &= \frac{t^{-2n}}{\pi^n}\int_{\mathbb{C}^n} \varphi(t^{-1}w)f(w)\overline{e}_\alpha(t^{-1}w)\exp(-|w|^2)dV(w) = t^{-n-|\alpha|}\int_{\mathbb{C}^n}(S_t\varphi)f\overline{e}_\alpha d\mu.\end{aligned}$$

For any $z \in \mathbb{C}^n$, by (4.1), we have

$$\begin{aligned}(\widetilde{T}_{V_t f}\varphi)(z) &= \sum_\alpha \Big(\int_{\mathbb{C}^n} \varphi(V_t f)\overline{e}_\alpha\, d\mu\Big)e_\alpha(z) = \sum_\alpha \Big(t^{-n-|\alpha|}\int_{\mathbb{C}^n}(S_t\varphi)f\overline{e}_\alpha d\mu\Big)e_\alpha(z)\\ &= t^{-n}\sum_\alpha \Big(\int_{\mathbb{C}^n}(S_t\varphi)f\overline{e}_\alpha d\mu\Big)e_\alpha(t^{-1}z) = (S_t\widetilde{T}_f S_t\varphi)(z).\end{aligned}$$

This shows that $\widetilde{T}_{V_t f}\varphi = S_t\widetilde{T}_f S_t\varphi$, which completes the proof of the lemma. $\square$

Now assume that f is a measurable radial function such that $f(z) = f(|z|)$ for a.e. $z \in \mathbb{C}^n$. Let φ be a function in $\mathcal{D}(\widetilde{T}_f)$. Write $\varphi = \sum_\beta \langle \varphi, e_\beta \rangle e_\beta$. For each multi-index α, integration in polar coordinates gives

$$\begin{aligned}\int_{\mathbb{C}^n} \varphi f \overline{e}_\alpha \, d\mu &= \frac{1}{\pi^n} \int_0^\infty 2nr^{2n-1} f(r) \Big(\int_{\mathbb{S}} \varphi(r\zeta) \overline{e}_\alpha(r\zeta) \, d\sigma(\zeta) \Big) e^{-r^2} \, dr \\ &= \frac{1}{\pi^n} \int_0^\infty 2nr^{2n-1} f(r) \Big\{ \int_{\mathbb{S}} \big(\sum_\beta \langle \varphi, e_\beta \rangle e_\beta(r\zeta) \big) \overline{e}_\alpha(r\zeta) \, d\sigma(\zeta) \Big\} e^{-r^2} \, dr \\ &= \langle \varphi, e_\alpha \rangle \frac{1}{\pi^n} \int_0^\infty 2nr^{2n-1} f(r) \Big(\int_{\mathbb{S}} |e_\alpha(r\zeta)|^2 \, d\sigma(\zeta) \Big) e^{-r^2} \, dr \\ &= \langle \varphi, e_\alpha \rangle \int_{\mathbb{C}^n} f |e_\alpha|^2 \, d\mu.\end{aligned}$$

As before, we put $\omega(f, \alpha) = \int_{\mathbb{C}^n} f|e_\alpha|^2 \, d\mu$. Then by (4.1), we have

$$\widetilde{T}_f \varphi = \sum_\alpha \Big(\int_{\mathbb{C}^n} \varphi f \overline{e}_\alpha \, d\mu \Big) e_\alpha = \sum_\alpha \omega(f, \alpha) \langle \varphi, e_\alpha \rangle e_\alpha. \tag{4.2}$$

This shows, in particular, that $\widetilde{T}_f$ is a diagonal operator on analytic polynomials $\mathbb{P}[z]$ whenever $\mathbb{P}[z]$ is contained in $\mathcal{D}(\widetilde{T}_f)$. This generalizes the fact (as we have seen in Section 3) that T_f is diagonal on $\mathbb{P}[z]$ when f is radial and $\mathbb{P}[z]$ is contained in the domain of T_f.

Using (4.2) , we now show that $\widetilde{T}_f$ commutes with S_t whenever f is a radial function.

Lemma 4.3. *Let f be a radial measurable function and let $t > 0$. For any $\varphi \in H^2(\mathbb{C}^n, d\mu)$ such that both φ and $S_t\varphi$ belong to the domain of $\widetilde{T}_f$, we have $S_t \widetilde{T}_f \varphi = \widetilde{T}_f S_t \varphi$.*

Proof. Write $\varphi = \sum_\alpha \langle \varphi, e_\alpha \rangle e_\alpha$. Then $S_t \varphi = \sum_\alpha \langle \varphi, e_\alpha \rangle t^{-|\alpha|-n} e_\alpha$. For $z \in \mathbb{C}^n$, using (4.2) we obtain

$$\big(\widetilde{T}_f S_t \varphi\big)(z) = \sum_\alpha \omega(f, \alpha) \langle \varphi, e_\alpha \rangle t^{-|\alpha|-n} e_\alpha(z) = t^{-n} \sum_\alpha \omega(f, \alpha) \langle \varphi, e_\alpha \rangle e_\alpha(t^{-1}z) = S_t\big(\widetilde{T}_f \varphi\big)(z).$$

The conclusion of the lemma now follows. □

Using Lemmas 4.2 and 4.3, we are able to strengthen Corollary 3.6 to include functions that have higher orders of growth at infinity. Recall that for any real number $c < 1$, $\mathcal{D}_c$ denotes the space of all measurable functions f such that the map $z \mapsto f(z)\exp(-c|z|^2)$ is bounded on $\mathbb{C}^n$.

Theorem 4.4. *Let f and g be two functions belonging to $\mathcal{D}_c$ for some $c < 1$ so that f is radial. If $\widetilde{T}_f \widetilde{T}_g = 0$ or $\widetilde{T}_g \widetilde{T}_f = 0$ on $\mathbb{P}[z]$, then either $f = 0$ or $g = 0$ a.e. on $\mathbb{C}^n$.*

Proof. Since f and g belong to $\mathcal{D}_c$ with $c < 1$, it follows from discussion after Definition 4.1 that the space $\mathbb{P}[z]$ of analytic polynomials is contained in the domains of $\widetilde{T}_f$ and $\widetilde{T}_g$.

Choose a sufficiently large number $t > 0$ such that both functions $V_t f$ and $V_t g$ are bounded. Then $T_{V_t f} = \widetilde{T}_{V_t f}$ and $T_{V_t g} = \widetilde{T}_{V_t g}$ and they are all bounded operators. Let p be a polynomial in $\mathbb{P}[z]$. By Lemmas 4.2 and 4.3, we have

$$\begin{aligned}T_{V_t f} T_{V_t g} p &= \big(S_t \widetilde{T}_f S_t\big)\big(S_t \widetilde{T}_g S_t p\big) = S_t^3 \widetilde{T}_f \widetilde{T}_g S_t p, \\ T_{V_t g} T_{V_t f} p &= \big(S_t \widetilde{T}_g S_t\big)\big(S_t \widetilde{T}_f S_t p\big) = S_t \widetilde{T}_f \widetilde{T}_g S_t^3 p.\end{aligned}$$

Since $\mathbb{P}[z]$ is invariant under S_t, we conclude that on $\mathbb{P}[z]$, $T_{V_t f} T_{V_t g} = 0$ (if $\widetilde{T}_f \widetilde{T}_g = 0$) or $T_{V_t g} T_{V_t f} = 0$ (if $\widetilde{T}_g \widetilde{T}_f = 0$). By Corollary 3.6, either $V_t f = 0$ or $V_t g = 0$, which implies that either $f = 0$ or $g = 0$. □

It turns out that there are zero products of non-zero Toeplitz operators in which the symbols are radial and belong to $\mathcal{D}_1$. We now construct explicit examples. With parameters $s \geq 0$, $a > 0$ and $t \in (0, 1)$, consider the radial function $g_{s,a,t} : \mathbb{C}^n \to \mathbb{C}$ defined by $g_{s,a,t}(w) = |w|^{2s} \sin\big(a|w|^{2t}\big) e^{|w|^2 - |w|^{2t}}$. Note that

$g_{s,a,t} \notin L^2(\mathbb{C}^n, d\mu)$ but $g_{s,a,t} \in \mathcal{D}_1$. For simplicity we only consider the case $n = 1$ in the calculation below. However, the idea can be generalized to higher dimensional situation.

For any integer $m \geq 0$,

$$\begin{aligned}
\omega(g_{s,a,t}, m) &= \frac{1}{\pi\ m!} \int_{\mathbb{C}} g_{s,a,t}(z) |z|^{2m} e^{-|z|^2} dV(z) = \frac{2}{m!} \int_0^\infty \sin\left(ar^{2t}\right) e^{-r^{2t}} r^{2(m+s)+1} dr \\
&= \frac{1}{m!\ t} \int_0^\infty \sin(au) e^{-u} u^{\frac{m+s+1}{t} - 1} du \quad \text{(by the change of variables } u = r^{2t}\text{)} \\
&= \frac{1}{m!\ t} \left(1 + a^2\right)^{-\frac{m+s+1}{2t}} \sin\left(\frac{m+s+1}{t} \arctan(a)\right) \Gamma\left(\frac{m+s+1}{t}\right) \text{ (by formula (3.8))}.
\end{aligned} \tag{4.3}$$

By (4.2), the monomial $e_m(z) = z^m/\sqrt{m!}$ belongs to $\mathcal{D}(\widetilde{T}_{g_{s,a,t}})$ and $\widetilde{T}_{g_{s,a,t}} e_m = \omega(g_{s,a,t}, m) e_m$ for any integer $m \geq 0$. This shows that the space of analytic polynomials $\mathbb{P}[z]$ is contained in $\mathcal{D}(\widetilde{T}_{g_{s,a,t}})$ and $\mathbb{P}[z]$ is invariant under $\widetilde{T}_{g_{s,a,t}}$. It also follows from (4.3) that $\omega(g_{s,a,t}, m) = 0$ if and only if $m = \frac{t\pi}{\arctan(a)} k - s - 1$ for some $k \in \mathbb{Z}$.

For a fixed number $t \in [\frac{1}{2}, 1)$ we choose $a > 0$ such that $\arctan(a) = t\pi/2$. Then $\omega(g_{0,a,t}, 2k+1) = \omega(g_{1,a,t}, 2k) = 0$ and $\omega(g_{0,a,t}, 2k), \omega(g_{1,a,t}, 2k+1) \neq 0$ for all $k \in \mathbb{N}_0$. Let φ be a function in the domain of the product $\widetilde{T}_{g_{0,a,t}} \widetilde{T}_{g_{1,a,t}}$. Applying (4.2) twice, we obtain

$$\widetilde{T}_{g_{0,a,t}} \widetilde{T}_{g_{1,a,t}} \varphi = \widetilde{T}_{g_{0,a,t}} \Big(\sum_{m=0}^\infty \omega(g_{1,a,t}, m) \langle \varphi, e_m \rangle e_m \Big) = \sum_{m=0}^\infty \omega(g_{0,a,t}, m) \omega(g_{1,a,t}, m) \langle \varphi, e_m \rangle e_m = 0.$$

Similarly, we can show that $\widetilde{T}_{g_{1,a,t}} \widetilde{T}_{g_{0,a,t}} \varphi = 0$ for all φ belonging to the domain of $\widetilde{T}_{g_{1,a,t}} \widetilde{T}_{g_{0,a,t}}$. Thus, we have shown the existence of zero-products of non-zero Toeplitz operators.

Proposition 4.5. *For each fixed t in $[\frac{1}{2}, 1)$, one can choose $a > 0$ such that $\widetilde{T}_{g_{0,a,t}} \widetilde{T}_{g_{1,a,t}} = \widetilde{T}_{g_{1,a,t}} \widetilde{T}_{g_{0,a,t}} = 0$ on their domains (which contain the space of analytic polynomials $\mathbb{P}[z]$).*

Now for a fixed number $t \in (0, \frac{1}{2})$ we choose $a > 0$ so that $\arctan(a) = t\pi$. Then for any integers $s, m \geq 0$, (4.3) shows that $\omega(g_{s,a,t}, m) = 0$. By (4.2), $\widetilde{T}_{g_{s,a,t}} \varphi = 0$ for all φ in the domain of $\widetilde{T}_{g_{s,a,t}}$. We have thus shown the existence of zero Toeplitz operators with non-zero symbols. In [9], Grudsky and Vasilevski showed the existence of such symbols using Fourier transform. Here we obtain concrete examples.

Proposition 4.6. *For any fixed number $t \in (0, \frac{1}{2})$, one can choose $a > 0$ such that for any integer $s \geq 0$, $\widetilde{T}_{g_{s,a,t}} = 0$.*

5. FINITE RANK TOEPLITZ OPERATORS

In this section we study the finite rank problem for Toeplitz operators on $H^2(\mathbb{C}^n, d\mu)$. More precisely, we are interested in the conjecture: *If the Toeplitz operator T_f with $f \in \mathrm{Sym}_{>0}(\mathbb{C}^n)$ has finite rank, then $f = 0$.* Starting from Toeplitz operators acting on the Bergman space over bounded domains, the discussion of the finite rank problem has a long history. The conjecture for finite rank Toeplitz operators on Bergman spaces was open for about thirty years. Only recently was the conjecture solved by D. Luecking in [14] (for dimension $n = 1$) and later on generalized to arbitrary dimensions in [1, 5, 16]. We state here the corresponding result in our setting.

Theorem 5.1 ([1, 5, 14, 16])**.** *Let $f \in L^1(\mathbb{C}^n, d\mu)$ be a function having a compact support such that $T_f(\mathbb{P}[z])$ has finite dimensions, then $f = 0$ a.e. on $\mathbb{C}^n$.*

More general versions of Theorem 5.1 were shown in the above papers for a larger class of operators whose symbols are measures (or even distributions) with compact supports. The corresponding results assert that the measure (or the distribution) must have a finite support. In [15] the recent progress on the finite rank problem of Toeplitz operators was described in a systematic way.

It turns out that the approach employed by Luecking and others does not immediately generalize to symbols having non-compact supports. We point out here that by Proposition 4.6, the conclusion of Theorem 5.1 fails if the support of f is not compact. However, the counterexample given in Proposition 4.6 has certain growth at infinity. In fact, it does not even belong to any $\mathcal{D}_c$ with $c < 1$. By this reason one still hopes to prove an affirmative result when appropriate restrictions are impulsed on the growth of the function f, for example, f belongs to $\mathrm{Sym}_{>0}(\mathbb{C}^n)$ as we stated above. In this section we offer several partial results which, we hope, will shed some light on the conjecture.

We first give a necessary condition for a finite rank operator to be a Toeplitz operator and derive some consequences. Let A be a finite rank operator in $H^2(\mathbb{C}^n, d\mu)$ given by the form $A(\cdot) = \sum_{j=1}^N \langle \cdot, f_j\rangle g_j$, where $f_1, \ldots, f_N$ and $g_1, \ldots, g_N$ belong to $H^2(\mathbb{C}^n, d\mu)$. Define the function $F(z) = \sum_{j=1}^N g_j(-iz)\overline{f_j(iz)}$ for $z \in \mathbb{C}^n$. Suppose there exists a function h in $\mathrm{Sym}_{>0}(\mathbb{C}^n)$ such that $T_h = A$ on $H^2(\mathbb{C}^n, d\mu)$. Using the reproducing property of the kernel functions $K_a(w) = \exp(w \cdot \bar{a})$, we get

$$\begin{aligned} F(z) = \sum_{j=1}^N \langle K_{iz}, f_j\rangle\langle g_j, K_{-iz}\rangle = \langle T_h K_{iz}, K_{-iz}\rangle = \langle hK_{iz}, K_{-iz}\rangle \\ = \int_{\mathbb{C}^n} h(w) \exp\{-iz\cdot\bar{w} - iw\cdot\bar{z}\}\, d\mu(w). \end{aligned}$$

For multi-indices α and β, let ∂_z^α and $\partial_{\bar{z}}^\beta$ denote the partial derivatives $\frac{\partial^{|\alpha|}}{\partial z_1^{\alpha_1}\cdots\partial z_n^{\alpha_n}}$ and $\frac{\partial^{|\beta|}}{\partial \bar{z}_1^{\beta_1}\cdots\partial \bar{z}_n^{\beta_n}}$, respectively. By differentiating under the integral sign we obtain

$$\begin{aligned} \partial_z^\alpha\partial_{\bar{z}}^\beta F(z) &= \partial_z^\alpha\partial_{\bar{z}}^\beta \Big(\int_{\mathbb{C}^n} h(w)\exp\{-iz\cdot\bar{w} - i\bar{z}\cdot w\}\, d\mu(w)\Big) \\ &= (-i)^{|\alpha|+|\beta|}\int_{\mathbb{C}^n} h(w)\bar{w}^\alpha w^\beta \exp\{-iz\cdot\bar{w} - i\bar{z}\cdot w\}\, d\mu(w) \\ &= (-i)^{|\alpha|+|\beta|}\int_{\mathbb{C}^n} h(w)\bar{w}^\alpha w^\beta \exp\{-i\mathrm{Re}\,(2z\cdot\bar{w})\}\, d\mu(w) \qquad (5.1) \\ &= (-i)^{|\alpha|+|\beta|}\cdot 2^n \cdot \mathfrak{F}\Big(h(w)\bar{w}^\alpha w^\beta e^{-|w|^2}\Big)(2z), \end{aligned}$$

where $\mathfrak{F}$ denotes the Fourier transform on $\mathbb{C}^n \cong \mathbb{R}^{2n}$. Since $w \mapsto h(w)\bar{w}^\alpha w^\beta e^{-|w|^2}$ is a function in $L^1(\mathbb{C}^n, dV)$, it follows from the mapping properties of the Fourier transform that $\partial_z^\alpha\partial_{\bar{z}}^\beta F$ belongs to $C_0(\mathbb{C}^n)$, the space of all continuous functions on $\mathbb{C}^n$ vanishing at infinity. Thus we have proved

Lemma 5.2. *Let $f_1, \ldots, f_N$ and $g_1, \ldots, g_N$ be in $H^2(\mathbb{C}^n, d\mu)$ such that the operator $A(\cdot) = \sum_{j=1}^N \langle\cdot, f_j\rangle g_j$ equals a Toeplitz operator T_h for some h in $\mathrm{Sym}_{>0}(\mathbb{C}^n)$. Then the function $F(z) = \sum_{j=1}^N g_j(-iz)\overline{f_j(iz)}$ and all its partial derivatives vanish at infinity.*

We suspect that any function F satisfying the conclusion of Lemma 5.2 must be identically zero but we have not found a proof.

We now consider two special cases for which we are able to show that F, and hence h, must be the zero function. The first (trivial) case is when $g_j(z) = f_j(-z)$ for all $1 \le j \le N$. Since $F(z) = \sum_{j=1}^N |f_j(iz)|^2$ and F vanishes at infinity, we conclude that $f_j = 0$ for all $1 \le j \le N$. Using this, we show now that certain finite rank "twisted" projections cannot be represented as Toeplitz operators. Let $\{0\} \neq V \subset H^2(\mathbb{C}^n, d\mu)$ be a subspace of finite dimension N and $P_V : H^2(\mathbb{C}^n, d\mu) \to V$ be the orthogonal projection. Define the operator U_{-1} by $(U_{-1}\varphi)(z) = \varphi(-z)$ for $\varphi \in H^2(\mathbb{C}^n, d\mu)$ and $z \in \mathbb{C}^n$. Then neither $U_{-1}P_V$ nor $P_V U_{-1}$ are Toeplitz operators with symbols in $\mathrm{Sym}_{>0}(\mathbb{C}^n)$. In fact, choose an orthonormal basis $\{\varphi_1, \ldots, \varphi_N\}$ for V. Then we can write $P_V(\cdot) = \sum_{j=1}^N \langle\cdot, \varphi_j\rangle\varphi_j$ and hence $U_{-1}P_V(\cdot) = \sum_{j=1}^N \langle\cdot, \varphi_j\rangle U_{-1}\varphi_j$ and $P_V U_{-1}(\cdot) = \sum_{j=1}^N \langle\cdot, U_{-1}\varphi_j\rangle\varphi_j$. If either $U_{-1}P_V$ or $P_V U_{-1}$ were a Toeplitz operator with symbol in $\mathrm{Sym}_{>0}(\mathbb{C}^n)$, then it would imply, from the above argument, that $\varphi_1 = \cdots = \varphi_N = 0$, a contradiction since $V \neq \{0\}$. We mention in passing here that it can be shown that the operator U_{-1} itself is also not a Toeplitz operator, see [7, Theorem 3].

In order to describe the second special case, we need to introduce some notation. For any polynomial Q in n variables with complex coefficients, we write $Q(\partial) = Q(\partial_{z_1}, \ldots, \partial_{z_n})$ and $Q(\overline{\partial}) = Q(\partial_{\overline{z}_1}, \ldots, \partial_{\overline{z}_n})$. It can be verified that for any polynomial Q, there is a polynomial Q_* such that for any analytic function f on $\mathbb{C}^n$, we have $\overline{Q(\partial)\Big(f(iz)\Big)} = Q_*(\overline{\partial})\Big(\overline{f(iz)}\Big)$ for all $z \in \mathbb{C}^n$. We define $\mathcal{E}$ to be the set of all functions f in $H^2(\mathbb{C}^n, d\mu)$ for which $Q(\partial)f \equiv 0$ for some polynomial Q. It is immediate that $\mathcal{E}$ is a linear subspace of $H^2(\mathbb{C}^n, d\mu)$ which contains all functions of the form $p(z)\exp(z \cdot \overline{a})$, where p is an analytic polynomial and a belongs to $\mathbb{C}^n$.

Now we assume that for each $j = 2, \ldots, N$, either f_j or g_j belongs to $\mathcal{E}$. Choose polynomials Q and R such that $Q(\partial)R_*(\overline{\partial})\Big(g_j(-iz)\overline{f_j(iz)}\Big) = Q(\partial)\Big(g_j(-iz)\Big)\overline{R(\partial)\Big(f_j(iz)\Big)} = 0$ for all $2 \leq j \leq n$ and all $z \in \mathbb{C}^n$. Put $H(z) = Q(\partial)\Big(g_1(-iz)\Big) \cdot R(\partial)\Big(f_1(iz)\Big)$. Then H is entire and

$$|H(z)| = \Big|Q(\partial)\Big(g_1(-iz)\Big) \cdot \overline{R(\partial)\Big(f_1(iz)\Big)}\Big| = \Big|Q(\partial)R_*(\overline{\partial})\Big(F(z, \overline{z})\Big)\Big| \to 0$$

as $|z| \to \infty$. This implies that H, and hence $Q(\partial)R_*(\overline{\partial})F$, is identically zero on $\mathbb{C}^n$. On the other hand, (5.1) shows that there is a polynomial G in the variables $w_1, \ldots, w_n$ and $\overline{w}_1, \ldots, \overline{w}_n$ for which $Q(\partial)R_*(\overline{\partial})F$ is the Fourier transform of $h(w)G(w)e^{-|w|^2}$. We then conclude that $h(w)G(w)e^{-|w|^2} = 0$ for a.e. w. Since the zero set of G has measure zero, $h(w)$ must be zero for a.e. w in $\mathbb{C}^n$. Using this, we now show

Theorem 5.3. *Let h be in $\mathcal{D}_c$ for some $c < 1$. Suppose there exists a function φ in $H^2(\mathbb{C}^n, d\mu)$ such that $\widetilde{T}_h(\mathbb{P}[z])$ is a finite dimensional vector subspace of $\mathcal{E} + \mathbb{C}\varphi$. Then $h(z) = 0$ for a.e. $z \in \mathbb{C}^n$.*

Proof. Consider first the case h is a bounded function. Then $\widetilde{T}_h = T_h$ is a bounded operator on $H^2(\mathbb{C}^n, d\mu)$. Since $\mathbb{P}[z]$ is dense in $H^2(\mathbb{C}^n, d\mu)$ and $T_h(\mathbb{P}[z])$ is a finite dimensional vector space, we conclude that T_h, as an operator on $H^2(\mathbb{C}^n, d\mu)$, has finite rank. Furthermore, the range of T_h is the same as $T_h(\mathbb{P}[z])$, which is contained in $\mathcal{E} + \mathbb{C}\varphi$. This implies that there exist functions $f_1, \ldots, f_N$ in $H^2(\mathbb{C}^n, d\mu)$ and functions $g_2, \ldots, g_N$ in $\mathcal{E}$ such that $T_h(\cdot) = \langle \cdot, f_1\rangle\varphi + \sum_{j=2}^{N}\langle \cdot, f_j\rangle g_j$. It now follows from the discussion preceding the theorem that $h(w) = 0$ for a.e. w in $\mathbb{C}^n$.

Now suppose h is not bounded but it belongs to $\mathcal{D}_c$ for some $c < 1$. Choose a positive number t sufficiently large so that the function $V_t h$ is bounded. Using Lemma 4.2 and the fact that the operator S_t preserves $\mathbb{P}[z]$ and $\mathcal{E}$, we see that $T_{V_t h}(\mathbb{P}[z]) = S_t\widetilde{T}_h S_t(\mathbb{P}[z])$ is a finite dimensional vector subspace of $\mathcal{E} + \mathbb{C}(S_t\varphi)$. It now follows from the case already considered that $V_t h = 0$ a.e., which implies that $h = 0$ a.e. on $\mathbb{C}^n$. This completes the proof of the theorem. □

Corollary 5.4. *If $h \in \mathcal{D}_c$ for some $c < 1$ and $\widetilde{T}_h$ has rank one on $\mathbb{P}[z]$, then $h = 0$ a.e. on $\mathbb{C}^n$.*

Corollary 5.5. *If p is a polynomial in z and $\bar{z}$ such that $\widetilde{T}_p$ has finite rank on $\mathbb{P}[z]$, then $p = 0$ a.e. on $\mathbb{C}^n$.*

Proof. It follows from Definition 4.1 that $\mathbb{P}[z]$ is invariant under $\widetilde{T}_p$ when p is a polynomial in z and $\bar{z}$. The corollary then follows immediately from Theorem 5.3. □

We conclude this section by showing that the existence (or non-existence) of non-trivial finite rank Toeplitz operators on $H^2(\mathbb{C}^n, d\mu)$ can be reduced to the complex one dimensional case, i.e., $H^2(\mathbb{C}, d\mu)$.

Proposition 5.6. *Let $n > 1$ and assume that there is a non-trivial function $f \in \mathcal{D}_c$ for some $c < 1$ such that $\mathbb{P}[z]$ belongs to the domain of $\widetilde{T}_f$ and $\widetilde{T}_f\big(\mathbb{P}[z]\big)$ has finite dimension. Then there is a non-trivial function $g \in L^\infty(\mathbb{C})$ such that T_g has finite rank on $H^2(\mathbb{C}, d\mu)$.*

Conversely, if there is a bounded non-trivial function g on $\mathbb{C}$ such that T_g has finite rank on $H^2(\mathbb{C}, d\mu)$, then there exists a bounded non-trivial function f on $\mathbb{C}^n$ such that T_f has finite rank on $H^2(\mathbb{C}^n, d\mu)$.

Proof. First assume that f is a non-trivial function in $\mathcal{D}_c$ with $c < 1$ such that $\widetilde{T}_f\big(\mathbb{P}[z]\big)$ is finite dimensional. Choose a sufficiently large real number t such that $V_t f$ is bounded. Since $\widetilde{T}_{V_t f} = S_t \widetilde{T}_f S_t$ by Lemma 4.2 and $\mathbb{P}[z]$ is invariant under S_t, the operator $T_{V_t f} = \widetilde{T}_{V_t f}$ also has finite rank on $\mathbb{P}[z]$. By replacing f by $V_t f$, we may assume that f is bounded.

Since f is not the zero function, there is an analytic monomial p such that $T_f p \neq 0$. We write $p(z) = p_1(z_1)p_2(z')$ for $z = (z_1, z')$ in $\mathbb{C}^n$ and choose $y = (y_1, y') \in \mathbb{C}^n$ such that $(T_f p)(y) \neq 0$. For any $w_1 \in \mathbb{C}$, define

$$g(w_1) = \int_{\mathbb{C}^{n-1}} f(w_1, w')p_2(w')e^{y'\cdot\overline{w'}}\, d\mu(w').$$

Since f is bounded on $\mathbb{C}^n$, we see that g is bounded on $\mathbb{C}$ and hence, the Toeplitz operator T_g is bounded on $H^2(\mathbb{C}, d\mu)$.

Put $V = T_f\big(\mathbb{P}[z]\big)$. For any function h in V, define $\tilde{h}$ on $\mathbb{C}$ by $\tilde{h}(z_1) = h(z_1, y')$ for $z_1 \in \mathbb{C}$. Let $\tilde{V} = \{\tilde{h} : h \in V\}$. Then $\dim(\tilde{V}) < \infty$ since $\dim(V) < \infty$. For any analytic polynomial q in one complex variable and any z_1 in $\mathbb{C}$, using (2.2) and Fubini's Theorem, we obtain

$$\begin{aligned}(T_g q)(z_1) &= \int_{\mathbb{C}} g(w_1)q(w_1)e^{z_1\cdot\overline{w}_1}\, d\mu(w_1)\\ &= \int_{\mathbb{C}^n} f(w_1, w')p_2(w')q(w_1)e^{y'\cdot\overline{w'}+z_1\cdot\overline{w}_1}\, d\mu(w_1, w')\\ &= T_f(q\otimes p_2)(z_1, y'),\end{aligned}$$

here $q\otimes p_2$ is the polynomial given by $(q\otimes p_2)(w) = q(w_1)p_2(w')$ for $w = (w_1, w')$. This shows that $T_g q$ belongs to $\tilde{V}$. Since q was arbitrary and $\dim(\tilde{V}) < \infty$, we conclude that $T_g(\mathbb{P}[z_1])$ is finite dimensional. Because $(T_g p_1)(y_1) = (T_f p)(y_1, y') = (T_f p)(y) \neq 0$ by our choice of y, we see also that g is a non-trivial function.

Now assume that g is bounded on $\mathbb{C}$ such that T_g has rank $M < \infty$. Put $f(w) = g(w_1)\cdots g(w_n)$ for $w = (w_1, \ldots, w_n) \in \mathbb{C}^n$. Then f is a bounded function. For any analytic monomial p of the form $p(w) = p_1(w_1)\cdots p_n(w_n)$ and $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$, we have

$$\begin{aligned}T_f p(z) = \int_{\mathbb{C}^n} f(w)p(w)\exp(z\cdot\overline{w})d\mu(w) &= \prod_{j=1}^{n}\int_{\mathbb{C}} g(w_j)p_j(w_j)\exp(z_j\cdot\overline{w}_j)\, d\mu(w_j)\\ &= (T_g p_1)(z_1)\cdots(T_g p_n)(z_n).\end{aligned}$$

Since the space $\{T_g(q) : q$ is an analytic polynomial in one complex variable$\}$ has dimension M, the above formula shows that $T_f(\mathbb{P}[z])$ has dimension M^n. It then follows from the density of $\mathbb{P}[z]$ in $H^2(\mathbb{C}^n, d\mu)$ that T_f has rank M^n. □

6. Open Problems

In this final section we collect and discuss some problems that we have not been able to solve with the hope that they will stimulate further investigation.

First of all, Theorem 4.4 shows that if f and g belong to $\mathcal{D}_c$ for some $c < 1$, $\widetilde{T}_f\widetilde{T}_g = 0$ on analytic polynomials, and one of the functions is radial, then either $f = 0$ or $g = 0$. We do not know if the conclusion still holds if the functions are not assumed to be radial. In fact the corresponding problem for Toeplitz operators on the Bergman space of the unit disc is still unsolved. Thus the following question is open and quite challenging.

Question A. *Let f, g be in $\mathcal{D}_c$ for some $c < 1$ such that $T_fT_g = 0$. Is it true that $f = 0$ or $g = 0$ a.e.?*

The restriction on the growth of the functions f and g in Question A is essential since Proposition 4.5 gives two non-trivial Toeplitz operators (with radial symbols belonging to $\mathcal{D}_1$) whose product is a zero operator.

Other unanswered questions we would like to discuss in this section are related to the existence of finite rank Toeplitz operators. By Proposition 5.6, we only need to consider Toeplitz operators with bounded symbols on $H^2(\mathbb{C}, d\mu)$. Also, by Corollary 5.4, the case of rank one has an affirmative answer.

Question B. *Let f be in $L^\infty(\mathbb{C})$ such that T_f has finite rank ≥ 2 on $\mathbb{P}[z]$. Is it true that $f = 0$ a.e.?*

We present here Luecking's approach [14] to Question B and discuss the difficulties when applied to our current settings. For any integer $N \geq 2$, let $\mathcal{S}_N$ be the permutation group on $\{1, \ldots, N\}$. Any $\pi \in \mathcal{S}_N$ acts on $\mathbb{C}^N$ by $\pi(z) = (z_{\pi(1)}, \ldots, z_{\pi(N)})$ for $z \in \mathbb{C}^N$. We call a function f on $\mathbb{C}^N$ symmetric if $f \circ \pi = f$ for all $\pi \in \mathcal{S}_N$.

On $L^2(\mathbb{C}^N, d\mu)$ we define the symmetrization S by

$$\big[Sf\big](z) = \frac{1}{N!} \sum_{\pi \in \mathcal{S}_N} f \circ \pi(z) = \frac{1}{N!} \sum_{\pi \in \mathcal{S}_N} f(z_{\pi(1)}, \ldots, z_{\pi(N)})$$

for $f \in L^2(\mathbb{C}^N, d\mu)$. It can be checked that S is an orthogonal projection and that $H^2(\mathbb{C}^N, d\mu)$ is invariant under S. This implies $PS = SP$ (recall that P is the orthogonal projection from $L^2(\mathbb{C}^N, d\mu)$ onto $H^2(\mathbb{C}^N, d\mu)$). We define $H^2_s(\mathbb{C}^N, d\mu) = S(H^2(\mathbb{C}^N, d\mu))$, which is the subspace of $H^2(\mathbb{C}^N, d\mu)$ consisting of symmetric functions.

Now assume there is a bounded function f such that T_f has rank less than N. As in [14], we obtain

$$0 = \int_{\mathbb{C}^N} f(z_1) f(z_2) \cdots f(z_N) F_1(z) \overline{F_2(z)} \, |V(z)|^2 \, d\mu(z) \tag{6.1}$$

for all symmetric analytic polynomials F_1 and F_2. Here $V(z) = \det(z_l^{j-1})_{1 \leq l, j \leq N}$ denotes the Vandermonde determinant (note that $|V(z)|^2$ is a symmetric polynomial in $z_1, \ldots, z_N$ and $\overline{z}_1, \ldots, \overline{z}_N$). Put $G(z) = f(z_1) \cdots f(z_N) |V(z)|^2$, which is a symmetric function in $\mathrm{Sym}_{>0}(\mathbb{C}^N)$. For any symmetric polynomial F_1 in $H^2_s(\mathbb{C}^N, d\mu)$ and any polynomial F in $H^2(\mathbb{C}^N, d\mu)$, using the fact that S is a projection, $GF_1 = S(GF_1)$ and (6.1) with $F_2 = S(F)$, we obtain

$$0 = \langle GF_1, S(F) \rangle = \langle S(GF_1), F \rangle = \langle GF_1, F \rangle = \langle T_G F_1, F \rangle.$$

This shows that the Toeplitz operator T_G vanishes on the space of symmetric analytic polynomials in $\mathbb{C}^N$. If it can be proved that $G = 0$ a.e., then it follows that $f = 0$ a.e. since the set of zeros of V has measure zero. Thus the following question is closely related to Question B.

Question C. *Let G be a symmetric function on $\mathbb{C}^N$ ($N \geq 2$) that has at most polynomial growth at infinity. Assume that $T_G : H^2_s(\mathbb{C}^N, d\mu) \longrightarrow H^2_s(\mathbb{C}^N, d\mu)$ vanishes on symmetric analytic polynomials. Does it follow that $G = 0$ a.e.?*

It was shown in [14] that Question C has an affirmative answer when G has compact support. In fact, it follows from the Stone-Weierstrass Theorem that the set $\{F_1 \overline{F}_2 : F_1, F_2 \text{ are analytic symmetric polynomials}\}$ is dense in the space of continuous symmetric functions on any bounded ball centered at the origin in $\mathbb{C}^N$. This then implies that any function G satisfying the hypothesis of Question C must be zero almost everywhere. This approach works even in the case the measure $G d\mu$ is replaced by any complex regular Borel measure with compact support. When G does not have compact support, the Stone-Weierstrass Theorem does not apply and it is, we believe, the main difficulty in this approach.

Finally, Question D below is also closely related to the finite rank problem for Toeplitz operators by Lemma 5.2. We showed in Section 5 that Question D has an affirmative answer under certain restrictions. However, we have not been able to resolve the general case.

Question D. *Let $N \geq 2$ be an integer and $f_1, \ldots, f_N, g_1, \ldots, g_N$ belong to $H^2(\mathbb{C}, d\mu)$ such that the function $F = f_1 \overline{g}_1 + \cdots + f_N \overline{g}_N$ and all of its partial derivatives vanish at infinity. Does it follow that $F = 0$ identically?*

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