# ALGEBRAIC PROPERTIES OF OPERATOR ROOTS OF POLYNOMIALS 

TRIEU LE


#### Abstract

Properties of $m$-selfadjoint and $m$-isometric operators have been investigated by several researchers. Particularly interesting to us are algebraic properties of nilpotent perturbations of such operators. McCullough and Rodman showed in the nineties that if $Q^{n}=0$ and $A$ is a selfadjoint operator commuting with $Q$ then the sum $A+Q$ is a $(2 n-1)$-selfadjoint operator. Very recently, Bermúdez, Martinón, and Noda proved a similar result for nilpotent perturbations of isometries. Via a new approach, we obtain simple proofs of these results and other generalizations to operator roots of polynomials.


## 1. Introduction

Throughout the paper, $H$ denotes a complex Hilbert space and $\mathfrak{L}(H)$ the algebra of all bounded linear operators on $H$. Let $m$ be a positive integer. An operator $T$ in $\mathfrak{L}(H)$ is said to be $m$-selfadjoint if it satisfies the operator equation

$$
\begin{equation*}
\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k} T^{* k} T^{m-k}=0 \tag{1.1}
\end{equation*}
$$

where $T^{*}$ is the adjoint operator of $T$. Here we use the convention that $T^{0}=T^{* 0}=I$, the identity operator on $H$. It is clear that any 1-selfadjoint operator is selfadjoint. This notion of $m$-selfadjoint operators was introduced and studied by Helton [14]. Following Helton, we call an operator $n$-Jordan (or Jordan of order $n$ ) if it can be written as $S+Q$, where $S$ is selfadjoint, $Q$ commutes with $S$ and $Q^{n}=0$. Helton [14] showed that an operator $T$ is 2 -Jordan if and only if $T$ and $T^{*}$ are 3 -selfadjoint.

In [15, McCullough and Rodman studied several algebraic and spectral properties of $m$-selfadjoint operators and they obtained the following result.
Theorem A. (15, Theorem 3.2]) Let $n$ be a positive integer. Suppose $S$ is selfadjoint and $Q^{n}=0$ such that $S Q=Q S$. Then the $n$-Jordan operator $S+Q$ is $(2 n-1)$-selfadjoint.

An operator $T$ is said to be $m$-isometric if

$$
\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k} T^{* k} T^{k}=0
$$

[^0]We say that $T$ is a strict $m$-isometric operator if $T$ is $m$-isometric but it is not ( $m-1$ )-isometric. It is clear that any 1 -isometric operator is isometric. Such $m$-isometric operators were introduced by Agler back in the early nineties and were studied in great detail by Agler and Stankus in a series of three papers [4-6]. In higher dimensions $(d \geq 1)$, the notion of $m$-isometries was introduced and studied by Gleason and Richter in [11]. Recently, researchers [8-10] have been interested in algebraic properties of $m$-isometries. In [8], Bermúdez, Martinón, and Noda proved a result analogous to Theorem A. We say that an operator $Q$ is nilpotent of order $n \geq 1$ if $Q^{n}=0$ and $Q^{n-1} \neq 0$.

Theorem B. ( [8, Theorem 2.2]) Suppose $S$ is an isometry and $Q$ is a nilpotent operator of order $n$ that commutes with $S$. Then the operator $S+Q$ is a strict $(2 n-1)$-isometry.

The proofs of Theorem A and Theorem B in the aforementioned papers rely heavily on combinatorial identities, which do not provide us with any hints why the results are true. The main purpose of this paper is to offer a new approach, which not only simplifies the proofs but also provides a unified treatment to the above results. In addition, our approach reveals a more general phenomenon for nilpotent perturbations of operators that are roots of polynomials.

## 2. Hereditary functional calculus in several variables

The approach that we take in this paper relies on pairs of commuting operators and polynomials in two complex variables. However, it is also convenient to discuss general tuples of commuting operators and polynomials in several variables. We begin with some definitions and notation. Fix an integer $d \geq 1$. Throughout the paper, we use $z$ to denote a single complex variable and the boldface letter $\mathbf{z}$ to denote a tuple of complex variables $\mathbf{z}=\left(z_{1}, \ldots, z_{d}\right) \in \mathbb{C}^{d}$. We write $\overline{\mathbf{z}}=\left(\bar{z}_{1}, \ldots, \bar{z}_{d}\right)$. Let $\mathbb{Z}_{+}^{d}$ be the set of all multiindices $\alpha=\left(\alpha_{1}, \ldots, \alpha_{d}\right)$ of non-negative integers. We shall use the standard multiindex notation $\mathbf{z}^{\alpha}=z_{1}^{\alpha_{1}} \cdots z_{d}^{\alpha_{d}}$ with the convention that $0^{0}=1$. Let $\mathbf{T}=\left(T_{1}, \ldots, T_{d}\right)$ be a $d$-tuple of commuting bounded linear operators on $H$. We write $\mathbf{T}^{\alpha}=T_{1}^{\alpha_{1}} \cdots T_{d}^{\alpha_{d}}$ and $\mathbf{T}^{* \alpha}=\left(\mathbf{T}^{\alpha}\right)^{*}$. Here for $A \in \mathfrak{L}(H)$, the operator $A^{0}$ is the identity operator on $H$.

In this paper, a function $f: \mathbb{C}^{d} \rightarrow \mathbb{C}$ is a polynomial if $f$ is a finite sum of the form

$$
f(\mathbf{z})=\sum_{\alpha, \beta} a_{\alpha, \beta} \overline{\mathbf{z}}^{\alpha} \mathbf{z}^{\beta} .
$$

The coefficients $a_{\alpha, \beta}$ are complex numbers. We shall use the standard notation $\mathbb{C}[\overline{\mathbf{z}}, \mathbf{z}]$ for the ring of all such polynomials. The dimension $d$ should be understood from the context. A polynomial is holomorphic if it is a holomorphic function. Equivalently, it is a polynomial in $\mathbf{z}$ only. A polynomial is anti-holomorphic if it is a polynomial in $\overline{\mathbf{z}}$ only. Equivalently, its conjugate is holomorphic.

For a polynomial $f(\mathbf{z})=\sum_{\alpha, \beta} a_{\alpha, \beta} \overline{\mathbf{z}}^{\alpha} \mathbf{z}^{\beta}$ on $\mathbb{C}^{d}$ and a $d$-tuple $\mathbf{T}$ of commuting operators in $\mathfrak{L}(H)$, we define

$$
\begin{equation*}
f(\mathbf{T})=\sum_{\alpha, \beta} a_{\alpha, \beta} \mathbf{T}^{* \alpha} \mathbf{T}^{\beta} \tag{2.1}
\end{equation*}
$$

This functional calculus was termed the hereditary functional calculus by Agler [1] and was studied in [1,2]. The following properties are immediate from the definition.

Lemma 2.1. Let $p, q, r$ be polynomials in $\mathbb{C}^{d}$ and $\mathbf{T}$ be a d-tuple of commuting operators in $\mathfrak{L}(H)$. Then the following statements hold.
(1) $(p+q)(\mathbf{T})=p(\mathbf{T})+q(\mathbf{T})$.
(2) If $p$ is anti-holomorphic and $r$ is holomorphic, then

$$
(p q r)(\mathbf{T})=p(\mathbf{T}) q(\mathbf{T}) r(\mathbf{T}) .
$$

(3) Define $\bar{p}(z)=\overline{p(z)}$. Then $\bar{p}(\mathbf{T})=(p(\mathbf{T}))^{*}$.

Remark 2.2. Since the adjoints $T_{1}^{*}, \ldots, T_{d}^{*}$ may not commute with $T_{1}, \ldots, T_{d}$, the identity $(p q)(\mathbf{T})=p(\mathbf{T}) q(\mathbf{T})$ does not hold in general.

We call a mapping $\varphi=\left(\varphi_{1}, \ldots, \varphi_{m}\right): \mathbb{C}^{d} \rightarrow \mathbb{C}^{m}$ a holomorphic polynomial mapping if each component $\varphi_{j}(1 \leq j \leq m)$ is a holomorphic polynomial over $\mathbb{C}^{d}$. The following fact concerns composition with holomorphic polynomial mappings.
Lemma 2.3. Suppose $f: \mathbb{C}^{m} \rightarrow \mathbb{C}$ is a polynomial and $\varphi: \mathbb{C}^{d} \rightarrow \mathbb{C}^{m}$ is a holomorphic polynomial mapping. Then for any d-tuple $\mathbf{T}$ of commuting operators in $\mathfrak{L}(H), \varphi(\mathbf{T})$ is an m-tuple of commuting operators and we have

$$
(f \circ \varphi)(\mathbf{T})=f(\varphi(\mathbf{T})) .
$$

Proof. For $\mathbf{w} \in \mathbb{C}^{m}$, write $f(\mathbf{w})=\sum_{\alpha, \beta} a_{\alpha, \beta} \overline{\mathbf{w}}^{\alpha} \mathbf{w}^{\beta}$. Then for $\mathbf{z} \in \mathbb{C}^{d}$,

$$
f \circ \varphi(\mathbf{z})=\sum_{\alpha, \beta} a_{\alpha, \beta} \overline{(\varphi(\mathbf{z}))^{\alpha}}(\varphi(\mathbf{z}))^{\beta}=\sum_{\alpha, \beta} a_{\alpha, \beta} \overline{\varphi^{\alpha}}(\mathbf{z}) \varphi^{\beta}(\mathbf{z}) .
$$

Using the fact that $\varphi$ is holomorphic together with Lemma 2.1, we obtain

$$
(f \circ \varphi)(\mathbf{T})=\sum_{\alpha, \beta} a_{\alpha, \beta} \overline{\varphi^{\alpha}}(\mathbf{T}) \varphi^{\beta}(\mathbf{T})=\sum_{\alpha, \beta} a_{\alpha, \beta}\left(\varphi(\mathbf{T})^{\alpha}\right)^{*} \varphi(\mathbf{T})^{\beta}=f(\varphi(\mathbf{T})) .
$$

Remark 2.4. It is clear that the conclusion of Lemma 2.3 remains valid if $\varphi$ is an anti-holomorphic mapping.

Definition 2.5. Let $f$ be a polynomial in $\mathbb{C}^{d}$ and $\mathbf{T}$ a $d$-tuple of commuting operators in $\mathfrak{L}(H)$. We say that $\mathbf{T}$ is a (hereditary) root of $f$ if $f(\mathbf{T})=0$.
Example 2.6. (1) An operator is $m$-selfadjoint if and only if it is a root of the polynomial $p(z)=(\bar{z}-z)^{m}$ in one variable $z \in \mathbb{C}$.
(2) An operator is $m$-isometric if and only if it is a root of the polynomial $q(z)=(\bar{z} z-1)^{m}$ in one variable $z \in \mathbb{C}$.

Hereditary roots have been studied by several researchers, see $1-6,14$, 15 and the references therein. In a recent paper, Stankus 16 studied spectrum pictures, maximal invariant subspaces, resolvent inequalities and other properties of roots of polynomials over $\mathbb{C}$. In this paper we focus on certain algebraic properties of roots. In particular, we generalize several results obtained in $8-10,15]$.

We begin with a simple but crucial result. Recall that $\mathbb{C}[\overline{\mathbf{z}}, \mathbf{z}]$ denotes the space of polynomials over $\mathbb{C}^{d}$.

Proposition 2.7. Let $\mathbf{T}$ be a d-tuple of commuting operators in $\mathfrak{L}(H)$. Then the set

$$
\mathfrak{J}(\mathbf{T})=\{p \in \mathbb{C}[\overline{\mathbf{z}}, \mathbf{z}]: p(\mathbf{T})=0\}
$$

is an ideal in $\mathbb{C}[\overline{\mathbf{z}}, \mathbf{z}]$.
Proof. Let $p, q$ be polynomials over $\mathbb{C}^{d}$ such that $p$ belongs to $\mathfrak{J}(\mathbf{T})$. Write $q(\mathbf{z})=\sum_{\alpha, \beta} b_{\alpha, \beta} \overline{\mathbf{z}}^{\alpha} \mathbf{z}^{\beta}$. Then

$$
p q(\mathbf{z})=\sum_{\alpha, \beta} b_{\alpha, \beta} \overline{\mathbf{z}}^{\alpha} p(\mathbf{z}) \mathbf{z}^{\beta}
$$

Using Lemma 2.1 and the fact that $p(\mathbf{T})=0$, we obtain

$$
(p q)(\mathbf{T})=\sum_{\alpha, \beta} b_{\alpha, \beta} \mathbf{T}^{* \alpha} p(\mathbf{T}) \mathbf{T}^{\beta}=0
$$

Thus, $p q$ belongs to $\mathfrak{J}(\mathbf{T})$. This shows that $\mathfrak{J}(\mathbf{T})$ is an ideal of $\mathbb{C}[\overline{\mathbf{z}}, \mathbf{z}]$.
Using Proposition 2.7 we obtain
Theorem 2.8. Let $\varphi: \mathbb{C}^{2} \rightarrow \mathbb{C}$ be a holomorphic polynomial. Suppose $p$ is a polynomial over $\mathbb{C}$ such that for all $z, w \in \mathbb{C}$,

$$
p(\varphi(z, w))=p(z) h(z, w)+p(w) g(z, w)
$$

where $g, h$ are polynomials over $\mathbb{C}^{2}$. Let $A$ and $B$ be two commuting operators in $\mathfrak{L}(H)$. Suppose there exist two positive integers $m, n \geq 1$ such that $p^{m}(A)=p^{n}(B)=0$. Then $p^{m+n-1}(\varphi(A, B))=0$.

Proof. By the binomial expansion, there are polynomials $p_{1}$ and $p_{2}$ over $\mathbb{C}^{2}$ such that

$$
q(z, w):=p^{m+n-1}(\varphi(z, w))=(p(z))^{m} p_{1}(z, w)+(p(w))^{n} p_{2}(z, w)
$$

Since $p^{m}(A)=p^{n}(B)=0$, Proposition 2.7 implies that $q(A, B)=0$. On the other hand, Lemma 2.3 gives $q(A, B)=p^{m+n-1}(\varphi(A, B))$. It then follows that $p^{m+n-1}(\varphi(A, B))=0$ as required.

Theorem 2.8 enjoys a number of interesting consequences that we now describe. These results have appeared in the literature with different proofs.

Corollary 2.9. Let $A$ and $B$ be two commuting operators in $\mathfrak{L}(H)$ such that $A$ is $m$-selfadjoint and $B$ is n-selfadjoint. Then for any integers $k, \ell \geq 1$, the operators $A^{k}+B^{\ell}$ and $A^{k} B^{\ell}$ are $(m+n-1)$-selfadjoint. In particular, if $B$ is selfadjoint $(n=1)$, then $A^{k}+B^{\ell}$ and $A^{k} B^{\ell}$ are $m$-selfadjoint.

Proof. Let $p(z)=\bar{z}-z$. Then for $z, w \in \mathbb{C}$, we have

$$
p\left(z^{k}+w^{\ell}\right)=p(z)\left(\bar{z}^{k-1}+\cdots+z^{k-1}\right)+p(w)\left(\bar{w}^{k-1}+\cdots+w^{k-1}\right),
$$

and

$$
\begin{aligned}
p\left(z^{k} w^{\ell}\right) & =\left(\bar{z}^{k}-z^{k}\right) \bar{w}^{\ell}+\left(\bar{w}^{\ell}-w^{\ell}\right) z^{k} \\
& =p(z)\left(\bar{z}^{k-1}+\cdots+z^{k-1}\right) \bar{w}^{\ell}+p(w)\left(\bar{w}^{\ell-1}+\cdots+w^{\ell-1}\right) z^{k} .
\end{aligned}
$$

Since $p^{m}(A)=p^{n}(B)=0$, the conclusion follows from Theorem 2.8 with the choice of $\varphi(z, w)=z^{k}+w^{\ell}$ and $\varphi(z, w)=z^{k} w^{\ell}$, respectively.

In the special case $k=1$ and $B=-\mu I$ with $\mu$ a real number, we recover [15. Proposition 2.1].

Corollary 2.10. Let $A$ and $B$ be two commuting operators in $\mathfrak{L}(H)$ such that $A$ is $m$-isometric and $B$ is $n$-isometric. Then for any integers $k, \ell \geq 1$, the product $A^{k} B^{\ell}$ is $(m+n-1)$-isometric. In particular, if $B$ is isometric ( $n=1$ ), then $A^{k} B^{\ell}$ is m-isometric.

Proof. Let $p(z)=\bar{z} z-1$. Then for $z, w \in \mathbb{C}$,

$$
\begin{aligned}
p\left(z^{k} w^{\ell}\right) & =\left(\bar{z}^{k} z^{k}-1\right) \bar{w}^{\ell} w^{\ell}+\left(\bar{w}^{\ell} w^{\ell}-1\right) \\
& =p(z)\left((\bar{z} z)^{k-1}+\cdots+1\right) \bar{w}^{\ell} w^{\ell}+p(w)\left((\bar{w} w)^{\ell-1}+\cdots+1\right) .
\end{aligned}
$$

The conclusion of the corollary now follows from Theorem 2.8 by the choice of $\varphi(z, w)=z^{k} w^{\ell}$.

A version of Corollary 2.10 was proved in [9, Theorem 3.3] by a combinatorial argument and in the Banach space setting.

## 3. Nilpotent perturbations of roots

We now study nilpotent perturbations of roots. Our main result in this section shows that if $A$ is a root of a polynomial and $Q$ is a nilpotent operator commuting with $A$, then the sum $A+Q$ is a root of a related polynomial. This generalizes Theorems A and B mentioned in the Introduction.

We begin with two elementary results about polynomials in two variables. Recall that for non-negative integers $\alpha, \alpha_{1}, \ldots, \alpha_{k} \geq 0$ with $\alpha=\alpha_{1}+\cdots+\alpha_{k}$, we have the multinomial coefficient defined by

$$
\binom{\alpha}{\alpha_{1}, \ldots, \alpha_{k}}=\frac{\alpha!}{\alpha_{1}!\cdots \alpha_{k}!} .
$$

Proposition 3.1. Suppose $F(z, w)=\bar{w} g_{1}(z, w)+w g_{2}(z, w)$, where $g_{1}$ and $g_{2}$ are polynomials over $\mathbb{C}^{2}$. Let $s \geq 1$ be an integer. Denote by $\mathcal{I}$ the ideal in $\mathbb{C}[\bar{z}, \bar{w}, z, w]$ generated by $\bar{w}^{s}$ and $w^{s}$. Then the polynomials

$$
\begin{equation*}
F^{2 s-2}(z, w)-\binom{2 s-2}{s-1, s-1}(\bar{w} w)^{s-1}\left(g_{1}(z, 0) g_{2}(z, 0)\right)^{s-1} \tag{3.1}
\end{equation*}
$$

and $F^{2 s-1}(z, w)$ belong to $\mathcal{I}$.

Proof. The binomial expansion gives

$$
F^{2 s-2}(z, w)=\sum_{j=0}^{2 s-2}\binom{2 s-2}{j, 2 s-2-j}\left(\bar{w} g_{1}(z, w)\right)^{j}\left(w g_{2}(z, w)\right)^{2 s-2-j} .
$$

Consider the terms on the right hand side. If $j \leq s-2$ then the exponent of $w$ is at least $2 s-2-(s-2)=s$. If $j \geq s$ then the exponent of $\bar{w}$ is at least $s$. This shows that the difference

$$
F^{2 s-2}(z, w)-\binom{2 s-2}{s-1, s-1}(\bar{w} w)^{s-1}\left(g_{1}(z, w) g_{2}(z, w)\right)^{s-1}
$$

is a sum of multiples of $\bar{w}^{s}$ and of $w^{s}$, which belongs to $\mathcal{I}$. On the other hand, the power expansion of the product $g_{1} g_{2}$ at the origin shows that $g_{1}(z, w) g_{2}(z, w)-g_{1}(z, 0) g_{2}(z, 0)$ is a sum of a multiple of $\bar{w}$ and a multiple of $w$. This implies that polynomial

$$
(\bar{w} w)^{s-1}\left(g_{1}(z, w) g_{2}(z, w)\right)^{s-1}-(\bar{w} w)^{s-1}\left(g_{1}(z, 0) g_{2}(z, 0)\right)^{s-1}
$$

belongs to $\mathcal{I}$. Consequently, the difference (3.1) is a polynomial in $\mathcal{I}$. Multiplying (3.1) by $F(z, w)$ immediately yields that $F^{2 s-1}(z, w)$ belongs to $\mathcal{I}$.

Proposition 3.2. Suppose $f$ is a polynomial over $\mathbb{C}$. Let $m, s \geq 1$ be two positive integers. Let $\mathcal{J}$ denote the ideal of $\mathbb{C}[\bar{z}, \bar{w}, z, w]$ generated by the set $\left\{f^{m}(z), \bar{w}^{s}, w^{s}\right\}$. Then the polynomials

$$
f^{m+2 s-3}(z+w)-\binom{m+2 s-3}{m-1, s-1, s-1}(\bar{w} w)^{s-1} f^{m-1}(z)\left(f_{z}(z) f_{\bar{z}}(z)\right)^{s-1}
$$

and $f^{m+2 s-2}(z+w)$ belong to $\mathcal{J}$. Here $f_{z}$ and $f_{\bar{z}}$ denote the partial derivatives $\frac{\partial f}{\partial z}$ and $\frac{\partial f}{\partial \bar{z}}$, respectively.
Proof. There are polynomials $g_{1}$ and $g_{2}$ in $\mathbb{C}[\bar{z}, \bar{w}, z, w]$ such that

$$
f(z+w)=f(z)+\bar{w} g_{1}(z, w)+w g_{2}(z, w) \quad \text { for all } z, w \in \mathbb{C} .
$$

Define $F(z, w)=\bar{w} g_{1}(z, w)+w g_{2}(z, w)$. The binomial expansion gives

$$
f^{m+2 s-3}(z+w)=\sum_{j=0}^{m+2 s-3}\binom{m+2 s-3}{j, m+2 s-3-j} f^{j}(z) F^{m+2 s-3-j}(z, w) .
$$

If $j \leq m-2$, then $m+2 s-3-j \geq 2 s-1$, Proposition 3.1 shows that $F^{m+2 s-3-j}(z, w)$ belongs to $\mathcal{J}$. On the other hand, if $j \geq m$, then $f^{j}(z)$ belongs to $\mathcal{J}$. Consequently, the difference

$$
f^{m+2 s-3}(z+w)-\binom{m+2 s-3}{m-1,2 s-2} f^{m-1}(z) F^{2 s-2}(z, w)
$$

belongs to $\mathcal{J}$. By Proposition 3.1 again,

$$
F^{2 s-2}(z, w)-\binom{2 s-2}{s-1, s-1}(\bar{w} w)^{s-1}\left(g_{1}(z, 0) g_{2}(z, 0)\right)^{s-1}
$$

belongs to $\mathcal{J}$. Since $g_{1}(z, 0)=f_{\bar{z}}(z)$ and $g_{2}(z, 0)=f_{z}(z)$, the conclusion of the proposition follows.

We are now in a position to prove the main result in this section.

Theorem 3.3. Let $p$ be a polynomial over $\mathbb{C}$ and let $A$ and $Q$ be commuting operators in $\mathfrak{L}(H)$. Suppose there are positive integers $m, s \geq 1$ such that $p^{m}(A)=0$ and $Q^{s}=0$. Then the following statements hold.
(a) $p^{m+2 s-2}(A+Q)=0$.
(b) Define $r(z)=p^{m-1}(z) p_{z}^{s-1}(z) p_{z}^{s-1}(z)$. Then $p^{m+2 s-3}(A+Q) \neq 0$ if and only if $\left(Q^{*}\right)^{s-1} r(A) Q^{s-1} \neq 0$.
Proof. Let $\mathcal{J}$ be the ideal in $\mathbb{C}[\bar{z}, \bar{w}, z, w]$ generated by $\left\{p^{m}(z), \bar{w}^{s}, w^{s}\right\}$. Proposition 3.2 shows that both polynomials $q_{1}(z, w)=p^{m+2 s-2}(z+w)$ and

$$
q_{2}(z, w)=p^{m+2 s-3}(z+w)-\binom{m+2 s-3}{m-1, s-1, s-1} \bar{w}^{s-1} r(z) w^{s-1}
$$

belong to $\mathcal{J}$. Since $p^{m}(A)=0$ and $\left(Q^{s}\right)^{*}=Q^{s}=0$, it follows from Proposition 2.7 that $f(A, Q)=0$ for any polynomial $f(z, w)$ in $\mathcal{J}$. In particular, $q_{1}(A, Q)=0$ and $q_{2}(A, Q)=0$. By Lemmas 2.1 and 2.3 we infer that $q_{1}(A, Q)=p^{m+2 s-2}(A+Q)$ and

$$
q_{2}(A, Q)=p^{m+2 s-3}(A+Q)-\binom{m+2 s-3}{m-1, s-1, s-1}\left(Q^{*}\right)^{s-1} r(A) Q^{s-1}
$$

The conclusions of the theorem then follow.
We now apply Theorem 3.3 to obtain several results on nilpotent perturbations of $m$-selfadjoint and $m$-isometric operators.
3.1. Perturbations of $m$-selfadjoint operators. Recall that an operator $T$ on $H$ is $m$-selfadjoint if $p^{m}(T)=0$, where $p(z)=\bar{z}-z$ for $z \in \mathbb{C}$. We say that $T$ is a strict $m$-selfadjoint operator if $p^{m-1}(T) \neq 0$.

Since $p_{z}=-1$ and $p_{\bar{z}}=1$, the polynomial $r(z)$ in Theorem 3.3 is

$$
r(z)=(-1)^{s-1} p^{m-1}(z) .
$$

We then obtain
Theorem 3.4. Let $A$ be m-selfadjoint and $Q$ be nilpotent with $Q^{s}=0$ for some integer $s \geq 1$. Suppose that $A$ and $Q$ commute. Then $A+Q$ is ( $m+2 s-1$ )-selfadjoint. Furthermore, $A+Q$ is strictly ( $m+2 s-2$ )-selfadjoint if and only if $\left(Q^{*}\right)^{s-1} p^{m-1}(A) Q^{s-1} \neq 0$.

In the case $A$ is selfadjoint (that is, $m=1$ ), we have

$$
\left(Q^{*}\right)^{s-1} p^{m-1}(A) Q^{s-1}=\left(Q^{*}\right)^{s-1} Q^{s-1} .
$$

This operator is not zero if and only if $Q^{s-1} \neq 0$. Consequently, we recover Theorem A as a corollary.

Corollary 3.5. Let $A$ be a selfadjoint operator and $Q$ be a nilpotent operator that commutes with $A$. Then $A+Q$ is strictly $(2 s-1)$-selfadjoint if and only if $Q$ is nilpotent of order $s$.
3.2. Perturbations of $m$-isometric operators. An operator $T$ on $H$ is $m$-isometric if $q^{m}(T)=0$, where $q(z)=\bar{z} z-1$. We say that $T$ is a strict $m$-isometric operator if $q^{m-1}(T) \neq 0$.

Since $q_{z}=\bar{z}$ and $q_{\bar{z}}=z$, the polynomial $r(z)$ in Theorem 3.3 is

$$
r(z)=q^{m-1}(z) q_{z}^{s-1}(z) q_{\bar{z}}^{s-1}(z)=\bar{z}^{s-1}(\bar{z} z-1)^{m-1} z^{s-1}
$$

As an application of Theorem 3.3, we have
Theorem 3.6. Let $A$ be m-isometric and $Q$ be nilpotent with $Q^{s}=0$ for some integer $s \geq 1$. Suppose that $A$ and $Q$ commute. Then $A+Q$ is an $(m+2 s-2)$-isometry. Furthermore, $A+Q$ is a strict $(m+2 s-2)$-isometry if and only if $\left(Q^{*}\right)^{s-1} r(A) Q^{s-1} \neq 0$.

In the case $A$ is isometric (that is, $m=1$ ), we have $A^{*} A=I$ and hence, $r(A)=\left(A^{*}\right)^{s-1} A^{s-1}=I$. Consequently, $\left(Q^{*}\right)^{s-1} r(A) Q^{s-1}=\left(Q^{*}\right)^{s-1} Q^{s-1}$. This operator is not zero if and only if $Q^{s-1} \neq 0$. We then recover Theorem B as a corollary.

Corollary 3.7. Let $A$ be an isometry and $Q$ be a nilpotent operator that commutes with $A$. Then $A+Q$ is strictly $(2 s-1)$-isometric if and only if $Q$ is nilpotent of order $s$.

Remark 3.8. It has been brought to our attention recently that the results in this subsection have been obtained by several researchers independently. In fact, Gu and Stankus [13] proved Theorem 3.6 (and Theorem 3.4 as well) using a similar but less general approach than ours here. In [7], Bermúdez et al. obtained Theorem 3.6 via a different method which makes use of arithmetic progressions. In addition, they provided examples which show that Theorem 3.6 no longer holds in the Banach space setting. Recently, Theorem 3.6 has also been used in the study of $m$-isometric elementary operators in 12].
3.3. Perturbations of $m$-isometric powers. We now describe an application that involves operators whose powers are $m$-isometric. Fix an integer $\ell \geq 1$. Suppose $A$ is a bounded operator on $H$ such that $A^{\ell}$ is $m$ isometric. There exist examples of such operators $A$ for which $A$ is not itself $m$-isometric. Theorem 3.3 offers a result concerning perturbations of $A$ by a nilpotent operator.

Theorem 3.9. Let $Q$ be an operator commuting with $A$ and $Q^{s}=0$ for some integer $s \geq 1$. Then $(A+Q)^{\ell}$ is $(m+2 s-2)$-isometric.

Proof. Let $h(z)=\bar{z}^{\ell} z^{\ell}-1$. Then $h^{m}(A)=0$. Theorem 3.3 shows that $h^{m+2 s-2}(A+Q)=0$, which means that $(A+Q)^{\ell}$ is $(m+2 s-2)$-isometric.

It is not clear whether one can find a combinatorial proof of Theorem 3.9, We leave this for the interested reader.

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Department of Mathematics and Statistics, Mail Stop 942, University of Toledo, Toledo, OH 43606

E-mail address: trieu.le2@utoledo.edu


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