ON A CLASS OF IDEALS OF THE TOEPLITZ ALGEBRA ON THE BERGMAN SPACE

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ABSTRACT. Let $\mathfrak{T}$ denote the full Toeplitz algebra on the Bergman space of the unit ball $\mathbb{B}_n$. For each subset G of L^∞, let $\mathfrak{CI}(G)$ denote the closed two-sided ideal of $\mathfrak{T}$ generated by all $T_fT_g - T_gT_f$ with $f, g \in G$. It is known that $\mathfrak{CI}(C(\overline{\mathbb{B}}_n)) = \mathcal{K}$ - the ideal of compact operators and $\mathfrak{CI}(C(\mathbb{B}_n) \cap L^\infty) = \mathfrak{T}$. Despite these "extreme cases", there are subsets G of L^∞ so that $\mathcal{K} \subsetneq \mathfrak{CI}(G) \subsetneq \mathfrak{T}$. This paper gives a construction of a class of such subsets.

1. INTRODUCTION

For any integer $n \geq 1$, let $\mathbb{C}^n$ denote the Cartesian product of n copies of $\mathbb{C}$. For $z = (z_1, \ldots, z_n)$ and $w = (w_1, \ldots, w_n)$ in $\mathbb{C}^n$, we write $\langle z, w\rangle = z_1\overline{w}_1 + \cdots + z_n\overline{w}_n$ and $|z| = \sqrt{|z_1|^2 + \cdots + |z_n|^2}$ for the inner product and the associated Euclidean norm. Let $\mathbb{B}_n$ denote the open unit ball which consists of all $z \in \mathbb{C}^n$ with $|z| < 1$. Let $\mathbb{S}_n$ denote the unit sphere which consists of all $z \in \mathbb{C}^n$ with $|z| = 1$. For any subset V of $\mathbb{B}_n$ we write $\mathrm{cl}(V)$ for the closure of V as a subset of $\mathbb{C}^n$ with respect to the Euclidean metric. We write $\overline{\mathbb{B}}_n$ for the closed unit ball which is also $\mathrm{cl}(\mathbb{B}_n)$. Let $C(\mathbb{B}_n)$ (respectively, $C(\overline{\mathbb{B}}_n)$) denote the space of all functions that are continuous in the open unit ball (respectively, the closed unit ball).

Let ν denote the Lebesgue measure on $\mathbb{B}_n$ normalized so that $\nu(\mathbb{B}_n) = 1$. Let $L^2 = L^2(\mathbb{B}_n, \mathrm{d}\nu)$ and $L^\infty = L^\infty(\mathbb{B}_n, \mathrm{d}\nu)$. The Bergman space L^2_a is the subspace of L^2 which consists of all analytic functions. The normalized reproducing kernels for L^2_a are of the form

$$k_z(w) = (1 - |z|^2)^{(n+1)/2}(1 - \langle w, z\rangle)^{-n-1},\ |z|, |w| < 1.$$

We have $\|k_z\| = 1$ and $\langle g, k_z\rangle = (1 - |z|^2)^{(n+1)/2} g(z)$ for all $g \in L^2_a, z \in \mathbb{B}_n$.

The orthogonal projection from L^2 onto L^2_a is given by

$$(Pg)(z) = \int_{\mathbb{B}_n} \frac{g(w)}{(1 - \langle z, w\rangle)^{n+1}}\ \mathrm{d}\nu(w),\ g \in L^2,\ z \in \mathbb{B}_n.$$

For any $f \in L^\infty$ the Toeplitz operator $T_f : L^2_a \longrightarrow L^2_a$ is defined by $T_f h = P(fh)$ for $h \in L^2_a$. We have

$$(T_f h)(z) = \int_{\mathbb{B}_n} \frac{f(w)h(w)}{(1 - \langle z, w\rangle)^{n+1}}\ \mathrm{d}\nu(w) \tag{1.1}$$

for $h \in L^2_a$ and $z \in \mathbb{B}_n$.

For all $f \in L^\infty$, $\|T_f\| \leq \|f\|_\infty$ and $T^*_f = T_{\overline{f}}$. In contrast with Toeplitz operators on the Hardy space of the unit sphere, there are functions $f \in L^\infty$ so that $\|T_f\| <$

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$\|f\|_\infty$. Since T_f is an integral operator by equation (1.1), we see that T_f is compact if f vanishes almost everywhere in the complement of a compact subset of $\mathbb{B}_n$.

Let $\mathfrak{B}(L^2_a)$ be the C$^*-$algebra of all bounded linear operators on L^2_a. Let $\mathcal{K}$ denote the ideal of $\mathfrak{B}(L^2_a)$ that consists of all compact operators. The full Toeplitz algebra $\mathfrak{T}$ is the C$^*-$subalgebra of $\mathfrak{B}(L^2_a)$ generated by $\{T_f : f \in L^\infty\}$. For any subset G of L^∞, let $\mathfrak{I}(G)$ denote the closed two-sided ideal of $\mathfrak{T}$ generated by all T_f with $f \in G$. Let $\mathfrak{CI}(G)$ denote the closed two-sided ideal of $\mathfrak{T}$ generated by all commutators $[T_f, T_g] = T_fT_g - T_gT_f$ with $f, g \in G$. A result of L. Coburn [1] in the 70's showed that $\mathfrak{CI}(C(\overline{\mathbb{B}}_n)) = \mathcal{K}$. In 2004 D. Suárez [5] showed that $\mathfrak{CI}(L^\infty) = \mathfrak{T}$ for the case $n = 1$. This result has been generalized by the author [2] to all $n \geq 1$. In fact, we are able to show that $\mathfrak{CI}(G) = \mathfrak{T}$ for certain subsets G of L^∞. We can take $G = \{f \in C(\mathbb{B}_n) \cap L^\infty : f \text{ vanishes on } \mathbb{B}_n \backslash E\}$ where $0 < \nu(E)$ can be as small as we please. We can also take $G = \{f \in L^\infty : f \text{ vanishes on } \mathbb{B}_n \backslash E\}$ where E is a closed nowhere dense subset of $\mathbb{B}_n$ with $0 < \nu(E)$ as small as we please. From these results, one may be interested in the question: is there any subset G of L^∞ so that $\mathcal{K} \subsetneq \mathfrak{CI}(G) \subsetneq \mathfrak{T}$? The purpose of this paper is to show that there are infinitely many such subsets. Our main result is the following theorem.

Theorem 1.1. *To every closed subset F of $\mathbb{S}_n$, there is a subset G_F of L^∞ so that the following statements hold true:*

1. $\mathfrak{I}(G_\emptyset) = \mathcal{K}$ *and* $\mathfrak{CI}(G_{\mathbb{S}_n} \cap C(\mathbb{B}_n)) = \mathfrak{T}$.
2. *If F_1, F_2 are closed subsets of $\mathbb{S}_n$ and $F_1 \subset F_2$ then $G_{F_1} \subset G_{F_2}$.*
3. *If F_1, F_2 are closed subsets of $\mathbb{S}_n$ and $F_2 \backslash F_1 \neq \emptyset$ then we have $\mathfrak{CI}(G_{F_2} \cap C(\mathbb{B}_n)) \backslash \mathfrak{I}(G_{F_1}) \neq \emptyset$. In particular, if $\emptyset \neq F \subsetneq \mathbb{S}_n$ then $\mathcal{K} \subsetneq \mathfrak{CI}(G_F \cap C(\mathbb{B}_n)) \subseteq \mathfrak{I}(G_F) \subsetneq \mathfrak{T}$.*

In Section 2 and Section 3, we provide some preliminaries and basic results. In Section 4, we give a proof for Theorem 1.1.

2. PRELIMINARIES

For any $z \in \mathbb{B}_n$, let φ_z denote the Mobius automorphism of $\mathbb{B}_n$ that interchanges 0 and z. For any $z, w \in \mathbb{B}_n$, let $\rho(z, w) = |\varphi_z(w)|$. Then ρ is a metric on $\mathbb{B}_n$ (called the pseudo-hyperbolic metric) which is invariant under the action of the automorphism group $\text{Aut}(\mathbb{B}_n)$ of $\mathbb{B}_n$. For any $w \in \mathbb{B}_n$, we have $\rho(z, w) \to 1$ as $|z| \to 1$. These properties and the following inequality can be proved by using the identities in [4, Theorem 2.2.2]. See [2, Section 2] for more details.

For any $z, w, u \in \mathbb{B}_n$,

$$\rho(z, w) \leq \frac{\rho(z, u) + \rho(u, w)}{1 + \rho(z, u)\rho(u, w)}. \tag{2.1}$$

For any a in $\mathbb{B}_n$ and any $0 < r < 1$, let $B(a, r) = \{z \in \mathbb{B}_n : |z - a| < r\}$ and $E(a, r) = \{z \in \mathbb{B}_n : \rho(z, a) < r\}$.

Inequality (2.1) shows that if $z, w \in \mathbb{B}_n$ so that $E(z, r_1) \cap E(w, r_2) \neq \emptyset$ for some $0 < r_1, r_2 < 1$ then

$$\begin{aligned}\rho(z, w) &\leq \frac{\rho(z, u) + \rho(u, w)}{1 + \rho(z, u)\rho(u, w)} \quad \text{(where } u \text{ is any element in } E(z, r_1) \cap E(w, r_2)\text{)} \\ &< \frac{r_1 + r_2}{1 + r_1 r_2}.\end{aligned}$$

This implies that if $\rho(z,w) \geq \dfrac{r_1+r_2}{1+r_1r_2}$ then $E(z,r_1)\cap E(w,r_2)=\emptyset$.

Lemma 2.1. *For any $0<r<1$ and $\zeta\in\mathbb{S}_n$ there is an increasing sequence $\{t_m\}_{m=1}^\infty\subset(0,1)$ so that $t_m\to 1$ as $m\to\infty$ and $E(t_k\zeta,r)\cap E(t_l\zeta,r)=\emptyset$ for all $k\neq l$.*

Proof. We will construct the required sequence $\{t_m\}_{m=1}^\infty$ by induction. We begin by taking any t_1 in $(0,1)$. Suppose we have chosen $t_1<\cdots<t_m$ so that $1-j^{-1}<t_j<1$ for all $1\leq j\leq m$ and $E(t_k\zeta,r)\cap E(t_l\zeta,r)=\emptyset$ for all $1\leq k<l\leq m$, where $m\geq 1$. Since $\rho(t\zeta,t_j\zeta)\to 1$ as $t\uparrow 1$ for all $1\leq j\leq m$, we can choose a t_{m+1} with $\max\{t_m,1-(m+1)^{-1}\}<t_{m+1}<1$ and $\rho(t_{m+1}\zeta,t_j\zeta)>\dfrac{2r}{1+r^2}$ for all $1\leq j\leq m$. It then follows that $E(t_{m+1}\zeta,r)\cap E(t_j\zeta,r)=\emptyset$ for all $1\leq j\leq m$. Also, since $1-m^{-1}<t_m<1$ for all m, $t_m\to 1$ as $m\to\infty$. □

Lemma 2.2. *For any $0<r<1$ and any $\epsilon>0$, there is a δ depending on r and ϵ so that for all $\zeta\in\mathbb{S}_n$ and all $a\in\mathbb{B}_n$ with $|a-\zeta|<\delta$, we have*

$$E(a,r)\subset\{z\in\mathbb{B}_n:|z-\zeta|<\epsilon\}.$$

As a consequence, if $b\in\mathbb{B}_n$ and $\zeta\in\mathbb{S}_n$ so that $E(b,r)\cap\{z\in\mathbb{B}_n:|z-\zeta|<\delta\}\neq\emptyset$ then $|b-\zeta|<\epsilon$.

Proof. From [4, Section 2.2.7], for any $a\neq 0$,

$$E(a,r)\subset\{z\in\mathbb{B}_n:\frac{|Pz-c|^2}{r^2s^2}+\frac{|Qz|^2}{r^2s}<1\},$$

where $Pz=\dfrac{\langle z,a\rangle}{\langle a,a\rangle}a$, $Qz=z-Pz$, $c=\dfrac{1-r^2}{1-r^2|a|^2}a$ and $s=\dfrac{1-|a|^2}{1-r^2|a|^2}$. It then follows that $E(a,r)\subset B(c,r\sqrt{s})$.

Since $|a-c|=r^2s|a|\leq r\sqrt{s}$, we get $B(c,r\sqrt{s})\subset B(a,2r\sqrt{s})$. Hence $E(a,r)\subset B(a,2r\sqrt{s})$. Note that the inclusion certainly holds true for $a=0$ (in this case $s=1$).

Now suppose $|a-\zeta|<\delta$. Then $|a|\geq|\zeta|-|a-\zeta|>1-\delta$. Hence,

$$s=\frac{1-|a|^2}{1-r^2|a|^2}\leq\frac{1-|a|^2}{1-r^2}\leq\frac{2(1-|a|)}{1-r^2}<\frac{2\delta}{1-r^2}.$$

So for any $z\in E(a,r)$,

$$|z-\zeta|\leq|z-a|+|a-\zeta|\leq 2r\sqrt{s}+\delta<2r\sqrt{\frac{2\delta}{1-r^2}}+\delta.$$

Choosing δ so that $2r\sqrt{\dfrac{2\delta}{1-r^2}}+\delta<\epsilon$, we then have the first conclusion of the lemma.

Now suppose $a,b\in\mathbb{B}_n$ so that $|a-\zeta|<\delta$ and $a\in E(b,r)$. Then since $b\in E(a,r)$, the first conclusion of the lemma implies $|b-\zeta|<\epsilon$. □

For any $z\in B_n$, the formula

$$U_z(f)=(f\circ\varphi_z)k_z,\ f\in L^2,$$

defines a bounded operator on L^2. It is well-known that U_z is a unitary operator with L_a^2 as a reducing subspace and $U_zT_fU_z^*=T_{f\circ\varphi_z}$ on L_a^2 for all $z\in B_n$ and all $f\in L^\infty$. See, for example, [3, Lemmas 7 and 8].

Lemma 2.3. *For any sequence* $\{z_m\}_{m=1}^{\infty} \subset \mathbb{B}_n$ *with* $|z_m| \to 1$, $U_{z_m} \to 0$ *in the weak operator topology of* $\mathfrak{B}(L^2_a)$.

Proof. Since $\operatorname{Span}(\{k_z : z \in \mathbb{B}_n\})$ is dense in L^2_a, it suffices to show that for all z, w in $\mathbb{B}_n$, we have $\lim\limits_{m\to\infty} \langle U_{z_m} k_z, k_w\rangle = 0$.

Fix such z and w. For each $m \geq 1$,

$$\begin{aligned}\langle U_{z_m} k_z, k_w\rangle &= (1-|w|^2)^{(n+1)/2}(U_{z_m}k_z)(w)\\ &= (1-|w|^2)^{(n+1)/2}k_z(\varphi_{z_m}(w))k_{z_m}(w)\\ &= \frac{\big((1-|w|^2)(1-|z|^2)(1-|z_m|^2)\big)^{(n+1)/2}}{\big((1-\langle\varphi_{z_m}(w),z\rangle)(1-\langle w,z_m\rangle)\big)^{n+1}}.\end{aligned}$$

Since $|\langle\varphi_{z_m}(w),z\rangle| \leq |z|$ and $|\langle w, z_m\rangle| \leq |w|$, we obtain

$$|\langle U_{z_m}k_z, k_w\rangle| \leq \frac{\big((1-|w|^2)(1-|z|^2)(1-|z_m|^2)\big)^{(n+1)/2}}{\big((1-|z|)(1-|w|)\big)^{n+1}}.$$

It then follows that $\lim\limits_{m\to\infty}\langle U_{z_m}k_z, k_w\rangle = 0$. □

Lemma 2.4. *Let* $\{z_m\}_{m=1}^{\infty} \subset \mathbb{B}_n$ *so that* $|z_m| \to 1$ *as* $m \to \infty$. *Let* S *be any non-zero positive operator on* L^2_a. *Suppose* $A = \sum\limits_{m=1}^{\infty} U_{z_m} S U^*_{z_m}$ *exists in the strong operator topology and is a bounded operator on* L^2_a. *Then there is a constant* $c > 0$ *and an* $f \in L^2_a$ *so that* $\|AU_{z_m} f\| \geq c > 0$ *for all* m.

Proof. Since S is non-zero and positive, there is an $f \in L^2_a$ with $\|f\| = 1$ so that $\langle Sf, f\rangle > 0$. For each $m \geq 1$,

$$\begin{aligned}\langle AU_{z_m}f, U_{z_m}f\rangle &\geq \langle U_{z_m}SU^*_{z_m}U_{z_m}f, U_{z_m}f\rangle\\ &\geq \langle SU^*_{z_m}U_{z_m}f, U^*_{z_m}U_{z_m}f\rangle\\ &\geq \langle Sf, f\rangle.\end{aligned}$$

Since $\|U_{z_m}f\| = 1$ it follows that $\|AUz_m f\| \geq \langle Sf, f\rangle > 0$ for all m. □

3. BASIC RESULTS

The first result in this section shows that for a certain class of subsets G of L^∞, $\mathfrak{I}(G)$ possesses a special property. This property will later help us distinguish $\mathfrak{I}(G_1)$ and $\mathfrak{I}(G_2)$ for $G_1 \neq G_2$.

Proposition 3.1. *Let* W *be a subset of* $\mathbb{B}_n$ *and let* $F = \operatorname{cl}(W) \cap \mathbb{S}_n$. *Let* f *be in* L^∞ *so that* f *vanishes almost everywhere in* $\mathbb{B}_n\backslash W$. *Let* $g_1, \ldots, g_l$ *be any functions in* L^∞. *Let* $\{z_m\}_{m=1}^{\infty}$ *be any sequence in* $\mathbb{B}_n$ *so that* $|z_m| \to 1$ *and* $|z_m - w| \geq \epsilon > 0$ *for all* $w \in F$, *all* $m \geq 1$, *where* ϵ *is a fixed constant. Then the sequence* $\{T_f T_{g_1}\cdots T_{g_l} U_{z_m}\}_{m=1}^{\infty}$ *converges to* 0 *in the strong operator topology of* $\mathfrak{B}(L^2_a)$. *Consequently, if we set*

$$G = \{f \in L^\infty : f \text{ vanishes almost everywhere in } \mathbb{B}_n\backslash W\}$$

then for any $T \in \mathfrak{I}(G)$, $TU_{z_m} \to 0$ *in the strong operator topology of* $\mathfrak{B}(L^2_a)$.

Proof. Let $V_1 = \{|z| \le 1 : |z-w| < \epsilon/3 \text{ for some } w \in F\}$ and $V_2 = \{|z| \le 1 : |z-w| < \epsilon/2 \text{ for some } w \in F\}$. Let η be a continuous function on $\overline{\mathbb{B}}_n$ so that $0 \le \eta \le 1,\ \eta(z) = 1$ if $z \in \mathrm{cl}(V_1)$ and $\eta(z) = 0$ if $z \notin V_2$. Let $Z = \mathrm{cl}(W) \cap (\overline{\mathbb{B}}_n \backslash V_1)$. Then $Z \subset \overline{\mathbb{B}}_n$ and Z is compact with respect to the Euclidean metric. We have

$$Z \cap \mathbb{S}_n = \mathrm{cl}(W) \cap \mathbb{S}_n \cap (\overline{\mathbb{B}}_n \backslash V_1) = F \cap (\overline{\mathbb{B}}_n \backslash V_1) = \emptyset.$$

Thus Z is a compact subset of $\mathbb{B}_n$. Since the function $f(1-\eta)$ vanishes almost everywhere in $(\mathbb{B}_n \backslash W) \cup \mathrm{cl}(V_1)$ which contains $\mathbb{B}_n \backslash Z$, the operator $T_{f(1-\eta)}$ is compact.

Since η is continuous on $\overline{\mathbb{B}}_n$, the operators $T_g T_{1-\eta} - T_{g(1-\eta)}$ and $T_{1-\eta} T_g - T_{g(1-\eta)}$ are compact for all $g \in L^\infty$ (see [1]). So we have

$$\begin{aligned} T_f T_{g_1} \cdots T_{g_l} &= T_f T_{g_1} \cdots T_{g_l} T_\eta + T_f T_{g_1} \cdots T_{g_l} T_{1-\eta} \\ &= T_f T_{g_1} \cdots T_{g_l} T_\eta + T_{f(1-\eta)} T_{g_1} \cdots T_{g_l} + K_1 \\ &= T_f T_{g_1} \cdots T_{g_l} T_\eta + K, \end{aligned} \tag{3.1}$$

where K_1 is a compact operator and $K = T_{f(1-\eta)} T_{g_1} \cdots T_{g_l} + K_1$ is also a compact operator.

For any $h \in L_a^2 \cap L^\infty$ and any $m \ge 1$ we have

$$\begin{aligned} \|T_\eta U_{z_m} h\|^2 &\le \|\eta U_{z_m} h\|^2 \\ &\le \int_{V_2} |(U_{z_m} h)(z)|^2 \, \mathrm{d}\nu(z) \\ &= \int_{V_2} |h(\varphi_{z_m}(z)) k_{z_m}(z)|^2 \, \mathrm{d}\nu(z) \\ &\le \|h\|_\infty^2 \int_{V_2} |k_{z_m}(z)|^2 \, \mathrm{d}\nu(z). \end{aligned}$$

Let $V_3 = \{|z| \le 1 : |z-w| < \epsilon \text{ for some } w \in F\}$. Since the map $(z,w) \mapsto |1 - \langle z, w\rangle|$ is continuous and does not vanish on the compact set $\mathrm{cl}(V_2) \times (\overline{\mathbb{B}}_n \backslash V_3)$, there is a $\delta > 0$ so that $|1 - \langle z, w\rangle| \ge \delta$ for all $z \in \mathrm{cl}(V_2)$ and $w \in (\overline{\mathbb{B}}_n \backslash V_3)$.

For each $m \ge 1,\ z_m \in (\mathbb{B}_n \backslash V_3)$, so for all $z \in V_2$,

$$|k_{z_m}(z)| \le \frac{(1-|z_m|^2)^{(n+1)/2}}{|1 - \langle z, z_m\rangle|^{n+1}} \le \frac{(1-|z_m|^2)^{(n+1)/2}}{\delta^{n+1}}.$$

Hence we have

$$\|T_\eta U_{z_m} h\| \le \|h\|_\infty \sqrt{\nu(V_2)} \frac{(1-|z_m|^2)^{(n+1)/2}}{\delta^{n+1}}.$$

This implies $\|T_\eta U_{z_m} h\| \to 0$ as $m \to \infty$. Since $L_a^2 \cap L^\infty$ is dense in L_a^2 and $\|T_\eta U_{z_m}\| \le \|T_\eta\| \le 1$ for all m, we conclude that $T_\eta U_{z_m} \to 0$ in the strong operator topology of $\mathfrak{B}(L_a^2)$. So $T_f T_{g_1} \cdots T_{g_l} T_\eta U_{z_m} \to 0$ in the strong operator topology of $\mathfrak{B}(L_a^2)$. Also by Lemma 2.3, $U_{z_m} \to 0$ in the weak operator topology, so $K U_{z_m} \to 0$ in the strong operator topology for any compact operator K. Combining these facts with (3.1), we conclude that $T_f T_{g_1} \cdots T_{g_l} U_{z_m} \to 0$ in the strong operator topology of $\mathfrak{B}(L_a^2)$. ☐

The following proposition was proved by Suárez for the case $n = 1$ (see [5, Proposition 2.9]). The case $n \ge 2$ is similar and can be proved with the same method. The point is that for all $n \geq 2$, the metric ρ and the reproducing kernel functions have all the properties needed for Suárez's proof. See [2, Section 2] for more details.

Proposition 3.2. *Let $0 < r < 1$ and $\{w_m\}_{m=1}^{\infty}$ be a sequence in $\mathbb{B}_n$ so that $E(w_k, r) \cap E(w_l, r) = \emptyset$ for all $k \neq l$. For each $m \in \mathbb{N}$, let $c_m^1, \ldots, c_m^l$, a_m, b_m, $d_m^1, \ldots, d_m^k \in L^\infty$ be functions of norm ≤ 1 that vanish almost everywhere on $\mathbb{B}_n \backslash E(w_m, r)$. Then*

$$\sum_{m \in \mathbb{N}} T_{c_m^1} \cdots T_{c_m^l} (T_{a_m} T_{b_m} - T_{b_m} T_{a_m}) T_{d_m^1} \cdots T_{d_m^k}$$

belongs to $\mathfrak{CI}(L^\infty)$.

Remark 3.3. In the proof of Proposition 3.2, we work only with Toeplitz operators with symbols in the set G which consists of functions of the form $\sum_{m \in F} f_m$, where F is a subset of $\mathbb{N}$ and f is one of the symbols $c^1, \ldots, c^l$, a, b, $d^1, \ldots, d^k$. So in the conclusion of the proposition, we may replace $\mathfrak{CI}(L^\infty)$ by $\mathfrak{CI}(G)$.

4. PROOF OF THE MAIN THEOREM

We are now ready for the proof of Theorem 1.1.

Fix $0 < r < 1$. Let $W_\emptyset = E(0, r)$. For any closed non-empty subset F of $\mathbb{S}_n$, let

$$W_F = \bigcup_{0<t<1} \bigcup_{\zeta \in F} E(t\zeta, r).$$

It is clear that $W_{\mathbb{S}_n} = \mathbb{B}_n$. We always have $F \subset \text{cl}(W_F) \cap \mathbb{S}_n$. We will show that in fact $F = \text{cl}(W_F) \cap \mathbb{S}_n$. Suppose $\zeta \in \text{cl}(W_F) \cap \mathbb{S}_n$. For any $m \geq 1$, applying Lemma 2.2 with $\epsilon = m^{-1}$, we get a $\delta_m > 0$ so that if $E(b, r) \cap V_m \neq \emptyset$, where $V_m = \{z \in \mathbb{B}_n : |z - \zeta| < \delta_m\}$, then $|b - \zeta| < m^{-1}$. Since $W_F \cap V_m \neq \emptyset$, there is $0 < t_m < 1$ and $\zeta_m \in F$ so that $E(t_m\zeta_m, r) \cap V_m \neq \emptyset$. Hence $|t_m\zeta_m - \zeta| < m^{-1}$. So $|t_m\zeta_m - \zeta| \to 0$ as $m \to \infty$. Since $t_m = |t_m\zeta_m| \to |\zeta| = 1$ as $m \to \infty$, we get $\zeta_m = t_m^{-1}(t_m\zeta_m) \to \zeta$ in the Euclidean metric as $m \to \infty$. This implies that $\zeta \in F$. Thus, $\text{cl}(W_F) \cap \mathbb{S}_n \subset F$ and hence, $\text{cl}(W_F) \cap \mathbb{S}_n = F$.

Now define

$$G_F = \{f \in L^\infty : f \text{ vanishes almost everywhere in } \mathbb{B}_n \backslash W_F\}.$$

It is clear that if $F_1 \subset F_2$ then $W_{F_1} \subset W_{F_2}$, hence $G_{F_1} \subset G_{F_2}$.

Since T_f is compact for all $f \in G_\emptyset$, $\mathfrak{I}(G_\emptyset) = \mathcal{K}$. Since $G_{\mathbb{S}_n} = L^\infty$, we have $\mathfrak{CI}(G_{\mathbb{S}_n} \cap C(\mathbb{B}_n)) = \mathfrak{CI}(L^\infty \cap C(\mathbb{B}_n)) = \mathfrak{T}$.

Now suppose F_1 and F_2 are two closed subsets of $\mathbb{S}_n$ so that $F_2 \backslash F_1 \neq \emptyset$. Let $\zeta \in F_2 \backslash F_1$. From Lemma 2.1 there is a sequence $\{t_m\}_{m=1}^{\infty} \subset (0, 1)$ with $t_m \uparrow 1$ and $E(t_k\zeta, r) \cap E(t_l\zeta, r) = \emptyset$ for all $k \neq l$. Let $z_m = t_m\zeta$ for all $m \geq 1$. Since $|z_m - \zeta| \to 0$ and $\zeta \notin F_1$ which is a closed subset of $\mathbb{S}_n$, there is an $\epsilon > 0$ so that $|z_m - w| \geq \epsilon$ for all $w \in F_1$, all $m \geq 1$. Since $\text{cl}(W_{F_1}) \cap \mathbb{S}_n = F_1$, Proposition 3.1 shows that $TU_{z_m} \to 0$ in the strong operator topology for all $T \in \mathfrak{I}(G_{F_1})$.

Take f to be any continuous function supported in $E(0, r)$ such that $[T_f, T_{\overline{f}}] \neq 0$. Any function of the form $f(z) = z_1\eta(|z|/r)$ where η is non-negative, continuous and supported in $[0, 1]$ with $\|\eta\|_\infty > 0$ will work. Let $S = [T_f, T_{\overline{f}}]^2$ then S is a non-zero, positive operator on L_a^2. Define

$$T = \sum_{m=1}^{\infty} U_{z_m} S U_{z_m}^*.$$

By Lemma 2.4, there is a constant $c > 0$ and an $h \in L_a^2$ so that $\|TU_{z_m}h\| \geq c$ for all m. This implies that T is not in $\mathfrak{I}(G_{F_1})$.

For each m,

$$\begin{aligned} U_{z_m}[T_f, T_{\overline{f}}]U_{z_m}^* &= U_{z_m}T_fT_{\overline{f}}U_{z_m}^* - U_{z_m}T_{\overline{f}}T_fU_{z_m}^* \\ &= T_{f\circ\varphi_{z_m}}T_{\overline{f}\circ\varphi_{z_m}} - T_{\overline{f}\circ\varphi_{z_m}}T_{f\circ\varphi_{z_m}} \\ &= [T_{f\circ\varphi_{z_m}}, T_{\overline{f}\circ\varphi_{z_m}}]. \end{aligned}$$

So $U_{z_m}SU_{z_m}^* = [T_{f\circ\varphi_{z_m}}, T_{\overline{f}\circ\varphi_{z_m}}]^2$. Hence $T = \sum_{m=1}^{\infty}[T_{f\circ\varphi_{z_m}}, T_{\overline{f}\circ\varphi_{z_m}}]^2$.

Since each $f \circ \varphi_{z_m}$ is continuous and supported in $\{w \in \mathbb{B}_n : |\varphi_{z_m}(w)| < r\} = E(z_m, r)$ and $E(z_k, r) \cap E(z_l, r) = \emptyset$ for all $k \neq l$, Proposition 3.2 and Remark 3.3 show that $T \in \mathfrak{CI}(G_{F_2} \cap C(\mathbb{B}_n))$. So $T \in \mathfrak{CI}(G_{F_2} \cap C(\mathbb{B}_n)) \backslash \mathfrak{I}(G_{F_1})$.

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