CLOSED RANGE POSINORMAL OPERATORS AND THEIR PRODUCTS

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Abstract. We focus on two problems relating to the question of when the product of two posinormal operators is posinormal, giving (1) necessary conditions and sufficient conditions for posinormal operators to have closed range, and (2) sufficient conditions for the product of commuting closed-range posinormal operators to be posinormal with closed range. We also discuss the relationship between posinormal operators and EP operators (as well as hypo-EP operators), concluding with a new proof of the Hartwig-Katz Theorem, which characterizes when the product of posinormal operators on $\mathbb{C}^n$ is posinormal.

1. Introduction

Throughout this paper the term *operator* means a bounded linear transformation of a Hilbert space into itself. An operator is *posinormal* if its range is included in the range of its adjoint. The class of posinormal operators includes the class of hyponormal operators. An operator is *quasiposinormal* if the closure of its range is included in the closure of the range of its adjoint—if ether of these ranges is closed, then they are both closed and the concepts of posinormal and quasiposinormal coincide.

A necessary and sufficient condition for the product of a pair of closed-range operators to have a closed range is given in Theorem 1 of [2], which also provides an example of a closed-range operator whose square does not have closed-range [2, Corollary 5]. A simpler example is given in [3, Example 1]. The square of a posinormal operator is not necessarily posinormal [15, Example 1] but every positive-integer power of a posinormal operator with closed range is posinormal with closed range [12, Corollary 14] (and so powers of a hyponormal operator with closed range have closed range). The closed-range assumption is crucial even if the operator and its adjoint are both posinormal: Proposition 4.3 of [4] describes examples of non-closed-range posinormal operators having posinormal adjoints for which all sufficiently large powers fail to be posinormal. The fact that powers of a closed-range posinormal operator is again a closed-range posinormal operator prompts the following question.

Question 1.1. *Is the product of two commuting posinormal operators, both with closed range, a posinormal operator with closed range?*

Motivated by the preceding (still open) question, we explore necessary conditions and sufficient conditions for posinormal operators to have closed range, as well as

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investigate the structure of matrix representations of a pair of commuting, closed range posinormal operators on a Hilbert space $\mathcal{H}$ relative to a natural orthogonal decomposition of $\mathcal{H}$. We note that Question 1.1 above has an affirmative answer if "posinormal" is replaced by "normal" because (i) the product of two commuting normal operators is normal, thanks to Fuglede's Theorem, and (ii) the product of two commuting normal operators A and B having closed range will also have closed range by, e.g., Proposition 2.3 below: the kernel of A will be reducing for B by Fuglede's Theorem, showing that the operator Y of part (a) of Proposition 2.3 must be the zero operator, so that the hypotheses part (b) of Proposition 2.3 hold. The product of two closed-range normal operators that do not commute need not have closed range—see Example 1.2 below.

A normal operator has closed range if and only if 0 *is not a limit point of its spectrum* (e.g,. set $\lambda = 0$ in [6, Proposition XI.4.5]). In Section 3, we identify three properties of Hilbert-space operators, each one of which normal operators possess, such that if T is an operator on a Hilbert space $\mathcal{H}$ having these three properties, then T has closed range if and only if 0 is not a limit point of the spectrum of T. As corollaries of this result, we show that

- if T is a hyponormal operator such that 0 is not a limit point of the spectrum of T, then the range of T is closed (Corollary 3.3),
- if T is a posinormal operator such that 0 is not a limit point of the spectrum of T and the restriction of T to the orthogonal complement of the kernel of T is isoloid, then the range of T is closed (Corollary 3.4), and
- if T is a posinormal operator with closed range, then 0 is not a limit point of the spectrum of T if and only if the adjoint of T is also posinormal (Proposition 3.5).

If A is a posinormal operator on a Hilbert space $\mathcal{H}$, then, by definition, the range of A is contained in the range of A^*, and, upon taking orthogonal complements, we see that the kernel A is a subset of the kernel of A^*. Thus, for a posinormal operator A on $\mathcal{H}$, the kernel $\mathcal{N}(A)$ of A is a subspace of $\mathcal{H}$ that reduces A. In general, if B is an operator in the commutant of A and if the kernel $\mathcal{N}(A)$ of A reduces A, then relative to the orthogonal decomposition $\mathcal{H} = \mathcal{N}(A)^\perp \oplus \mathcal{N}(A)$, the operators A and B have the following matrix representations:

$$A = \begin{pmatrix} A' & O \\ O & O \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} B' & O \\ Y & Z \end{pmatrix}.$$

In Section 4 (Corollary 4.3), we give a sufficient condition for the product of closed-range commuting posinormal operators to be posinormal with closed range:

If A and B are commuting posinormal operators with closed range, then AB is posinormal with closed range if B' and Z^* are posinormal.

Moreover, using the matrix representations described above, we show (Theorem 4.6) that if A and B are commuting posinormal operators with closed range and one of A and B has posinormal adjoint, then AB is posinormal with closed range, a result that generalizes [7, Theorem 3].

Of course, products of noncommuting posinormal operators can be posinormal. One can check that for the posinormal operators $G = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ on $\mathbb{C}^2$, the operator GP is posinormal, but PG is not (and all these operators have closed range because all linear operators on $\mathbb{C}^n$ have closed range). Another example: on any complex Hilbert space a pair of non-commuting unitary operators will be a pair of non-commuting, closed-range normal operators whose product is normal with closed range (in fact, unitary).

The product of two closed-range normal operators need not have closed range:

Example 1.2. *There exist normal operators A and B having closed range such that AB does not have closed range.*

Let $(e_j)_{j=0}^{\infty}$ be the natural basis of ℓ^2 so that the sequence e_j has 1 as its j-th term and zeros elsewhere. Let $\mathcal{M}_1$ be the closure of the span of $\{e_{2k} : k = 0, 1, ...\}$ and $\mathcal{M}_2$ be the closure of the span of $\{g_k := e_{2k} + e_{2k+1}/(2k + 1) : k = 0, 1, \ldots\}$. (Note that $\{g_k : k \geq 0\}$ is orthogonal.) Set $A = (I - P_{\mathcal{M}_1})$ (i.e. orthogonal projection onto the orthogal complement of $\mathcal{M}_1$) and $B = P_{\mathcal{M}_2}$. We use Bouldin's criterion, stated below, to establish that AB does not have a closed range.

> Bouldin's Criterion [2]: *If S and T are operators on $\mathcal{H}$ having closed range then ST also has closed range if and only if the angle between* $\operatorname{ran} T$ *and* $\ker S \cap (\ker S \cap \operatorname{ran} T)^{\perp}$ *is positive.*

Observe that $\ker A = \mathcal{M}_1$ and $\operatorname{ran} B = \mathcal{M}_2$, so that $\ker A \cap \operatorname{ran} B = \mathcal{M}_1 \cap \mathcal{M}_2 = \{0\}$. Thus, $(\ker A \cap \operatorname{ran} B)^{\perp} = \mathcal{H}$. We show that the angle between $\operatorname{ran} B = \mathcal{M}_2$ and $\ker A \cap (\ker A \cap \operatorname{ran} B)^{\perp} = \ker A = \mathcal{M}_1$ is 0, showing the range of AB is not closed by Bouldin's Criterion.

The angle θ between $\mathcal{M}_1$ and $\mathcal{M}_2$ of $\mathcal{H}$ is given by

$$\theta = \cos^{-1}\Big(\sup\{|\langle f, g\rangle| : f \in \mathcal{M}_1, g \in \mathcal{M}_2, \|f\| = 1 = \|g\|\}\Big),$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product of ℓ^2. For $n \geq 0$, let

$$f_n = e_{2n} \quad \text{and} \quad g_n = \frac{\left(e_{2n} + \frac{e_{2n+1}}{2n+1}\right)}{\sqrt{1 + 1/(2n+1)^2}}$$

and observe that (f_n) and (g_n) are sequences of unit vectors such that $\langle f_n, g_n \rangle = 1/\sqrt{1 + 1/(2n+1)^2} \to 1$, as $n \to \infty$. We see the angle between $\mathcal{M}_1$ and $\mathcal{M}_2$ is 0, as desired. $\square$

For the normal operators A and B of the preceding example, it's easy to show that AB is not normal. In general, it's possible to show that for orthogonal projections A and B, the product AB is normal if and only if A and B commute (and the projections A and B of Example 1.2 do not commute).

The paper is organized into four more sections. Notation, terminology and auxiliary results are considered in Section 2. The results summarized above are treated in Sections 3 and 4. Section 5 brings a detailed discussion of EP operators and matrices and how they relate to posinormal operators and matrices, concluding with a discussion of, as well as a new proof of, the Hartwig–Katz Theorem, which characterizes when the product of two posinormal matrices is a posinormal matrix.

2. Notation, Terminology, and Auxiliary Results

Let $\mathcal{H}$ be an infinite-dimensional complex Hilbert space. If $\mathcal{M}$ is a subspace of $\mathcal{H}$, then we let $\mathcal{M}^-$ denote its closure and $\mathcal{M}^{\perp}$ its orthogonal complement. The algebra of all operators on $\mathcal{H}$ will be denoted by $\mathcal{B}[\mathcal{H}]$. For any operator $T \in \mathcal{B}[\mathcal{H}]$, let $\mathcal{N}(T)$ stand for the kernel of T, which is a closed subspace of $\mathcal{H}$, and let $\mathcal{R}(T)$

stand for the range of T. Let $T^* \in \mathcal{B}[\mathcal{H}]$ denote the adjoint of $T \in \mathcal{B}[\mathcal{H}]$. Posinormal operators are defined as follows.

An operator $T \in \mathcal{B}[\mathcal{H}]$ is *posinormal* if

$$\mathcal{R}(T) \subseteq \mathcal{R}(T^*) \qquad \text{(which implies } \mathcal{N}(T) \subseteq \mathcal{N}(T^*)\text{)},$$

and *quasiposinormal* if

$$\mathcal{R}(T)^- \subseteq \mathcal{R}(T^*)^- \qquad \text{(equivalently, } \mathcal{N}(T) \subseteq \mathcal{N}(T^*)\text{)}.$$

Posinormal operators are quasiposinormal and the concepts coincide if $\mathcal{R}(T)$ is closed. If T is injective, then it is quasiposinormal; if T^* is surjective (equivalently, if T is injective with closed range), then T is posinormal. For equivalent definitions of posinormal operators, see, e.g., [17, Theorem 2.1], [9, Theorem B], [11, Theorem 1], [14, Proposition 1], [15, Definition 1]). An operator $T \in \mathcal{B}[\mathcal{H}]$ is called *coposinormal* or *coquasiposinormal* if its adjoint $T^* \in \mathcal{B}[\mathcal{H}]$ is posinormal or quasiposinormal, respectively.

Posinormal operators were introduced and systematically investigated by Rhaly in [17], which appeared in 1994. The class of posinormal includes the hyponormal operators but is not included in the class of normaloid operators. For a comprehensive exposition on posinormal operators see, e.g., [17] and [14]. For basic properties of posinormal operators, see, e.g., [17, Corollary 2.3], [11, Propositions 3 and 4], [14, Lemma 1, Remark 2], and [15, Proposition 1]. Those properties required in this paper are summarized below.

Proposition 2.1. *Let T be a Hilbert-space operator.*

(a) *If T is quasiposinormal (in particular, posinormal), then $\mathcal{N}(T)$ reduces T.*

(b) *The restriction of a posinormal (quasiposinormal) operator to a closed invariant subspace is posinormal (quasiposinormal).*

(c) *If T is quasiposinormal (in particular, posinormal), then $\mathcal{N}(T^2) = \mathcal{N}(T)$.*

Regarding part (a) of the preceding proposition, we note that, in fact, a Hilbert space operator T is quasiposinormal if and only if $\mathcal{N}(T)$ reduces T:

$$T \text{ is quasiposinormal} \iff \mathcal{N}(T) \subseteq \mathcal{N}(T^*) \iff \mathcal{N}(T) \text{ reduces } T.$$

The next lemma facilitates our exploration of properties of matrix representations of commuting posinormal operators.

Lemma 2.2. *Let B be an operator with closed range on $\mathcal{H}$. Suppose that with respect to an orthogonal decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$,*

$$B = \begin{bmatrix} B' & 0 \\ Y & Z \end{bmatrix}.$$

Then

(a) *$\mathcal{R}(B'^*)^- \subseteq \mathcal{R}(B'^*) + \mathcal{R}(Y^*|_{\mathcal{N}(Z^*)})$. As a result, if $\mathcal{R}(Y^*|_{\mathcal{N}(Z^*)}) \subseteq \mathcal{R}(B'^*)$, then $\mathcal{R}(B'^*)$ is closed and hence, $\mathcal{R}(B')$ is closed.*

(b) *If B is also assumed to be posinormal, then*

$$\mathcal{R}(Z)^- \subseteq \mathcal{R}(Z^*).$$

As a result, if Z^* is quasiposinormal, then $\mathcal{R}(Z^*)^- \subseteq \mathcal{R}(Z)^-$ and hence $\mathcal{R}(Z^*) = \mathcal{R}(Z)^-$, which implies that $\mathcal{R}(Z^*)$ is closed.

Proof. (a) Let $x \in \mathcal{R}(B'^*)^-$ and let (x_n) be a sequence in $\mathcal{R}(B'^*)$ that converges to x. For each n, there exists $u_n \in \mathcal{H}_1$ such that $x_n = {B'}^* u_n$ and we have $B^*(u_n, 0) = ({B'}^* u_n, 0) = (x_n, 0)$. So $(x_n, 0) \in \mathcal{R}(B^*)$ for each n and $(x_n, 0) \to (x, 0)$. Since $\mathcal{R}(B^*)$ is closed (because $\mathcal{R}(B)$ is closed) it follows that $(x, 0) \in \mathcal{R}(B^*)$. Thus there exists $(u, v) \in \mathcal{H}_1 \oplus \mathcal{H}_2$ such that

$$(x, 0) = B^*(u, v) = ({B'}^* u + Y^* v, Z^* v).$$

This implies $Z^* v = 0$, which shows that $Y^* v \in \mathcal{R}(Y^*|_{\mathcal{N}(Z^*)})$. Since $x = B'^* u + Y^* v$, we conclude that

$$x \in \mathcal{R}(B'^*) + \mathcal{R}(Y^*|_{\mathcal{N}(Z^*)}).$$

(b) Now assume that B is posinormal. Let $y \in \mathcal{R}(Z)^-$. Then $(0, y) \in \mathcal{R}(B)^- = \mathcal{R}(B) \subseteq \mathcal{R}(B^*)$ because B is posinormal with closed range. Then there exists $(u, v) \in \mathcal{H}_1 \oplus \mathcal{H}_2$ such that

$$(0, y) = B^*(u, v) = ({B'}^* u + Y^* v, Z^* v).$$

This shows that $y \in \mathcal{R}(Z^*)$. $\square$

Let A and B be operators on a Hilbert space $\mathcal{H}$. According to Proposition 2.1(a), if A is quasiposinormal (or posinormal), then $\mathcal{N}(A)$ reduces A.

Proposition 2.3. *Suppose that A and B commute and that $\mathcal{N}(A)$ reduces A.*

(a) *With respect to the decomposition $\mathcal{H} = \mathcal{N}(A)^\perp \oplus \mathcal{N}(A)$,*

$$A = \left(\begin{smallmatrix} A' & O \\ O & O \end{smallmatrix}\right) \quad \textit{and} \quad B = \left(\begin{smallmatrix} B' & O \\ Y & Z \end{smallmatrix}\right) \quad \textit{with} \quad YA' = O,$$

and $Y = O$ if and only if $\mathcal{N}(A)$ reduces B.

(b) *If $\mathcal{R}(A)$ and $\mathcal{R}(B)$ are closed and $\mathcal{R}(Y^*|_{\mathcal{N}(Z^*)}) \subseteq \mathcal{R}({B'}^*)$, then $\mathcal{R}(AB)$ is closed.*

Proof. (a) Consider the decomposition $\mathcal{H} = \mathcal{N}(A)^\perp \oplus \mathcal{N}(A)$. Since $\mathcal{N}(A)$ reduces A,

$$A = \left(\begin{smallmatrix} A' & O \\ O & O \end{smallmatrix}\right) = A' \oplus O, \quad B = \left(\begin{smallmatrix} B' & X \\ Y & Z \end{smallmatrix}\right), \quad BA = \left(\begin{smallmatrix} B'A' & O \\ Y'A & O \end{smallmatrix}\right), \quad AB = \left(\begin{smallmatrix} A'B' & A'X \\ O & O \end{smallmatrix}\right)$$

with $A' = A|_{\mathcal{N}(A)^\perp}$ and B' in $\mathcal{B}[\mathcal{N}(A)^\perp]$. Thus if A and B commute, then

$$AB = BA = \left(\begin{smallmatrix} B'A & O \\ O & O \end{smallmatrix}\right) = \left(\begin{smallmatrix} A'B' & O \\ O & O \end{smallmatrix}\right) = B'A' \oplus O = A'B' \oplus O,$$

with $A'X = O$ and $YA' = O$. Since A' is injective and $A'X = O$ we get $X = O$. So

$$B = \left(\begin{smallmatrix} B' & O \\ Y & Z \end{smallmatrix}\right) \quad \text{and so} \quad B^* = \left(\begin{smallmatrix} B'^* & Y^* \\ O & Z^* \end{smallmatrix}\right),$$

and hence $Y = O$ if and only if $\mathcal{N}(A)$ also reduces B.

(b) Suppose $\mathcal{R}(B)$ is closed in $\mathcal{H}$ and $\mathcal{R}(Y^*|_{\mathcal{N}(Z^*)}) \subseteq \mathcal{R}({B'}^*)$. By Lemma 2.2(a), we conclude that $\mathcal{R}(B')$ is closed in $\mathcal{N}(A)^\perp$.

Next suppose $\mathcal{R}(A)$ is closed in $\mathcal{H}$. Since $\mathcal{R}(A) = \mathcal{R}(A') \oplus \{0\}$ and $\mathcal{R}(A)$ is closed,

$$\mathcal{R}(A') \text{ is closed in } \mathcal{N}(A)^\perp.$$

Then A' is injective with closed range, which means it is bounded below—there is a positive constant c such that $\|A'x\| \geq c\|x\|$ for all $x \in \mathcal{N}(A)^\perp$.

Now let (x_n) be an arbitrary convergent sequence in $\mathcal{R}(A'B')$. Then, for each n, $x_n = A'B'u_n$ for some $u_n \in \mathcal{N}(A)^\perp$. Because (x_n) is convergent $(A'B'u_n)$ is Cauchy, which implies, because A' is bounded below, that $(B'u_n)$ is Cauchy. Thus $(B'u_n)$ converges and, because $\mathcal{R}(B')$ is closed, there is a $u \in \mathcal{R}(B')$ such that

$\lim(B'u_n) = B'u$. We have $A'B'u = \lim(A'B'u_n) = \lim(x_n)$ and we see that $\mathcal{R}(A'B')$ is closed, as desired. □

3. Closed-Range Posinormal Operators

In this section, we identify three properties of Hilbert-space operators, each one of which normal operators possess, such that if T is an operator on a Hilbert space $\mathcal{H}$ having these three properties, then T has closed range if and only if 0 is not a limit point of the spectrum of T.

Let T be a bounded linear operator on a complex Hilbert space, let $\sigma(T)$ and $\rho(T) = \mathbb{C}\backslash\sigma(T)$ denote the spectrum and the resolvent set of T, respectively, and consider the standard partition of the spectrum into $\sigma_P(T)$, the point spectrum, $\sigma_R(T)$, the residual spectrum, and $\sigma_C(T)$, the continuous spectrum. An operator T is *isoloid* is every isolated point of the spectrum $\sigma(T)$ is an eigenvalue. In particular, if the spectrum $\sigma(T)$ is a singleton $\{\lambda\}$, then T is isoloid if and only if $\sigma_P(T) = \{\lambda\}$. Vacuously, every operator whose spectrum has no isolated point is isoloid. There exist posinormal operators that are not isoloid, e.g., an injective weighted unilateral shift T_+ on ℓ^2_+ with weight sequence $\left(\frac{1}{k}\right)$ is a (compact, quasinilpotent) posinormal operator (cf. [14, Section 3]) whose spectrum $\sigma(T_+) = \{0\}$ coincides with the residual spectrum $\sigma_R(T_+)$.

A normal operator has closed range if and only if zero is not a limit point of its spectrum. This is a particular case of [6, Proposition XI.4.5], whose proof is based on the Spectral Theorem as well as the Open Mapping Theorem. The preceding characterization of closed range normal operators doesn't extend to posinormal operators; in fact, it doesn't extend to hyponormal operators. For example, the forward shift operator on ℓ^2 is hyponormal with closed range but its spectrum is the entire closed unit disk (so that 0 is a limit point of the spectrum). We seek additional conditions that ensure a posinormal operator has closed range if and if 0 is not a limit point of its spectrum.

If T is a posinormal operator (or even a quasiposinormal operator), we have $\mathcal{N}(T) \subseteq \mathcal{N}(T^*)$, which ensures that $\mathcal{N}(T)$ reduces T. Suppose that for some $T \in \mathcal{B}[\mathcal{H}]$, we know $\mathcal{N}(T)$ reduces T. Then, we have

$$T = T' \oplus 0 \quad \text{on} \quad \mathcal{H} = \mathcal{N}(T)^{\perp} \oplus \mathcal{N}(T),$$

where $T' = T|_{\mathcal{N}(T)^{\perp}} \in \mathcal{B}[\mathcal{N}(T)^{\perp}]$. Assuming that T has the representation $T = T' \oplus 0$ above, we will show that if we want the condition "T has closed range" to imply 0 is not a limit point of the spectrum of T, then it's sufficient to assume that 0 does not belong to the residual spectrum of T', i.e., $0 \notin \sigma_R(T')$. We will also show that if we want the condition "0 is not a limit point of $\sigma(T)$" to imply $\mathcal{R}(T)$ is closed, then it's sufficient to assume that T' is isoloid.

If T is a normal operator, observe that

(i) $\mathcal{N}(T)$ reduces T,
(ii) $0 \notin \sigma_R\left(T|_{\mathcal{N}(T)^{\perp}}\right)$,
(iii) $T|_{\mathcal{N}(T)^{\perp}}$ is isoloid.

As for property (i), not only does $\mathcal{N}(T)$ reduce T when T is normal, we have $\mathcal{N}(T) = \mathcal{N}(T^*)$. To see that normal operators satisfy (ii), let T be normal and observe $T|_{\mathcal{N}(T)^{\perp}}$ is normal because $\mathcal{N}(T)^{\perp}$ reduces T; thus, 0 is either an eigenvalue of $T|_{\mathcal{N}(T)^{\perp}}$ (so that $0 \notin \sigma_R\left(T|_{\mathcal{N}(T)^{\perp}}\right)$) or fails to be eigenvalue of both $T|_{\mathcal{N}(T)^{\perp}}$ and its adjoint, and 0's failing to be an eigenvalue of $T|_{\mathcal{N}(T)^{\perp}}$ means $T|_{\mathcal{N}(T)^{\perp}}$

has dense range (so that $0 \notin \sigma_R\left(T|_{\mathcal{N}(T)^\perp}\right)$). We have already noted that if T is normal, then $T|_{\mathcal{N}(T)^\perp}$ is also normal and because all normal operators are isoloid, we see $T|_{\mathcal{N}(T)^\perp}$ is isoloid; i.e., (iii) holds. That normal operators are isoloid is a consequence of the Spectral Theorem, and we note that with the help of the Riesz Decomposition Theorem isoloidness can be extended to hyponormal operators [19, Theorem 2]. We say a Hilbert-space operator T is of *class* $\mathcal{NL}$ ("normal like") provided T satisfies conditions (i)–(iii) above.

We have already pointed out that property (i) of $\mathcal{NL}$ operators is satisfied by any quasiposinormal operator (and hence by any posinormal and hyponormal operator). Also, hyponormal operators satisfy property (iii) of $\mathcal{NL}$ operators (because the restriction of a hyponormal operator to a reducing subspace is hyponormal and, as we noted in the preceding paragraph, hyponormal operators are isoloid).

Theorem 3.1. *An operator of class $\mathcal{NL}$ has closed range if and only if zero is not a limit point of its spectrum.*

Proof. Suppose $T \in \mathcal{B}[\mathcal{H}]$ is a closed-range operator on $\mathcal{H}$ of class $\mathcal{NL}$. Because T is of class $\mathcal{NL}$, we know (i) $\mathcal{N}(T)$ reduces T and (ii) $0 \notin \sigma_R(T|_{\mathcal{N}(T)^\perp})$. Because $\mathcal{N}(T)$ reduces T, we have the decomposition

$$T = T' \oplus 0 \quad \text{on} \quad \mathcal{H} = \mathcal{N}(T)^\perp \oplus \mathcal{N}(T),$$

where $T' = T|_{\mathcal{N}(T)^\perp} \in \mathcal{B}[\mathcal{N}(T)^\perp]$, so that $\mathcal{N}(T') = \{0\}$ and $\mathcal{R}(T')$ is closed because $\mathcal{R}(T)$ is closed. Also observe that the representation $T = T' \oplus 0$ on $\mathcal{H} = \mathcal{N}(T)^\perp \oplus \mathcal{N}(T)$ shows that for $\lambda \neq 0$, the operator $T - \lambda I$ is invertible on $\mathcal{H}$ if and only if $T' - \lambda I'$ is invertible on $\mathcal{N}(T)^\perp$, where I is the identity on $\mathcal{H}$ and I' is the identity on $\mathcal{N}(T)^\perp$.

Because 0 is not an eigenvalue of T' and $0 \notin \sigma_R(T|_{\mathcal{N}(T)^\perp})$, the range of T' must be dense in $\mathcal{N}(T)^\perp$. But the range of T' is closed, so that T' is surjective. Hence, T' is invertible; that is, $0 \in \rho(T')$. Because $\rho(T')$ is open, there is an $\epsilon > 0$ such that $T' - \lambda I'$ is invertible on $\mathcal{N}(T)^\perp$ whenever $|\lambda| < \epsilon$. Thus, $T - \lambda I$ is invertible whenever $0 < |\lambda| < \epsilon$ and we see 0 is not a limit point of the spectrum of T.

Conversely, suppose that 0 is not a limit point of the spectrum of T where T is of class $\mathcal{NL}$. In particular, we know that (i) $\mathcal{N}(T)$ reduces T and (iii) $T|_{N(T)^\perp}$ is isoloid. As we discussed in the first paragraph of the proof, because (i) holds, we have the representation $T = T' \oplus 0$ on $\mathcal{H} = \mathcal{N}(T)^\perp \oplus \mathcal{N}(T)$, where $\mathcal{N}(T') = \{0\}$ and for nonzero λ, the operator $T - \lambda I$ is invertible on $\mathcal{H}$ if and only if $T' - \lambda I'$ is invertible on $\mathcal{N}(T)^\perp$. Because 0 is not a limit point of the spectrum of T, we see that 0 is not a limit point of the spectrum of T'. Thus 0 is either not in the spectrum of T' or it's an isolated point of the spectrum; however, the latter is not a possibility—because $T|_{N(T)^\perp}$ is isoloid, if 0 were an isolated spectral point, then it would be an eigenvalue but we know $\mathcal{N}(T') = \{0\}$. Thus T' is invertible and it follows that $\mathcal{R}(T) = \mathcal{N}(T)^\perp$ is closed. □

The class $\mathcal{NL}$ is constructed so that the following holds.

Corollary 3.2. *If T is a posinormal operator on $\mathcal{H}$ such that $T|_{\mathcal{N}(T)^\perp}$ is a isoloid operator whose residual spectrum does not contain* 0*, then $\mathcal{R}(T)$ is closed if and only if* 0 *is not a limit point of $\sigma(T)$.*

Let T be hyponormal, then for every $\lambda \in \mathbb{C}$, the operator $T - \lambda I$ is hyponormal. By Proposition 2.1(a) we know $\mathcal{N}(T - \lambda I)$ reduces $T - \lambda I$ and by [19, Theorem 2], we know $T - \lambda I$ is isoloid. Thus $T - \lambda I$ satisfies (i) and (iii) of class $\mathcal{NL}$. Hence, the last paragraph of the proof of Theorem 3.1 yields the following.

Corollary 3.3. *If T is a hyponormal operator on $\mathcal{H}$, then whenever λ is not a limit point of $\sigma(T)$ the range of $T - \lambda I$ is closed.*

Similarly, the hypotheses of the next corollary imply that T satisfies (i) and (iii) of class $\mathcal{NL}$.

Corollary 3.4. *If T is a posinormal or quasiposinormal operator such that 0 is not a limit point of $\sigma(T)$ and $T|_{\mathcal{N}(T)^\perp}$ is isoloid, then $\mathcal{R}(T)$ is closed.*

We now characterize when a posinormal operator with closed range satisfies condition (ii) of class $\mathcal{NL}$.

Proposition 3.5. *Let T be a posinormal operator on $\mathcal{H}$ having closed range. The following are equivalent:*

(a) $0 \notin \sigma_R\left(T|_{\mathcal{N}(T)^\perp}\right)$*;*
(b) $T|_{\mathcal{N}(T)^\perp}$ *is invertible;*
(c) T *is coposinormal;*
(d) *0 is not a limit point of* $\sigma(T)$.

Proof. (a) $\Longrightarrow$ (b): Let T be a posinormal operator on $\mathcal{H}$ having closed range such that $0 \notin \sigma_R\left(T|_{\mathcal{N}(T)^\perp}\right)$. Because $\mathcal{N}(T)$ reduces T, recall that we have the decomposition

$$T = T' \oplus 0 \quad \text{on} \quad \mathcal{H} = \mathcal{N}(T)^\perp \oplus \mathcal{N}(T),$$

where $T' = T|_{\mathcal{N}(T)^\perp} \in \mathcal{B}[\mathcal{N}(T)^\perp]$, so that $\mathcal{N}(T') = \{0\}$ and $\mathcal{R}(T')$ is closed because $\mathcal{R}(T)$ is closed. Because 0 is not an eigenvalue of T' and we are assuming $0 \notin \sigma_R(T')$, the range of T' must be dense. However the range of T' is closed and thus T' is surjective as well as injective—it is invertible.

(b) $\Longrightarrow$ (c): Because T' is invertible, we have $\mathcal{R}(T') = \mathcal{N}(T)^\perp = \mathcal{R}(T^*)$, with the latter equality holding because $\mathcal{R}(T^*)$ is closed (because $\mathcal{R}(T)$ is closed). Because $\mathcal{R}(T') = \mathcal{R}(T^*)$, we see $\mathcal{R}(T) = \mathcal{R}(T^*)$, which, by definition, yields T is coposinormal.

(c) $\Longrightarrow$ (d): Because T is coposinormal as well as posinormal, and the range of T^* is closed, we have $\mathcal{R}(T) = \mathcal{R}(T^*) = \mathcal{N}(T)^\perp = \mathcal{R}(T')$. Thus T' is both injective and surjective–it is invertible. Recall that for $\lambda \neq 0$, $T - \lambda I$ is invertible if and only if $T' - \lambda I'$ is invertible. Because $\rho(T')$ is open, there is an $\epsilon > 0$ such that $T' - \lambda I'$ is invertible on $\mathcal{N}(T)^\perp = \mathcal{R}(T^*)$ whenever $|\lambda| < \epsilon$. Thus, $T - \lambda I$ is invertible whenever $0 < |\lambda| < \epsilon$ and we see 0 is not a limit point of the spectrum of T.

(d) $\Longrightarrow$ (a): We establish the contrapositive implication. Suppose that $0 \in \sigma_R(T')$. Because we are assuming $\mathcal{R}(T)$ is closed, $\mathcal{R}(T')$ is also closed; moreover, it's injective. Thus, T' is bounded below. Because 0 is in the spectrum of T' (in fact in the residual spectrum), it cannot be in the boundary of $\sigma(T')$ because that would put 0 in the approximate point spectrum of T' (implying T' is not bounded below). Thus, 0 is an interior point of $\sigma(T')$, so that there is an $\epsilon > 0$ such that $T' - \lambda I'$ is not invertible whenever $|\lambda| < \epsilon$. Hence, $T - \lambda I$ is not invertible whenever $0 < |\lambda| < \epsilon$. Thus 0 is a limit point of $\sigma(T)$, completing the proof. □

4. Posinormal Product of Posinormal Operators

Recall the matrix representations developed in Section 2, for a pair of commuting operators A and B on a Hilbert space $\mathcal{H}$ for which $\mathcal{N}(A)$ reduces A:

With respect to the decomposition $\mathcal{H} = \mathcal{N}(A)^{\perp} \oplus \mathcal{N}(A)$,

$$(\ddagger) \qquad A = \begin{pmatrix} A' & O \\ O & O \end{pmatrix} \quad \textit{and} \quad B = \begin{pmatrix} B' & O \\ Y & Z \end{pmatrix} \quad \textit{with} \quad YA' = O,$$

and $Y = O$ if and only if $\mathcal{N}(A)$ reduces B.

In this section, we use these matrix representations to obtain necessary conditions and sufficient conditions ensuring the product of two posinormal operators will be posinormal.

Lemma 4.1. *If A and B are commuting quasiposinormal operators on $\mathcal{H}$ having the matrix representations $(\ddagger)$ with respect to the decomposition $\mathcal{H} = \mathcal{N}(A)^{\perp} \oplus \mathcal{N}(A)$, then*

(a) $\mathcal{N}(Z) \subseteq \mathcal{N}(Z^*) \cap \mathcal{N}(Y^*)$ *and* (b) $\mathcal{N}(B') \cap \mathcal{N}(Y) \subseteq \mathcal{N}({B'}^*)$.

(c) *If B^* is also quasiposinormal, then the above inclusions become identities.*

Proof. Suppose that a quasiposinormal A commutes with B, then A and B have matrix representations $(\ddagger)$ and $B^* = \begin{pmatrix} {B'}^* & Y^* \\ O & Z^* \end{pmatrix}$. Because B is quasiposinormal, $\mathcal{N}(B) \subseteq \mathcal{N}(B^*)$, so that for an arbitrary $(u, v) \in \mathcal{N}(A)^{\perp} \oplus \mathcal{N}(A)$,

$$(B'u, Yu + Zv) = (0,0) \quad \Longrightarrow \quad ({B'}^* u + Y^* v, Z^* v) = (0,0).$$

(a) Set $u = 0$ in $\mathcal{N}(A)^{\perp}$. Then $(0, Zv) = (0,0)$ implies $(Y^*v, Z^*v) = (0,0)$ for any $v \in \mathcal{N}(A)$. Thus $\mathcal{N}(Z) \subseteq \mathcal{N}(Z^*) \cap \mathcal{N}(Y^*)$.

(b) Now set $v = 0$ in $\mathcal{N}(A)$. Then $(B'u, Yu) = (0,0)$ implies $({B'}^* u, 0) = (0,0)$ for any $u \in \mathcal{N}(A)^{\perp}$. Thus $\mathcal{N}(B') \cap \mathcal{N}(Y) \subseteq \mathcal{N}({B'}^*)$.

(c) If $\mathcal{N}(B) = \mathcal{N}(B^*)$, then the above argument shows that

$$\mathcal{N}(Z) = \mathcal{N}(Z^*) \cap \mathcal{N}(Y^*) \qquad \text{and} \qquad \mathcal{N}(B') \cap \mathcal{N}(Y) = \mathcal{N}({B'}^*). \qquad \square$$

Theorem 4.2. *Let A and B be commuting quasiposinormal operators having the matrix representations $(\ddagger)$.*

(a) *If B' is quasiposinormal, then AB is quasiposinormal.*

(b) *If Z^* is quasiposinormal, then AB has closed range whenever A and B have closed range.*

(c) *If Z^* quasiposinormal and A and B have closed range, then B' and Z have closed range.*

In other words,

(a$'$) *If $B^*|_{\mathcal{N}(A)^{\perp}}$ is coquasiposinormal, then AB is quasiposinormal.*

(b$'$) *If $B|_{\mathcal{N}(A)}$ is coquasiposinormal, then AB has closed range whenever A and B have closed range.*

(c$'$) *If $B|_{\mathcal{N}(A)}$ is coquasiposinormal and A and B have closed range, then the above coquasiposinormal restrictions have closed range.*

Proof. Suppose that a quasiposinormal A commutes with B, then A and B have matrix representations $(\ddagger)$:

$$A = \begin{pmatrix} A' & O \\ O & O \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} B' & O \\ Y & Z \end{pmatrix} \quad \text{with} \quad B^* = \begin{pmatrix} {B'}^* & Y^* \\ O & Z^* \end{pmatrix},$$
$$\text{where} \quad {B'}^* = B^*|_{\mathcal{N}(A)^{\perp}} \quad \text{and} \quad Z = B|_{\mathcal{N}(A)}.$$

Thus (a), (b) and (c) are equivalent to (a′), (b′) and (c′), respectively. Also,

$$A \text{ is quasiposinormal } \iff A' \text{ is quasiposinormal.}$$

(a) Note that $AB = BA = B'A' \oplus O = A'B' \oplus O$. Moreover, since $\mathcal{N}(A') = \{0\}$,

$$\mathcal{N}(B'A') = \mathcal{N}(B').$$

Indeed $x \in \mathcal{N}(B'A') \Leftrightarrow x \in \mathcal{N}(A'B') \Leftrightarrow A'B'x = 0 \Leftrightarrow B'x = 0 \Leftrightarrow x \in \mathcal{N}(B')$. Now suppose B' is quasiposinormal, that is,

$$\mathcal{N}(B') \subseteq \mathcal{N}(B'^*).$$

In this case, since $\mathcal{N}(B') \subseteq \mathcal{N}(B'^*)$ and $\mathcal{N}(B'A') = \mathcal{N}(B')$,

$$\mathcal{N}(B'A') = \mathcal{N}(B') \subseteq \mathcal{N}(B'^*) \subseteq \mathcal{N}(A'^*B'^*) = \mathcal{N}((B'A')^*),$$

so that $B'A'$ is quasiposinormal. Thus $A'B'$ is quasiposinormal and so is AB. Hence

$$B' \text{ quasiposinormal } \Longrightarrow A'B' \text{ quasiposinormal } \iff AB \text{ quasiposinormal.}$$

(The assumption "B is quasiposinormal" was not used in item (a).)

(b) Now suppose B is also quasiposinormal. Then Lemma 4.1(a) ensures that $\mathcal{N}(Z) \subseteq \mathcal{N}(Z^*) \cap \mathcal{N}(Y^*)$, which means

$$Z \text{ is quasiposinormal} \quad \text{and} \quad Y^*|_{\mathcal{N}(Z)} = O.$$

Thus if the quasiposinormal $Z = B|_{\mathcal{N}(A)}$ is coquasiposinormal as well, then $\mathcal{N}(Z) = \mathcal{N}(Z^*)$ so that $Y^*|_{\mathcal{N}(Z^*)} = O$. Therefore AB has closed range whenever A and B have closed range according to Proposition 2.3(b).

(c) Let A and B have closed range. Because A and B are commuting posinormal operators Lemma 4.1 yields $\mathcal{N}(Z) \subseteq \mathcal{N}(Z^*) \cap \mathcal{N}(Y^*) \subseteq \mathcal{N}(Y^*)$. Thus, $Y^*|_{\mathcal{N}(Z^*)} = O$ because $\mathcal{N}(Z^*) \subseteq \mathcal{N}(Z)$ given our assumption that Z^* is quasiposinormal. Lemma 2.2(a) now shows that B' has closed range. Since B is posinormal (i.e., quasiposinormal with closed range) and Z^* is quasiposinormal, Lemma 2.2(b) shows that $Z = B|_{\mathcal{N}(A)}$ has closed range. □

A rewriting of Theorem 4.2 gives a partial answer to Question 1.1.

Corollary 4.3. *If A and B are commuting posinormal operators with closed range, then AB is posinormal with closed range if B' is posinormal and Z is coposinormal.*

Remark 4.4. If A and B are commuting quasiposinormal operators, then

$$\mathcal{N}(B') \cap \mathcal{N}(Y) \subseteq \mathcal{N}(B'^*)$$

according to Lemma 4.1(b). Thus Theorem 4.2(a) ensures that

$$\mathcal{N}(B') \subseteq \mathcal{N}(Y) \Longrightarrow B' \text{ is quasiposinormal } \Longrightarrow AB \text{ is quasiposinormal.}$$

(The above holds, in particular, if $Y = O$; i.e., if $\mathcal{N}(A)$ also reduces B). Thus (cf. Proposition 2.3(b)), *if A and B commute and $\mathcal{N}(B') \subseteq \mathcal{N}(Y)$, then*

$$A \text{ and } B \text{ posinormal with closed range } \Longrightarrow AB \text{ posinormal with closed range.}$$

We can replace the commuting assumption with coincident kernels.

Proposition 4.5. *If A and B are posinormal operators with closed range, and if they have the same kernel, then their product is posinormal with closed range.*

Proof. Let A and B be closed-range posinormal operators on $\mathcal{H}$ such that $\mathcal{N}(A) = \mathcal{N}(B)$. Then $\mathcal{N}(A)$ is reducing for both A and B and by Proposition 2.3(a), A and B have the following matrix representations with respect to the decomposition $\mathcal{H} = \mathcal{N}(A)^{\perp} \oplus \mathcal{N}(A)$:

$$A = \begin{pmatrix} A' & O \\ O & O \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} B' & O \\ O & Z \end{pmatrix}.$$

Because A and B have closed range, the same is true of A' and B'. Moreover both A' and B' are injective, which means A' and B' are bounded below. Thus, their products $A'B'$ and $B'A'$ are bounded below and we see both $A'B'$ and $B'A'$ have closed range. Moreover, as $\mathcal{N}(A') = \mathcal{N}(B') = \{0\}$, we get $\mathcal{N}(A'B') = \mathcal{N}(B'A') = \{0\}$ so that $A'B'$ and $B'A'$ are quasiposinormal with closed range, thus posinormal. It follows that AB and BA are posinormal; e.g.,

$$\mathcal{R}(AB) = \mathcal{R}(A'B') \subseteq \mathcal{R}((A'B')^*) = \mathcal{R}(B'^*A'^*) = \mathcal{R}(B^*A^*) = \mathcal{R}((AB)^*). \qquad \square$$

If a pair of closed-range commuting posinormal operators is such that at least one of them is coposinormal, then their product is closed-range posinormal.

Theorem 4.6. *If a closed-range operator A that is both posinormal and coposinormal commutes with a closed-range posinormal operator B, then AB is posinormal with closed range.*

Proof. Let A and B be commuting, closed-range operators such that A is both posinormal and coposinormal and B is posinormal. Because $\mathcal{N}(A)$ reduces A (Proposition 2.1(a)), A and B have the matrix representations (‡) relative to the decomposition $\mathcal{H} = \mathcal{N}(A)^{\perp} \oplus \mathcal{N}(A)$. Because A is both posinormal and coposinormal $\mathcal{N}(A) = \mathcal{N}(A^*)$. Therefore $\mathcal{H} = \mathcal{N}(A)^{\perp} \oplus \mathcal{N}(A) = \mathcal{N}(A^*)^{\perp} \oplus \mathcal{N}(A^*)$. Hence $A'^* = A^*|_{\mathcal{N}(A^*)^{\perp}}$ is injective (as well as A'). Because A and B commute, Proposition 2.3(a) ensures that $YA' = O$ so that $A'^*Y^* = O$; equivalently, $A^*Y^* = 0$. Hence, because A^* is injective, $Y^* = O$ and therefore $Y = O$. Hence $\mathcal{N}(A)$ reduces B by Proposition 2.3(a). This implies that B' is posinormal because B is posinormal. By Theorem 4.2(a), AB is quasiposinormal; however, $\mathcal{R}(AB)$ is closed by Proposition 2.3(b) because $Y = O$, and thus AB is posinormal. $\square$

The preceding theorem generalizes Theorem 3 of [7], a result stated in language different from ours:

Theorem (Djordjević). *If $A, B \in \mathcal{B}[\mathcal{H}]$ are EP operators with closed ranges and $AB = BA$, then AB is the EP operator with a closed range also.*

Djordjevic's theorem arises in a line of investigation distinct from that started by Rhaly in 1994 when he introduced the notion of posinormality. In the final section of this paper, we briefly explore the connections between posinormal operators and EP operators, presenting a new proof of the Hartwig-Katz Theorem [8] characterizing "EP matrices".

The following corollary of our Theorem 4.6 above is equivalent to Djordjević's theorem:

Corollary 4.7. *Suppose that A and B are commuting posinormal and coposinormal operators with closed range; then AB is posinormal and coposinormal with closed range.*

Proof. Applying Theorem 4.6, we see that AB has closed range and is posinormal. Applying Theorem 4.6 with B^* playing the role of A and A^* playing the role of B, we obtain B^*A^* is posinormal; i.e., AB is coposinormal. $\square$

5. Posinormal Operators, EP Operators, and the Hartwig–Katz Theorem

If T is a posinormal operator on $\mathbb{C}^n$, then the range of T is necessarily closed and the inclusion $\mathcal{R}(T) \subseteq \mathcal{R}(T^*)$ implies that $\mathcal{R}(T) = \mathcal{R}(T^*)$ because $\mathcal{R}(T)$ and $\mathcal{R}(T^*)$ have the same dimension. Thus, in the finite-dimensional setting, an operator is posinormal if and only if its range equals that of its adjoint; that is, in finite dimensions, a posinormal operator is necessary both posinormal and coposinormal. Switching to matrix language, we see that an $n \times n$ matrix (with complex entries) is posinormal provided its range (column space) is the same as the range of its conjugate transpose. Such matrices are known as "EP matrices."

The modern definition of "EP matrix" evolved from Schwerdtfeger's notion of "EP_r matrix" ([18, p. 130], 1950). For an $n \times n$ matrix A, let $A_{(j)}$ denote the j-th column of A and $A^{(j)}$ denote the corresponding j-th row $(1 \le j \le n)$. Schwerdtfeger defines an $n \times n$ matrix A of rank r to be a P_r matrix provided there exist indices $i_1, i_2, \ldots i_r$ with $1 \le i_1 < i_2 < \ldots < i_r \le r$, such that $\{A_{(i_1)}, A_{(i_2)}, \ldots A_{(i_r)}\}$ and $\{A^{(i_1)}, A^(i_2)\}, \ldots A^{(i_r)}\}$ are both linearly independent sets of vectors. When these sets are appropriately chosen (see p. 130 of [18]), there is reason to call the vectors these sets contain *principal vectors* (because they correspond to an $r \times r$ rank r principal submatrix of A). Schwerdtfeger points out that every symmetric and every skew symmetric matrix A is a P_r matrix. Schwerdtfeger defines "EP_r matrix" in the two sentences following Theorem 18.1 on page 130 of [18]:

> The notion of P_r matrix may be further restricted so that it still covers the symmetric and skew symmetric matrices as well as other types of matrices to be mentioned later on. An n-matrix A [an n-matrix is an $n \times n$ matrix] may be called and EP_r matrix if it is a P_r matrix and the linear relations among its rows are the same as those among its corresponding columns.

Schwerdtfeger then explains what "same linear relations" means. In modern notation, it means that $\mathcal{N}(A) = \mathcal{N}(A^T)$. In fact, Schwerdtfeger's definition of EP_r matrix may be expressed: An $n \times n$ matrix A is an EP_r matrix provided that $\mathcal{N}(A) = \mathcal{N}(A^T)$. We see immediately, that Schwerdtfeger has "further restricted so that it still covers the symmetric and skew symmetric matrices." Note well that, e.g.,

$$A = \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix},$$

being symmetric, is an EP_r matrix. However, A is not an EP matrix because its range, the one-dimensional subspace of $\mathbb{C}^2$ spanned by $\begin{pmatrix} 1 \\ i \end{pmatrix}$, is *not* the same as the range of A^*, which is the one-dimensional subspace of $\mathbb{C}^2$ spanned by $\begin{pmatrix} 1 \\ -i \end{pmatrix}$. Of course, if an EP_r matrix A has real entries, then $\mathcal{N}(A) = \mathcal{N}(A^T) = \mathcal{N}(A^*)$ and, upon taking orthogonal complements, we have $\mathcal{R}(A) = \mathcal{R}(A^*)$. Thus, an EP_r matrix with real entries is an EP matrix.

Pearl ([16, p. 674], 1966) mischaracterized Schwerdtfeger's definition of EP_r matrix, stating that

> Schwerdtfeger ([18, p. 130]) has called a square matrix of rank r an EP_r matrix if it satisfies the condition:
>
> $$AX = 0 \text{ if and only if } A^*X = 0 \quad [\text{i.e., } \mathcal{N}(A) = \mathcal{N}(A^*)].$$

It seems that since Pearl's paper appeared, most have assumed that Schwerdtfeger's EP_r matrices are precisely today's EP matrices, but that's not quite true.

Generalizing the notion of "EP matrix," Campbell and Meyer ([5], 1975) introduced EP operators, which may be characterized as follows: a Hilbert space operator T is EP provided that T has closed range and $\mathcal{R}(T) = \mathcal{R}(T^*)$. Thus, according to the Campbell–Meyer definition, an EP operator in $\mathcal{B}[\mathcal{H}]$ is a closed-range operator that is both posinormal and coposinormal. It's not clear why Schwerdtfeger chose the E in his "EP_r" designation. Fortunately, there is a useful interpretation of "EP": an EP-operator is naturally associated with "Equal Projectors."

We now discuss why EP operators may be viewed as "equal-projector" operators. Let $T \in \mathcal{B}[\mathcal{H}]$ have closed range so that $\mathcal{H}$ has orthogonal decompositions $\mathcal{H} = \mathcal{R}(T^*) \oplus \mathcal{N}(T)$ and $\mathcal{H} = \mathcal{R}(T) \oplus \mathcal{N}(T^*)$. Thus, in particular we have

$$\mathcal{R}(T) = T(\mathcal{H}) = T\left(\mathcal{R}(T^*) \oplus \mathcal{N}(T)\right) = T(\mathcal{R}(T^*)),$$

and we see that the restriction $T|_{\mathcal{R}(T^*)}$ is an invertible operator mapping $\mathcal{R}(T^*)$ onto $\mathcal{R}(T)$. The *generalized inverse* $T^\dagger$ of T, which is also called its pseudoinverse or its Moore-Penrose inverse, is the operator that takes an element of $\mathcal{R}(T)$ to its unique T preimage in $\mathcal{R}(T^*)$ and takes elements of $\mathcal{N}(T^*)$ to zero; thus,

$$T^\dagger = (T|_{\mathcal{R}(T^*)})^{-1} P_{\mathcal{R}(T)}.$$

Observe that

$$T^\dagger T = P_{\mathcal{R}(T^*)} \quad \text{and} \quad TT^\dagger = P_{\mathcal{R}(T)}.$$

Hence, if T is an operator with closed range, then T is an EP operator ($\mathcal{R}(T) = \mathcal{R}(T^*)$) if and only if the two projectors $T^\dagger T$ and $TT^\dagger$ are equal; that is, T is an EP-operator if and only if T commutes with its generalized inverse. Pearl ([16], 1965), working in the matrix setting, was the first to provide this characterization; in addition, Pearl provides an explicit formula for $T^\dagger$ when T is a matrix—see Lemma 1 and Corollary 1 of [16].

We have seen that a closed-range operator $T \in \mathcal{B}(\mathcal{H})$ is EP if and only if $T^\dagger T - TT^\dagger = 0$. Itoh ([10], 2005) introduced hypo-EP operators, defining them as follows: $T \in \mathcal{B}[\mathcal{H}]$ is hypo-EP provided that T has closed range and $T^\dagger T - TT^\dagger \geq 0$. Itoh [10, Proposition 2.1] then provides several conditions equivalent to an operator's being hypo-EP, including $T \in \mathcal{B}(\mathcal{H})$ is hypo-EP if and only if T has closed range and $\mathcal{R}(T) \subseteq \mathcal{R}(T^*)$. Thus a hypo-EP operator is a posinormal operator with closed range.

In [12], Johnson and Vinoth provide the following sufficient condition for a product of operators to be EP:

Theorem (Johnson–Vinoth). *Let A be a hypo-EP operator and $B \in \mathcal{B}[\mathcal{H}]$ have closed range. If $\mathcal{R}(B) \subseteq \mathcal{R}(A)$ and $\mathcal{N}(B) \subseteq \mathcal{N}(A)$, then AB is hypo-EP.*

As an immediate corollary [12, Corollary 14], Johnson and Vinoth obtain the following result, which was described in the introduction of this paper in the following equivalent way "Every positive-integer power of a posinormal operator with closed range is posinormal with closed range":

Corollary (Johnson–Vinoth). *Let A be a hypo-EP operator on $\mathcal{H}$. Then A^n is hypo-EP for any integer $n \geq 1$.*

Djordjević's paper [7], whose Theorem 3 is Corollary 4.7 of the preceding section, contains a generalization of the well-known Hartwig–Katz Theorem, which characterizes when the product of two EP matrices is EP. Here's Djordjević's generalization ([7, Theorem 1], 2000):

Theorem 5.1 (Djordjević's generalization of the Hartwig–Katz Theorem)**.** *Let $A, B \in \mathcal{B}[\mathcal{H}]$ be EP operators. Then the following statements are equivalent.*

(a) *AB is an EP operator;*
(b) *$\mathcal{R}(AB) = \mathcal{R}(A) \cap \mathcal{R}(B)$ and $\mathcal{N}(AB) = \mathcal{N}(A) + \mathcal{N}(B)$;*
(c) *$\mathcal{R}(AB) = \mathcal{R}(A) \cap \mathcal{R}(B)$ and $\mathcal{N}(AB)$ is the closure of $\mathcal{N}(A) + \mathcal{N}(B)$.*

Keep in mind that we are assuming EP (and hypo-EP) operators have closed range.

We now turn our attention to the Hartwig–Katz Theorem. Our goal is to provide a new, elementary proof of it based on a characterization of $n \times n$ matrices A such that for every $n \times n$ matrix satisfying $\mathcal{R}(AB) \subseteq \mathcal{R}(B)$,

$$\mathcal{R}(AB) = \mathcal{R}(A) \cap \mathcal{R}(B).$$

Solving a problem that had been open for 25 years (see [1, p. 98], 1969), Hartwig and Katz proved the following result ([8], 1997) in which "RS" denotes *row space*:

Theorem (Hartwig–Katz). *Let A and B be $n \times n$ EP matrices. The following are equivalent:*

(a) *$\mathcal{R}(AB) = \mathcal{R}(A) \cap R(B)$ and $RS(AB) = RS(A) \cap RS(B)$;*
(b) *$\mathcal{R}(AB) \subseteq \mathcal{R}(B)$ and $RS(AB) \subseteq \mathcal{R}(A)$.*
(c) *AB is EP.*

Upon taking complex conjugates of all elements in a row space of an $n \times n$ matrix A and then transposing, we obtain the column space of A^*, the conjugate-transpose of A. Thus the condition $RS(AB) = RS(A) \cap RS(B)$ of part (a) of the Hartwig-Katz Theorem is equivalent to $\mathcal{R}((AB)^*) = \mathcal{R}(A^*) \cap \mathcal{R}(B^*)$. Taking orthogonal complements, we see $\mathcal{R}((AB)^*) = \mathcal{R}(A^*) \cap \mathcal{R}(B^*)$ is equivalent to $\mathcal{N}(AB) = \mathcal{N}(A) + \mathcal{N}(B)$. Similarly, we see that $RS(AB) \subseteq \mathcal{R}(A)$ is equivalent to $\mathcal{N}(A) \subseteq \mathcal{N}(AB)$. Thus, the Hartwig–Katz equivalent conditions may be restated as follows for EP matrices A and B:

(a) $\mathcal{R}(AB) = \mathcal{R}(A) \cap \mathcal{R}(B)$ and $\mathcal{N}(AB) = \mathcal{N}(A) + \mathcal{N}(B)$;
(b) $\mathcal{R}(AB) \subseteq \mathcal{R}(B)$ and $\mathcal{N}(A) \subseteq \mathcal{N}(AB)$;
(c) AB is EP.

We see that Djordjević in Theorem 5.1 has generalized to infinite dimensional Hilbert space the equivalence of (a) and (c) of the Hartwig–Katz Theorem but did not speak to the issue of generalizing the equivalence of (b) and (c). Here's an example illustrating that the equivalence of (b) and (c) does not generalize to infinite dimensions.

Proposition 5.2. *There exist EP operators A and B on a Hilbert space $\mathcal{H}$ for which* (i) $\mathcal{R}(AB) \subseteq \mathcal{R}(B)$ *and* (ii) $\mathcal{N}(A) \subseteq \mathcal{N}(AB)$ *such that AB is not EP.*

Proof. Let $(e_j)_{j=0}^{\infty}$ be the natural basis of ℓ^2 so that the sequence e_j has 1 as its j-th term and zeros elsewhere: $(a_0, a_1, a_2, \ldots) = \sum_{n=0}^{\infty} a_j e_j$. Let $\mathcal{M}$ be the one dimensional subspace of ℓ^2 spanned by e_0 and let $P_{\mathcal{M}}$ be the orthogonal projection of ℓ^2 onto $\mathcal{M}$. Let F be the forward shift on ℓ^2, $F\left(\sum_{j=0}^{\infty} a_j e_j\right) = \sum_{j=0}^{\infty} a_j e_{j+1}$, so that F^* is the backward shift, $F^*\left(\sum_{j=0}^{\infty} a_j e_j\right) = \sum_{j=1}^{\infty} a_j e_{j-1}$. Note that $F^*F = I$, while $FF^* = P_{\mathcal{M}^\perp}$.

Let $\mathcal{H} = \ell^2 \oplus \ell^2$, let I be the identity on ℓ^2, and define A and B on $\mathcal{H}$ by

$$A = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} F & P_{\mathcal{M}} \\ 0 & F^* \end{pmatrix}.$$

Observe that B is unitary (hence surjective):

$$BB^* = \begin{pmatrix} F & P_{\mathcal{M}} \\ 0 & F^* \end{pmatrix}\begin{pmatrix} F^* & 0 \\ P_{\mathcal{M}} & F \end{pmatrix} = \begin{pmatrix} P_{\mathcal{M}^\perp}+P_{\mathcal{M}} & P_{\mathcal{M}}F \\ F^*P_{\mathcal{M}} & I \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix},$$

while

$$B^*B = \begin{pmatrix} F^* & 0 \\ P_{\mathcal{M}} & F \end{pmatrix}\begin{pmatrix} F & P_{\mathcal{M}} \\ 0 & F^* \end{pmatrix} = \begin{pmatrix} I & F^*P_{\mathcal{M}} \\ P_{\mathcal{M}}F & P_{\mathcal{M}}+P_{\mathcal{M}^\perp} \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}.$$

Because B is surjective, clearly (i) $\mathcal{R}(AB) \subseteq \mathcal{R}(B)$. Also, $\mathcal{N}(A)$, which equals $\ell^2 \oplus 0$ is clearly contained in the kernel of $AB = \begin{pmatrix} 0 & 0 \\ 0 & F^* \end{pmatrix}$. The operator B is EP because it's invertible, and A is EP because it's self-adjoint. However, $AB = \begin{pmatrix} 0 & 0 \\ 0 & F^* \end{pmatrix}$ is not EP, because its range is $0 \oplus \ell^2$, but the range of its adjoint is $0 \oplus \mathcal{M}^\perp$, a proper subset of $\mathcal{R}(AB)$ (making AB coposinormal, but not posinormal). $\square$

Note that the condition $\mathcal{R}(AB) = \mathcal{R}(A) \cap \mathcal{R}(B)$ holds whenever A, B, and AB are all EP operators. We rely on the following to yield the Hartwig–Katz Theorem as a straightforward corollary.

Theorem 5.3. *Let $A\colon \mathbb{C}^n \to \mathbb{C}^n$ be linear. The following are equivalent.*

(a) $\mathcal{N}(A^2) = \mathcal{N}(A)$;

(b) *for each linear $B\colon \mathbb{C}^n \to \mathbb{C}^n$ satisfying $\mathcal{R}(AB) \subseteq \mathcal{R}(B)$,*

$$\mathcal{R}(AB) = \mathcal{R}(A) \cap \mathcal{R}(B);$$

(c) $\mathcal{R}(A^2) = \mathcal{R}(A)$;

(d) $r(A^2) = r(A)$, *where r denotes rank.*

Proof. (a) $\Longrightarrow$ (b): Suppose that $\mathcal{N}(A^2) = \mathcal{N}(A)$ and that $B\colon \mathbb{C}^n \to \mathbb{C}^n$ is a linear mapping whose range is invariant for A; i.e., $\mathcal{R}(AB) \subseteq \mathcal{R}(B)$.

Observe that $M := \mathcal{R}(A) \cap \mathcal{R}(B)$ is also invariant for A. Consider

$$A|_M\colon \mathcal{R}(A) \cap \mathcal{R}(B) \to \mathcal{R}(A) \cap \mathcal{R}(B).$$

Suppose that $v \in \mathcal{R}(A) \cap \mathcal{R}(B)$ is such that $Av = 0$. Then, because $v = Aw$ for some $w \in \mathbb{C}^n$, we see $0 = AAw = A^2w$, so that $w \in \mathcal{N}(A^2) = \mathcal{N}(A)$ and $0 = Aw = v$. Thus, $A|_M$ is injective and hence surjective. We see

$$\mathcal{R}(A) \cap \mathcal{R}(B) = A|_M(\mathcal{R}(A) \cap \mathcal{R}(B)) = A(\mathcal{R}(A) \cap \mathcal{R}(B)) \subseteq A(\mathcal{R}(B)) = \mathcal{R}(AB).$$

The reverse inclusion, $\mathcal{R}(AB) \subseteq \mathcal{R}(A) \cap \mathcal{R}(B)$ holds because $\mathcal{R}(AB)$ is clearly contained in $\mathcal{R}(A)$ while $\mathcal{R}(AB) \subseteq \mathcal{R}(B)$ by hypothesis. Hence, (a) $\Longrightarrow$ (b).

To see that (b) $\Longrightarrow$ (c), apply (b) with $B = A$. That (c) implies (d) is clear. By the rank-nullity theorem, condition (d) implies that $\mathcal{N}(A^2)$ and $\mathcal{N}(A)$ have the same dimension, but $\mathcal{N}(A) \subseteq \mathcal{N}(A^2)$, so that (a) holds. We've shown (d) implies (a), completing the proof. □

Recall that for all posinormal operators T, we have $\mathcal{N}(T^2) = \mathcal{N}(T)$ by Proposition 2.1(c) (which is Proposition 3 of [11]). Thus, condition (a) of Theorem 5.3 holds whenever A is EP.

Corollary 5.4 (Hartwig–Katz Theorem)**.** *Suppose that A and B are EP operators on $\mathbb{C}^n$. Then AB is EP if and only if* (i) $\mathcal{R}(AB) \subseteq \mathcal{R}(B)$ *and* (ii) $\mathcal{N}(A) \subseteq \mathcal{N}(AB)$.

Proof. Suppose that AB is EP. Then

$$\mathcal{R}(AB) = \mathcal{R}(B^*A^*) \subseteq \mathcal{R}(B^*) = \mathcal{R}(B),$$

where the final equality holds because B is EP. Thus, (i) holds. Similarly

$$\mathcal{R}(B^*A^*) = \mathcal{R}(AB) \subseteq \mathcal{R}(A) = \mathcal{R}(A^*).$$

Thus, $\mathcal{R}(A^*)^\perp \subseteq \mathcal{R}(B^*A^*)^\perp$; that is, $\mathcal{N}(A) \subseteq \mathcal{N}(AB)$, so that (ii) holds.

Now suppose that (i) and (ii) hold for EP operators A and B.

Because A is EP, $\mathcal{N}(A^2) = \mathcal{N}(A)$; thus, since we assuming (i) holds, Theorem 5.3, (a) $\Longrightarrow$ (b) yields

$$\mathcal{R}(AB) \subseteq \mathcal{R}(A) \cap \mathcal{R}(B). \tag{1}$$

Because B^* is EP, $\mathcal{N}(B^{*2}) = \mathcal{N}(B^*)$; taking orthogonal complements, (ii) yields $\mathcal{R}(B^*A^*) \subseteq \mathcal{R}(A^*)$. Hence, by Theorem 5.3, (a) $\Longrightarrow$ (b),

$$\mathcal{R}(B^*A^*) = \mathcal{R}(B^*) \cap \mathcal{R}(A^*). \tag{2}$$

Because $\mathcal{R}(A) = \mathcal{R}(A^*)$ and $\mathcal{R}(B) = \mathcal{R}(B^*)$ for the EP operators A and B, (1) and (2) yield

$$\mathcal{R}(AB) = \mathcal{R}((AB)^*),$$

so that AB is EP, as desired. □

We note that Koliha ([13], 1999) obtained a simple proof of the Hartwig–Katz Theorem quite different from ours.

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