COMMUTANT LIFTING
FOR COMMUTING ROW CONTRACTIONS

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Abstract. If $T = \begin{bmatrix} T_1 & \dots & T_n \end{bmatrix}$ is a row contraction with commuting entries, and the Arveson dilation is $\tilde{T} = \begin{bmatrix} \tilde{T}_1 & \dots & \tilde{T}_n \end{bmatrix}$, then any operator X commuting with each T_i dilates to an operator Y of the same norm which commutes with each $\tilde{T}_i$.

1. Introduction

The commutant lifting theorem of Sz.Nagy and Foiaş [**28, 27**] is a central result in the dilation theory of a single contraction. It states that if $T \in \mathcal{B}(\mathcal{H})$ is a contraction with isometric dilation V acting on $\mathcal{K} \supset \mathcal{H}$, and $TX = XT$, then there is an operator Y with $\|Y\| = \|X\|$, $VY = YV$ and $P_{\mathcal{H}}Y = XP_{\mathcal{H}}$. This result is equivalent to Ando's Theorem that two commuting contractions have a joint (power) dilation to commuting isometries. However the Ando dilation is not unique, and Varopoulos [**29**] showed that it fails for a triple of commuting contractions. In particular, the commutant lifting result does not generalize if one replaces T by a commuting pair T_1 and T_2 [**18**]. See Paulsen's book [**21**] for a nice treatment of these issues.

There is a multivariable context where the commutant lifting theorem does hold. This is the case of a row contraction $T = \begin{bmatrix} T_1 & \dots & T_n \end{bmatrix}$, i.e. $\|T\| \le 1$ considered as an operator in $\mathcal{B}(\mathcal{H}^{(n)}, \mathcal{H})$. The operators T_i are not assumed to commute. The Frazho–Bunce dilation theorem [**14, 6**] shows that there is a minimal dilation to a row isometry. Popescu [**22**] establishes uniqueness of this dilation. Frazho [**15**] and Popescu [**22**] establish the corresponding commutant lifting theorem. It works because the proofs of Bunce and Popescu follow the original Sz.Nagy–Foiaş proofs almost verbatim.

There is a related multivariable context of commuting row contractions. This area has received a lot of interest recently beginning with several papers of Arveson [**2, 3, 4**]. The analogue of the von Neumann inequality in this context was established by Drury [**13**]. The extension to an appropriate dilation theorem is due to Muller–Vasilescu [**19**] and Arveson [**2**]. We follow Arveson's approach. The model row contraction is the n-tuple $M = \begin{bmatrix} M_{z_1} & \dots & M_{z_n} \end{bmatrix}$ of multipliers on

2000 *Mathematics Subject Classification.* 47A20.

First author partially supported by an NSERC grant.

symmetric Fock space $\mathcal{H}^2_n$, sometimes called the Drury–Arveson space. Every commuting row contraction T dilates to a canonical operator $\tilde{T} \simeq M^{(\alpha)} \oplus U$, where U is a spherical unitary, i.e. $U = \begin{bmatrix} U_1 & \dots & U_n \end{bmatrix}$ where U_i are commuting normal operators satisfying $\sum U_i U_i^* = I$. (See the next section for details.)

A commutant lifting theorem in a related context was established by Ball, Trent and Vinnikov [**5**]. They only consider n-tuples with a pure dilation; that is, T dilates to some $M^{(\alpha)}$ with no spherical unitary part. They work more generally in the framework of complete Nevanlinna–Pick kernel Hilbert spaces. The connection is made because of the characterization of complete Nevanlinna–Pick kernels by McCullough [**16, 17**], Quiggin [**25**] and Agler–McCarthy [**1**]. In the last section of this paper, we will discuss their result and show how it follows from ours. Popescu [**24**] has a generalization of the Ball–Trent–Vinnikov result for pure row contractions satisfying relations other than commutativity.

In this paper, we establish the commutant lifting theorem for commuting row contractions.

Theorem 1.1. *Suppose that $T = \begin{bmatrix} T_1 & \dots & T_n \end{bmatrix}$ is a commuting row contraction on a Hilbert space $\mathcal{H}$, and that X is an operator on $\mathcal{H}$ which commutes with $T_1, \dots, T_n$. Let $\tilde{T} = \begin{bmatrix} \tilde{T}_1 & \dots & \tilde{T}_n \end{bmatrix}$ be the Arveson dilation of T on $\mathcal{K}$. Then there is an operator Z on $\mathcal{K}$ that commutes with each $\tilde{T}_i$ for $1 \le i \le n$, which dilates X in the sense $P_{\mathcal{H}} Z = X P_{\mathcal{H}}$ and satisfies $\|Z\| = \|X\|$.*

The minimal isometric dilation of a contraction is actually a co-extension. That is, given a contraction T, the minimal isometric dilation is an isometry V on a Hilbert space $\mathcal{K} \supset \mathcal{H}$ such that $\mathcal{H}$ is co-invariant for V and $P_{\mathcal{H}} V|_{\mathcal{H}} = T$. It follows that $P_{\mathcal{H}} V^k|_{\mathcal{H}} = T^k$ for all $k \ge 0$. The Arveson dilation is also a co-extension.

The minimal unitary dilation U of a contraction T is of a more general type. It is still true that $P_{\mathcal{H}} U^k|_{\mathcal{H}} = T^k$ for all $k \ge 0$. This implies [**26**] that $\mathcal{H}$ is semi-invariant, and we can decompose $\mathcal{K}$ into a direct sum $\mathcal{K} = \mathcal{H}_- \oplus \mathcal{H} \oplus \mathcal{H}_+$ so that U has the form $U = \begin{bmatrix} * & 0 & 0 \\ * & T & 0 \\ * & * & * \end{bmatrix}$. The restriction of U to $\mathcal{H} \oplus \mathcal{H}_+$ is the minimal isometric dilation V. The commutant lifting theorem of Sz.Nagy and Foiaş is still valid for this unitary dilation, or indeed, for any unitary dilation of T.

In the case of a commuting row contraction, the Arveson dilation is unique, but a maximal dilation of a row contraction of the type considered in the previous paragraph is not. However Arveson [**2**] showed that the C*-envelope of the universal commuting row contraction is just $\mathrm{C}^*(M) = \mathrm{C}^*(\{M_{z_1}, \dots, M_{z_n}\})$. The Arveson dilation is already maximal in the sense that any further dilation (either an extension or co-extension) to a commuting row contraction will just be the addition of a direct summand of another commuting row contraction.

Therefore the Arveson dilation is canonical and plays a special role, and the more general dilations are not really needed.

Nevertheless, one can ask whether the commutant lifting theorem is valid if one allows these more general dilations. It should not be surprising that the lack of uniqueness makes such a result impossible, as an easy example shows.

We add that there is a context in which one can deal with lack of uniqueness of unitary dilations and still retain commutant lifting. The tree algebras of Davidson–Paulsen–Power [**10**] have such a result. Given a completely contractive representation commuting with a contraction X, one can find *some* unitary dilation of the algebra commuting with a contractive dilation of X. If one restricts one's attention to the smaller class that Muhly and Solel [**20**] call tree algebras, the minimal unitary dilation is unique and one obtains the more traditional form of commutant lifting.

2. Background

We start with the canonical model for a row contraction. Let $\mathbb{F}_n^+$ denote the free semigroup on n letters, and form Fock space $\mathcal{F}_n = \ell^2(\mathbb{F}_n^+)$ with basis $\{\xi_w : w \in \mathbb{F}_n^+\}$. Define the left regular representation of $\mathbb{F}_n^+$ on $\mathcal{F}_n$ by $L_v \xi_w = \xi_{vw}$ for $v, w \in \mathbb{F}_n^+$. The algebra $\mathfrak{L}_n$ is the WOT-closed span of $\{L_v : v \in \mathbb{F}_n^+\}$. The commutant of $\mathfrak{L}_n$ is $\mathfrak{R}_n$, the algebra generated by the right regular representation [**7**, Theorem 1.2].

In particular, $L = \begin{bmatrix} L_1 & \dots & L_n \end{bmatrix}$ is a row isometry with range $(\mathbb{C}\xi_\varnothing)^\perp$. This row isometry L is the canonical model for a row contraction because of Popescu's von Neumann inequality [**23**] which states that for any $A = \begin{bmatrix} A_1 & \dots & A_n \end{bmatrix}$ which is a row contraction, i.e. $\|A\| \le 1$, and any polynomial p in n non-commuting variables, one has

$$\|p(A_1, \dots, A_n)\| \le \|p(L_1, \dots, L_n)\|.$$

Now we consider the commuting case. Let $\mathbb{N}_0$ be the set of non-negative integers. For $\lambda \in \mathbb{C}^n$ and each n-tuple $k = (k_1, \dots, k_n) \in \mathbb{N}_0^n$, let us write $\lambda^k := \lambda_1^{k_1} \dots \lambda_n^{k_n}$. For $k \in \mathbb{N}_0^n$, let $\mathcal{P}_k := \{w \in \mathcal{F}_n : w(\lambda) = \lambda^k \text{ for all } \lambda \in \mathbb{C}^n\}$. Define vectors in $\mathcal{F}_n$ by

$$\zeta^k := \frac{1}{|\mathcal{P}_k|} \sum_{w \in \mathcal{P}_k} \xi_w.$$

Note that $|\mathcal{P}_k| = \binom{|k|!}{k_1!\, k_2! \cdots k_n!}$ and $\|\zeta^k\| = |\mathcal{P}_k|^{-1/2}$. This set of symmetric words forms an orthogonal basis for *symmetric Fock space* $\mathcal{H}_n^2$. We consider $\mathcal{H}_n^2$ as a space of analytic functions on the unit ball $\mathbb{B}_n$ of $\mathbb{C}^n$ by identifying an element $f = \sum_{k \in \mathbb{N}^n} a_k \zeta^k$ with the function

$$f(\lambda) = \sum_{k \in \mathbb{N}^n} a_k \lambda^k.$$

This series converges uniformly for $\|\lambda\| \le r$ for any $r < 1$.

For each $\lambda \in \mathbb{B}_n$, let

$$k_\lambda = \sum_{w\in\mathcal{F}_n} \overline{w(\lambda)}\xi_w = \sum_{k\in\mathbb{N}_0^n} \overline{\lambda}^k |\mathcal{P}_k| \zeta^k.$$

Then

$$\langle \zeta^k, k_\lambda \rangle = \lambda^k |\mathcal{P}_k| \|\zeta^k\|^2 = \lambda^k.$$

Therefore $\langle f, k_\lambda \rangle = f(\lambda)$ for all $\lambda \in \mathbb{B}_n$. It is evident that each k_λ belongs to $\mathcal{H}^2_n$; and they span the whole space because any element $f \in \mathcal{H}^2_n$ orthogonal to all k_λ satisfies $f(\lambda) = 0$ for $\lambda \in \mathbb{B}_n$, and hence $f = 0$. Therefore $\mathcal{H}^2_n$ becomes a reproducing kernel Hilbert space.

A *multiplier* on $\mathcal{H}^2_n$ is a function h on $\mathbb{B}_n$ so that $M_h f = hf$ determines a well defined map of $\mathcal{H}^2_n$ into itself. A standard argument shows that such maps are continuous. It is easy to see that h must be an analytic function, and the set of all multipliers forms a WOT-closed algebra of operators. A routine calculation shows that $M_h^* k_\lambda = \overline{h(\lambda)} k_\lambda$ for $\lambda \in \mathbb{B}_n$; and conversely, any bounded operator T such that $T^* k_\lambda = \overline{h(\lambda)} k_\lambda$ determines a multiplier h.

Now, [**7**, Theorem 2.6] shows that the vectors k_λ are precisely the set of eigenvectors for $\mathfrak{L}_n^*$, and $L_i^* k_\lambda = \overline{\lambda_i} k_\lambda$. In particular,

$$P_{\mathcal{H}^2_n} L_i|_{\mathcal{H}^2_n} = M_{z_i} \quad \text{for} \quad 1 \le i \le n$$

are the multipliers by the coordinate functions. Indeed, $\mathcal{H}^2_n$ is co-invariant for $\mathfrak{L}_n$ and the compression to $\mathcal{H}^2_n$ yields a complete quotient of $\mathfrak{L}_n$ by its commutator ideal onto the space of all multipliers of $\mathcal{H}^2_n$ [**9**, Corollary 2.3 and § 4]. Here $A \in \mathfrak{L}_n$ is sent to $P_{\mathcal{H}^2_n} A|_{\mathcal{H}^2_n} = M_{\hat{A}}$ where $\hat{A}(\lambda) := \langle Ak_\lambda, k_\lambda \rangle / \|k_\lambda\|^2$.

The row operator $M = \begin{bmatrix} M_{z_1} & \dots & M_{z_n} \end{bmatrix}$ is the canonical model for a commuting row contraction. This is because of Drury's von Neumann inequality [**13**] which states that for any $T = \begin{bmatrix} T_1 & \dots & T_n \end{bmatrix}$ which is a commuting row contraction, i.e. $\|T\| \le 1$ and $T_iT_j = T_jT_i$ for $1 \le i, j \le n$, and any polynomial p in n commuting variables, one has

$$\|p(T_1, \dots, T_n)\| \le \|p(M_{z_1}, \dots, M_{z_n})\|.$$

If A is an operator on a Hilbert space $\mathcal{H}$, we call an operator B on a Hilbert space $\mathcal{K} \supset \mathcal{H}$ an *extension* of A if $\mathcal{H}$ is invariant for B and $B|_\mathcal{H} = A$, i.e., $B = \left[\begin{smallmatrix} A & * \\ 0 & * \end{smallmatrix}\right]$ with respect to the decomposition $\mathcal{K} = \mathcal{H} \oplus \mathcal{H}^\perp$. Likewise, B is a *co-extension* of A if $\mathcal{H}$ is co-invariant for B and $P_\mathcal{H} B = AP_\mathcal{H}$, i.e. $B = \left[\begin{smallmatrix} A & 0 \\ * & * \end{smallmatrix}\right]$. Finally, we say that B is a *dilation* of A if $\mathcal{H}$ is semi-invariant and $P_\mathcal{H} B|_\mathcal{H} = A$, i.e. $B = \left[\begin{smallmatrix} * & * & * \\ 0 & A & * \\ 0 & 0 & * \end{smallmatrix}\right]$ with respect to a decomposition $\mathcal{K} = \mathcal{K}_+ \oplus \mathcal{H} \oplus \mathcal{K}_-$. By a result of Sarason [**26**], this latter spatial condition is equivalent to the algebraic condition $P_\mathcal{H} B^k|_\mathcal{H} = A^k$ for $k \ge 0$. This is sometimes called a power dilation.

The Sz. Nagy dilation theory shows that every contraction A has a unique minimal isometric co-extension, and a unique minimal unitary dilation. There is an analogue of the Sz. Nagy isometric dilation in both of the multivariable contexts given above. In the case of a row contraction $A = \begin{bmatrix} A_1 & \dots & A_n \end{bmatrix}$, the Frazho–Bunce dilation [**14, 6**] states that there is a unique minimal isometric co-extension of A. That is, there is a unique minimal row isometry $W = \begin{bmatrix} W_1 & \dots & W_n \end{bmatrix}$ on a Hilbert space $\mathcal{K} \supset \mathcal{H}$ so that each W_i is a co-extension of A_i. Bunce's proof is to apply Sz. Nagy's isometric dilation to the row operator A and observe that this suffices.

A row isometry W on $\mathcal{H}$ must consist of n isometries with pairwise orthogonal ranges, and thus satisfies the Cuntz–Toeplitz relation $\sum_{i=1}^n W_i W_i^* \le I$. It is said to be Cuntz type if $\sum_{i=1}^n W_i W_i^* = I$. It can be canonically decomposed using the Wold decomposition [**22**] into $W \simeq L^{(\alpha)} \oplus S$ on $\mathcal{H} = {\mathcal{F}_n}^{(\alpha)} \oplus \mathcal{K}$, where S is a Cuntz row isometry, simply by setting $\mathcal{M}$ to be the range of $I - \sum_{i=1}^n W_i W_i^*$, identifying the invariant subspace it generates with $\mathcal{M} \otimes \mathcal{F}_n$, and letting $\mathcal{K}$ be the orthogonal complement. Popescu's von Neumann inequality is a consequence of the Frazho–Bunce dilation and the fact that there is a canonical $*$-homomorphism from the Cuntz–Toeplitz C*-algebra $\mathcal{E}_n = \mathrm{C}^*(L_1, \dots, L_n)$ onto the $\mathrm{C}^*(W_1, \dots, W_n)$ for any row isometry W.

In the case of a commuting row contraction, Drury did not extend his von Neumann inequality to a dilation theorem. This was done by Muller and Vasilescu [**19**] in a very general but combinatorial way, and by Arveson [**2**] in the context of symmetric Fock space. He first notes that $\mathrm{C}^*(M_{z_1}, \dots, M_{z_n})$ contains the compact operators, and the quotient is $\mathrm{C}(\mathbb{S}_n)$, the space of continuous functions on the unit sphere $\mathbb{S}_n$ in $\mathbb{C}^n$. A *spherical unitary* is a row operator $U = \begin{bmatrix} U_1 & \dots & U_n \end{bmatrix}$ where the U_i are commuting normal operators such that $\sum_{i=1}^n U_i U_i^* = I$, i.e. the U_i are the images of z_i for $1 \le i \le n$ under a $*$-representation of $\mathrm{C}(\mathbb{S}_n)$. Arveson's dilation theorem states that any commuting row isometry $T = \begin{bmatrix} T_1 & \dots & T_n \end{bmatrix}$ has a unique minimal co-extension to an operator of the form $M^{(\alpha)} \oplus U$, where $M = \begin{bmatrix} M_{z_1} & \dots & M_{z_n} \end{bmatrix}$ is the multiplier on symmetric Fock space and U is a spherical unitary.

Next, we consider the commutant lifting theorem. Consider the case of a single contraction A on $\mathcal{H}$ with minimal isometric dilation V on $\mathcal{K}$ and minimal unitary dilation U on $\mathcal{K}'$. Any contraction X commuting with A has a co-extension to a contraction Y on $\mathcal{K}$ which commutes with V, and this has an extension to a contraction Z on $\mathcal{K}'$ commuting with U.

Popescu [**22**] observes that exactly the same proof yields a commutant lifting theorem for a row contraction $A = \begin{bmatrix} A_1 & \dots & A_n \end{bmatrix}$ with minimal row isometric dilation W. If there is a contraction X which commutes with each A_i, then there is a contraction Y which is a co-extension of X that commutes with $W_1, \dots, W_n$. Frazho [**15**] did it for the case $n = 2$.

In **[9]**, the first author and Pitts gave a new proof of Drury's von Neumann inequality by observing that when T is a commuting row contraction of norm less than 1 (so that the isometric dilation has the form $L^{(\alpha)}$), the row isometric dilation on ${\mathcal{F}_n}^{(\alpha)}$ identifies the original space $\mathcal{H}$ with a subspace of ${\mathcal{H}_n^2}^{(\alpha)}$.

We have a similar approach to the commutant lifting theorem for a commuting row contraction. The plan is to apply Popescu's commutant lifting theorem and restrict down to the space on which Arveson's dilation lives.

3. Commutant Lifting

To set the stage, let $T = \begin{bmatrix} T_1 & \dots & T_n \end{bmatrix}$ be a commuting row contraction on a Hilbert space $\mathcal{H}$, and let X be an operator of norm 1 which commutes with each T_i, $1 \le i \le n$. Let the Arveson dilation of T be $\tilde{T} = \begin{bmatrix} \tilde{T}_1 & \dots & \tilde{T}_n \end{bmatrix}$. Decompose $\tilde{T} \simeq M^{(\alpha)} \oplus U$, where U is a spherical unitary, on $\mathcal{K} \simeq {\mathcal{H}_n^2}^{(\alpha)} \oplus \mathcal{N}$. By the Frazho–Bunce dilation theorem, we can dilate $\tilde{T}$ to a row isometry $\hat{T}$ on $\mathcal{K}' \simeq {\mathcal{F}_n}^{(\alpha)} \oplus \mathcal{N}'$ which is unitarily equivalent to $L^{(\alpha)} \oplus S$, where S is the isometric dilation of U and is of Cuntz type because $\sum_{i=1}^n U_i U_i^* = I$.

The starting point of our proof is to invoke Popescu's commutant lifting theorem to obtain an operator Y on $\mathcal{K}$ commuting with $\hat{T}_i$ for $1 \le i \le n$ such that $\|Y\| = 1$ and $P_{\mathcal{H}} Y = X P_{\mathcal{H}}$, so Y is a co-extension of X. Define $Z = P_{\mathcal{K}} Y|_{\mathcal{K}}$. We will establish that Z is the desired lifting.

The first lemma deals with the structure of the commutant of a row isometry which dilates $M^{(\alpha)} \oplus U$. Let $\mathcal{A}_n = \mathcal{F}_n \ominus \mathcal{H}_n^2$ denote the antisymmetric part of Fock space.

Lemma 3.1. *Let U be a spherical unitary, and let $L^{(\alpha)} \oplus S$ be the minimal row isometric dilation of $M^{(\alpha)} \oplus U$ as above. Suppose that Y is an operator commuting with $L^{(\alpha)} \oplus S$. Write $Y = \left[\begin{smallmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{smallmatrix}\right]$ with respect to the decomposition $\mathcal{K}' = {\mathcal{F}_n}^{(\alpha)} \oplus \mathcal{N}'$. Then*

(i) Y_{11} *commutes with* $L_1^{(\alpha)}, \dots, L_n^{(\alpha)}$.
(ii) Y_{22} *commutes with* $S_1, \dots, S_n$.
(iii) $Y_{12} = 0$.
(iv) $P_{\mathcal{N}} Y_{21}|_{{\mathcal{A}_n}^{(\alpha)}} = 0$ *and* $U_j P_{\mathcal{N}} Y_{21}|_{{\mathcal{H}_n^2}^{(\alpha)}} = P_{\mathcal{N}} Y_{21}|_{\mathcal{H}_n^2)^{(\alpha)}} M_j^{(\alpha)}$ *for* $1 \le j \le n$.

Proof. We have $(L^{(\alpha)} \oplus S) Y^{(n)} = Y (L^{(\alpha)} \oplus S)$, which yields

$$\begin{bmatrix} L^{(\alpha)} Y_{11}^{(n)} & L^{(\alpha)} Y_{12}^{(n)} \\ S Y_{21}^{(n)} & S Y_{22}^{(n)} \end{bmatrix} = \begin{bmatrix} Y_{11} L^{(\alpha)} & Y_{12} S \\ Y_{21} L^{(\alpha)} & Y_{22} S \end{bmatrix}.$$

Thus (i) and (ii) are immediate from the equality of the 11 and 22 entries, respectively. The 12 entry yields

$$\sum_{i=1}^{n} L_i^{(\alpha)} Y_{12} S_i^* = L^{(\alpha)} Y_{12}^{(n)} S^* = Y_{12} S S^* = Y_{12}.$$

Repeated application yields a sum over all words in $\mathbb{F}_n^+$ of length k:

$$Y_{12} = \sum_{|w|=k} L_w^{(\alpha)} Y_{12} S_w^*.$$

Thus Y_{12} has range contained in the space $\bigwedge_{k\geq 1} \sum_{|w|=k} (L_w \mathcal{F}_n)^{(\alpha)} = \{0\}$. Therefore, $Y_{12} = 0$; so (iii) holds.

Also, $SY_{21}^{(n)} = Y_{21} L^{(\alpha)}$. Let $T = P_{\mathcal{N}} Y_{21}$. Then

$$U_j T = U_j P_{\mathcal{N}} Y_{21} = P_{\mathcal{N}} S_j Y_{21} = P_{\mathcal{N}} Y_{21} L_j^{(\alpha)} = T L_j^{(\alpha)}.$$

Therefore $U_w T = T L_w^{(\alpha)}$ for all words $w \in \mathbb{F}_n^+$. (Here, if $w = i_1 \dots i_k$, we set $U_w = U_{i_1} \dots U_{i_k}$.) In particular,

$$T L_v (L_i L_j - L_j L_i) L_w = U_v (U_i U_j - U_j U_i) U_w T = 0$$

for all $1 \leq i, j \leq n$ and $u, v \in \mathbb{F}_n^+$. By [**8**, Prop. 2.4], the commutator ideal of $\mathfrak{L}_n$ is spanned by the words $L_v(L_iL_j - L_jL_i)L_w$, and the range of this ideal is $\mathcal{A}_n$. Hence $P_{\mathcal{N}} Y_{21} P_{\mathcal{A}_n{}^{(\alpha)}} = T P_{\mathcal{A}_n{}^{(\alpha)}} = 0$; or $T = T P_{\mathcal{H}_n^2{}^{(\alpha)}}$. Now,

$$U_j T|_{\mathcal{H}_n^2{}^{(\alpha)}} = T L_j^{(\alpha)}|_{\mathcal{H}_n^2{}^{(\alpha)}} = T P_{\mathcal{H}_n^2{}^{(\alpha)}} L_j^{(\alpha)}|_{\mathcal{H}_n^2{}^{(\alpha)}} = T|_{\mathcal{H}_n^2{}^{(\alpha)}} M_j^{(\alpha)}.$$

Thus (iv) follows. ∎

This next lemma is taken from Davie–Jewell [**11**, Prop. 2.4]. We provide a more conceptual proof.

Lemma 3.2. *Let $U = \begin{bmatrix} U_1 & \dots & U_n \end{bmatrix}$ be a spherical unitary on a Hilbert space $\mathcal{H}$. Suppose that A is a bounded operator on $\mathcal{H}$ satisfying $\sum_{j=1}^n U_j A U_j^* = A$. Then A lies in the commutant $\mathrm{C}^*(\{U_1, \dots, U_n\})'$.*

Proof. Let E be the spectral measure for U. Observe that E is supported on the unit sphere $\mathbb{S}_n$ of $\mathbb{C}^n$; and $U_j = \int_{\mathbb{S}_n} z_j \, dE$ for $j = 1, \dots, n$. We will show that $E(X)AE(Y) = 0$ if X, Y are disjoint Borel subsets of $\mathbb{S}_n$. Set $B = E(X)AE(Y)$.

Define

$$\rho(X, Y) = \inf\{|1 - \langle x, y\rangle| : x \in X, y \in Y\}$$

and

$$\mathrm{diam}(X) = \sup\{|a - b| : a, b \in X\}.$$

First consider the case in which

$$\max\{\mathrm{diam}(X), \mathrm{diam}(Y)\} < \rho(X, Y)/2n.$$

Fix points $x = (x_1, \dots, x_n) \in X$ and $y = (y_1, \dots, y_n) \in Y$. Since $E(X)$ and $E(Y)$ commute with the U_j's, we have

$$\begin{aligned} B &= \sum_{j=1}^n U_j B U_j^* = \sum_{j=1}^n U_j E(X) B E(Y) U_j^* \\ &= \sum_{j=1}^n x_j \bar{y}_j B + (U_j - x_j) E(X) B E(Y) U_j^* + x_j E(X) B E(Y) (U_j^* - \bar{y}_j). \end{aligned}$$

Since $|z_j - x_j| \le \operatorname{diam}(X)$ for all $z \in X$, we have

$$\|(U_j - x_j) E(X)\| \le \operatorname{diam}(X).$$

Similarly, $\|E(Y)(U_j^* - \bar{y}_j)\| \le \operatorname{diam}(Y)$. Therefore,

$$\begin{aligned} &|1 - \langle x, y\rangle| \|B\| \le \\ &\quad \sum_{j=1}^n \|(U_j - x_j)E(X)\| \, \|BE(Y)U_j^*\| + \|x_j E(X) B\| \, \|E(Y)(U_j^* - \bar{y}_j)\| \\ &\quad \le n \|B\| (\operatorname{diam}(X) + \operatorname{diam}(Y)) < \rho(X, Y) \|B\|. \end{aligned}$$

But $|1 - \langle x, y\rangle| \ge \rho(X, Y)$, so $\|B\| = 0$.

For X and Y with $\rho(X, Y) > 0$, partition X as a disjoint union of Borel subets $X_1, \dots, X_k$ and Y as a disjoint union of Borel subets $Y_1, \dots, Y_l$ so that

$$\max\{\operatorname{diam}(X_i), \operatorname{diam}(Y_j) : 1 \le i \le k,\ 1 \le j \le l\} < \rho(X, Y)/2n.$$

Since $\rho(X_i, Y_j) \ge \rho(X, Y)$, it then follows that $E(X_i) A E(Y_j) = 0$. Thus, $E(X) A E(Y) = \sum_{i,j} E(X_i) A E(Y_j) = 0$.

Finally suppose X and Y are disjoint Borel subsets of $\mathbb{S}_n$. By the regularity of the spectral measure E, we have $E(X) = \bigvee E(K)$ as K runs over all compact subsets of X. For any compact sets $K_1 \subset X$ and $K_2 \subset Y$, we have $\rho(K_1, K_2) > 0$ and hence $E(K_1) A E(K_2) = 0$. It follows that $E(X) A E(Y) = 0$.

In particular $E(X) A E(\mathbb{S}_n \backslash X) = 0 = E(\mathbb{S}_n \backslash X) A E(X)$. So

$$E(X) A = E(X) A E(X) = A E(X).$$

Since A commutes with all the spectral projections, we see that A commutes with U_j and U_j^* for $1 \le j \le n$, and thus lies in $\mathrm{C}^*(\{U_1, \dots, U_n\})'$. ∎

Lemma 3.3. *Let $U = \begin{bmatrix} U_1 & \dots & U_n \end{bmatrix}$ be a spherical unitary on a Hilbert space $\mathcal{N}$. Let $S = \begin{bmatrix} S_1 & \dots & S_n \end{bmatrix}$ be the minimal isometric dilation of U on $\mathcal{N}' \supset \mathcal{N}$. Suppose that Y is an operator on $\mathcal{N}'$ which commutes with $S_1, \dots, S_n$. Then $A = P_{\mathcal{N}} Y|_{\mathcal{N}}$ lies in $\mathrm{C}^*(\{U_1, \dots, U_n\})'$ and $P_{\mathcal{N}} Y P_{\mathcal{N}}^{\perp} = 0$.*

Proof. For $1 \le j \le n$, write $S_j = \begin{bmatrix} U_j & 0 \\ D_j & E_j \end{bmatrix}$ with respect to the decomposition $\mathcal{N}' = \mathcal{N} \oplus \mathcal{N}^{\perp}$. Let us write $D = \begin{bmatrix} D_1 & \dots & D_n \end{bmatrix}$ and $E = \begin{bmatrix} E_1 & \dots & E_n \end{bmatrix}$. Then we may write $S = \left[\begin{smallmatrix} U & 0 \\ D & E \end{smallmatrix}\right]$. Since S is a row isometry, $S^*S = I_{\mathcal{N}'}^{(n)}$, which implies $U^*U + D^*D = I_{\mathcal{N}}^{(n)}$ (note that U^* and D^* are column operators). This together with $UU^* = I_{\mathcal{N}}$ implies

$$UD^*DU^* = UU^* - UU^*UU^* = 0.$$

Thus, $DU^* = 0$.

Since Y commutes with the S_j's, we have $SY^{(n)} = YS$. Write $Y = \left[\begin{smallmatrix} A & B \\ * & * \end{smallmatrix}\right]$. Then $UA^{(n)} = AU + BD$ and $UB^{(n)} = BE$. Therefore,

$$\sum_{j=1}^{n} U_j A U_j^* = UA^{(n)}U^* = AUU^* + BDU^* = A.$$

By Lemma 3.2, A belongs to $\mathrm{C}^*(\{U_1, \dots, U_n\})'$.

Since $AU = UA^{(n)} = AU + BD$, we obtain $BD = 0$, i.e. $BD_j = 0$ for $1 \le j \le n$. The 12 entry of $SY^{(n)} = YS$ yields $U_jB = BE_j$. Since S is the minimal dilation, $\mathcal{N}' = \bigvee_{w \in \mathbb{F}_n^+} S_w \mathcal{N}$. Thus $\mathcal{N}^\perp = \bigvee_{w \in \mathbb{F}_n^+} E_w D \mathcal{N}^{(n)}$. However $BE_w D\mathcal{N}^{(n)} = U_w BD\mathcal{N}^{(n)} = 0$ for all $w \in \mathbb{F}_n^+$. Hence $B = 0$. ∎

We now have the tools to complete the proof of the commutant lifting theorem.

Proof of Theorem 1.1. We proceed as indicated at the beginning of this section, dilating $\tilde{T} \simeq M^{(\alpha)} \oplus U$ to a row isometry on $\mathcal{K}'$ unitarily equivalent to $L^{(\alpha)} \oplus S$. Let Y be the commuting lifting of X, and set $Z = P_{\mathcal{K}} Y|_{\mathcal{K}}$. Since $P_{\mathcal{H}} Y = XP_{\mathcal{H}}$, it follows that $P_{\mathcal{H}} Z = XP_{\mathcal{H}}$. So Z dilates X. Also,

$$\|X\| \le \|Z\| \le \|Y\| = \|X\|,$$

and hence $\|Z\| = \|X\|$. So it remains to verify that Z commutes with $\tilde{T}_j$ for $1 \le j \le n$.

Write $Y = \left[\begin{smallmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{smallmatrix}\right]$ with respect to the direct sum ${\mathcal{F}_n}^{(\alpha)} \oplus \mathcal{N}'$. By Lemma 3.1, $Y_{12} = 0$ and Y_{11} commutes with $L_1^{(\alpha)}, \dots, L_n^{(\alpha)}$. Since the commutant of $\mathfrak{L}_n$ is $\mathfrak{R}_n$, we see that Y_{11} can be written as an $\alpha \times \alpha$ matrix with coefficients in $\mathfrak{R}_n$. Since the symmetric Fock space $\mathcal{H}_n^2$ is co-invariant for $\mathfrak{R}_n$ and $\mathfrak{L}_n$, we see that ${\mathcal{H}_n^2}^{(\alpha)}$ is co-invariant for Y_{11} and for $L_1^{(\alpha)}, \dots, L_n^{(\alpha)}$. Therefore, $P_{{\mathcal{H}_n^2}^{(\alpha)}} Y_{11}|_{{\mathcal{A}_n}^{(\alpha)}} = 0$ and $P_{{\mathcal{H}_n^2}^{(\alpha)}} Y_{11}|_{{\mathcal{H}_n^2}^{(\alpha)}}$ commutes with $M_j^{(\alpha)} = P_{{\mathcal{H}_n^2}^{(\alpha)}} L_j^{(\alpha)}|_{{\mathcal{H}_n^2}^{(\alpha)}}$ for $1 \le j \le n$.

By Lemma 3.1 again, we have

$$P_{\mathcal{N}} Y_{21}|_{{\mathcal{A}_n}^{(\alpha)}} = 0$$

and

$$U_j P_{\mathcal{N}} Y_{21}|_{{\mathcal{H}_n^2}^{(\alpha)}} = P_{\mathcal{N}} Y_{21}|_{{\mathcal{H}_n^2}^{(\alpha)}} M_j^{(\alpha)} \quad \text{for} \quad j = 1, \dots, n$$

and Y_{22} commutes with each S_j. Lemma 3.3 shows that $P_{\mathcal{N}} Y_{22}|_{\mathcal{N}^\perp} = 0$ and $P_{\mathcal{N}} Y_{22}|_{\mathcal{N}}$ commutes with $U_1, \dots, U_n$.

Since

$$Z = \begin{bmatrix} P_{\mathcal{H}^2_n(\alpha)} Y_{11}|_{\mathcal{H}^2_n(\alpha)} & P_{\mathcal{H}^2_n(\alpha)} Y_{12}|_{\mathcal{N}} \\ P_{\mathcal{N}} Y_{21}|_{\mathcal{H}^2_n(\alpha)} & P_{\mathcal{N}} Y_{22}|_{\mathcal{N}} \end{bmatrix},$$

Z commutes with $M_1^{(\alpha)} \oplus U_1, \dots, M_n^{(\alpha)} \oplus U_n$ as claimed. ∎

We obtain the following corollary by the standard trick of applying the commutant lifting theorem to $S \oplus T$ and $\left[\begin{smallmatrix} 0 & X \\ 0 & 0 \end{smallmatrix}\right]$.

Corollary 3.4. *Suppose that* $S = \begin{bmatrix} S_1 & \dots & S_n \end{bmatrix}$ *and* $T = \begin{bmatrix} T_1 & \dots & T_n \end{bmatrix}$ *are commuting row contractions on Hilbert spaces* $\mathcal{H}_1$ *and* $\mathcal{H}_2$*, respectively, and that* X *is an operator in* $\mathcal{B}(\mathcal{H}_2, \mathcal{H}_1)$ *such that* $S_i X = X T_i$ *for* $1 \le i \le n$*. Let* $\tilde{S} = \begin{bmatrix} \tilde{S}_1 & \dots & \tilde{S}_n \end{bmatrix}$ *and* $\tilde{T} = \begin{bmatrix} \tilde{T}_1 & \dots & \tilde{T}_n \end{bmatrix}$ *be the Arveson dilations of* S *and* T *on* $\mathcal{K}_1$ *and* $\mathcal{K}_2$*, respectively. Then there is an operator* Z *in* $\mathcal{B}(\mathcal{K}_2, \mathcal{K}_1)$ *such that* $\tilde{S}_i Z = Z \tilde{T}_i$ *for* $1 \le i \le n$*, which dilates* X *in the sense* $P_{\mathcal{H}_1} Z = X P_{\mathcal{H}_2}$ *and satisfies* $\|Z\| = \|X\|$*.*

4. Failure of General Commutant Lifting

In this section, we consider more general dilations rather than the canonical co-extension. Since this is not unique, it should not be surprising that there is no commutant lifting theorem.

Before doing so, we wish to make the point that this is more natural in the single variable case. The reason is that one wants to obtain the maximal dilations, i.e. dilations which can only be further dilated by the adjoining of another direct summand. These are the elements which determine representations of the underlying operator algebra which extend to $*$-representations of the enveloping C*-algebra, and factor through the C*-envelope **[12]**. In the case of a single contraction, the operator algebra generated by a universal contraction is the disk algebra, as evidenced by (the matrix version of) the von Neumann inequality. The C*-envelope of $\mathrm{A}(\mathbb{D})$ is the abelian algebra $\mathrm{C}(\mathbb{T})$. To dilate a contraction to a maximal dilation, one must go to the unitary dilation.

Note that in the case of a single contraction, the uniqueness of the minimal unitary dilation means that one can dilate in either direction, or alternate at randon, since once one obtains a maximal dilation, it will be the minimal unitary dilation plus a direct summand of some other arbitrary unitary.

However in the case of a commuting row contraction, one is done after the co-extension to the Arveson dilation. No further extension is possible if the maximal co-extension is done first. So one could argue that the extension, being unnecessary and non-canonical, should never be considered.

The following example adds further evidence.

Example 4.1. Fix $0 < r < 1$. Let

$$T_1 = \begin{bmatrix} r & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad T_2 = \begin{bmatrix} 0 & 0 \\ 0 & r \end{bmatrix}.$$

Clearly $T = \begin{bmatrix} T_1, T_2 \end{bmatrix}$ is a commuting row contraction on $\mathcal{H}_1 = \mathbb{C}^2$. For $0 < \varepsilon \neq r$, define

$$B_1 = \begin{bmatrix} \varepsilon & 0 & 0 \\ \varepsilon(\varepsilon - r) & r & 0 \\ \varepsilon^2 & 0 & 0 \end{bmatrix} \quad \text{and} \quad B_2 = \begin{bmatrix} \varepsilon & 0 & 0 \\ \varepsilon^2 & 0 & 0 \\ \varepsilon(\varepsilon - r) & 0 & r \end{bmatrix}$$

Then B_1 and B_2 commute and if ε is small enough, $B = \begin{bmatrix} B_1, B_2 \end{bmatrix}$ is a row contraction on $\mathcal{H}_2 \oplus \mathcal{H}_1 = \mathbb{C}^3$. Let $S = \begin{bmatrix} S_1 \ S_2 \end{bmatrix}$ be the Arveson minimal dilation of B, acting on $\mathcal{H}_2 \oplus \mathcal{H}_1 \oplus \mathcal{H}_3$. Then S is a maximal dilation of T.

Let $X = \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$. Observe that X is a unitary operator which commutes with T_1 and T_2. We will show that there does not exist an operator Y on $\mathcal{H}_2 \oplus \mathcal{H}_1 \oplus \mathcal{H}_3$ which is a dilation of X such that $\|Y\| = 1$ and Y commutes with both S_1 and S_2. Suppose there were such an operator Y. Write

$$Y = \begin{bmatrix} y_{11} & y_{12} & y_{13} & y_{14} \\ y_{21} & -1 & 0 & y_{24} \\ y_{31} & 0 & 1 & y_{34} \\ y_{41} & y_{42} & y_{43} & y_{44} \end{bmatrix}$$

Since $\|Y\| = 1$ and X is a unitary, we see that

$$y_{12} = y_{13} = y_{21} = y_{24} = y_{31} = y_{34} = y_{42} = y_{43} = 0.$$

To be a dilation, $y_{14} = 0$ also; but one could also look at the 24 entry of $S_1 Y - Y S_1$ to see that this is necessary. But then $S_1 Y - Y S_1$ equals

$$\begin{bmatrix} \varepsilon & 0 & 0 & 0 \\ \varepsilon(\varepsilon - r) & r & 0 & 0 \\ \varepsilon^2 & 0 & 0 & 0 \\ * & * & * & * \end{bmatrix} \begin{bmatrix} y_{11} & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ y_{41} & 0 & 0 & y_{44} \end{bmatrix} - \begin{bmatrix} y_{11} & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ y_{41} & 0 & 0 & y_{44} \end{bmatrix} \begin{bmatrix} \varepsilon & 0 & 0 & 0 \\ \varepsilon(\varepsilon - r) & r & 0 & 0 \\ \varepsilon^2 & 0 & 0 & 0 \\ * & * & * & * \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ (y_{11} - 1)\varepsilon(\varepsilon - r) & 0 & 0 & 0 \\ (y_{11} + 1)\varepsilon^2 & 0 & 0 & 0 \\ * & * & * & * \end{bmatrix}$$

It follows that no choice of y_{11} makes both the 21 and 31 entries 0. Therefore there is no commuting lifting of X.

5. The Ball–Trent–Vinnikov CLT

A reproducing kernel Hilbert space (RKHS) is a Hilbert space $\mathcal{H}$ of functions on a set X which separates points such that point evaluations are continuous. There are functions $\{k_x \in \mathcal{H} : x \in X\}$ such that $f(x) = \langle h, k_x \rangle$ for all $f \in \mathcal{H}$ and $x \in X$. The kernel function k on $X \times X$ given by $k(x, y) = k_y(x)$ is positive definite, meaning that the matrix $\big[k(x_i, x_j)\big]_{i,j=1}^n$ is positive definite for every finite subset $x_1, \dots, x_n$ of X. The kernel is called irreducible if $k(x, y) \neq 0$ for all $x, y \in X$.

The space of multipliers $\operatorname{Mult}(\mathcal{H})$ consists of the (bounded) functions h on X such that $M_h f(x) = h(x)f(x)$ determines a bounded operator. $\operatorname{Mult}(\mathcal{H})$ is endowed with the operator norm, and it forms a WOT-closed, maximal abelian subalgebra of $\mathcal{B}(\mathcal{H})$.

The kernel k (or the space $\mathcal{H}$) is called a complete Nevanlinna–Pick kernel if for every finite subset $x_1, \dots, x_n$ of X and $d \times d$ matrices $W_1, \dots, W_n$, there is an operator F in $\mathfrak{M}_d(\operatorname{Mult}(\mathcal{H}))$ such that $F(x_i) = W_i$ for $1 \leq i \leq n$ and $\|M_F\| \leq 1$ if and only if the Pick matrix $\big[(I_d - W_i W_j^*)k(x_i, x_j)\big]_{i,j=1}^n$ is non-negative. This is motivated by the classical Nevanlinna–Pick Theorem for H^∞, where the corresponding RKHS is H^2 with the Szego kernel $k_z(w) = \frac{1}{1 - w\bar{z}}$ for $z \in \mathbb{D}$. The necessary and sufficient condition for the existence of an $\mathfrak{M}_d$-valued H^∞ function F with $F(z_i) = W_i$ for $1 \leq i \leq n$ and $\|F\|_\infty \leq 1$ is precisely that $\Big[\frac{I_k - W_i W_j^*}{1 - z_i \bar{z}_j}\Big]$ is non-negative.

These kernels were characterized by McCullough **[16, 17]** and Quiggin **[25]**. Agler and McCarthy **[1]** showed that every irreducible complete Nevanlinna–Pick kernel is just the restriction of symmetric Fock space on countably many generators to the span of some subset of the kernel functions. That is, one can always identify X with a subset of the ball $\mathbb{B}_\infty$.

Now we can state the Ball–Trent–Vinnikov theorem **[5]**.

Theorem 5.1 (Ball–Trent–Vinnikov)**.** *Let $\mathcal{H}$ be an irreducible complete Nevanlinna–Pick kernel Hilbert space, and let $\mathcal{M}$ be a subspace of $\mathcal{H}^{(\alpha)}$ which is co-invariant for* $\operatorname{Mult}(\mathcal{H})^{(\alpha)}$*. Suppose that $X \in \mathcal{B}(\mathcal{M})$ commutes with the algebra $P_\mathcal{M} \operatorname{Mult}(\mathcal{H})^{(\alpha)}|_\mathcal{M}$. Then there is an operator Y on $\mathcal{H}^{(\alpha)}$ which commutes with* $\operatorname{Mult}(\mathcal{H})^{(\alpha)}$ *such that $P_\mathcal{M} Y = X P_\mathcal{M}$ and $\|Y\| = \|X\|$.*

Using the Agler–McCarthy theorem, we identify $\mathcal{H}$ with a coinvariant subspace of H^2_n spanned by kernel functions, for some $n \leq \infty$. Saying that X commutes with $P_\mathcal{M} \operatorname{Mult}(\mathcal{H})^{(\alpha)}|_\mathcal{M}$ is equivalent to saying that it commutes with the commuting row contraction T where $T_i = P_\mathcal{M} M_{z_i}|_\mathcal{M}$ for $1 \leq i \leq n$. However, in this case, T is a *pure contraction*, meaning that the Arveson dilation does not have a spherical unitary part.

In this case, our theorem becomes much easier to prove. One can dilate $M^{(\alpha)}$ to $L^{(\alpha)}$, and use Popescu's CLT **[22]** to dilate X to an operator Z of the same norm which commutes with $\mathfrak{L}_n^{(\alpha)}$. Thus Z belongs to $\mathfrak{M}_\alpha(\mathfrak{R}_n)$. Restricting Z to the co-invariant subspace $(H^2_n)^{(\alpha)}$ yields a matrix Y of multipliers of H^2_n. This dilates X, has the same norm and commutes with all multipliers. Now a further compression to $\mathcal{H}^{(\alpha)}$ yields a matrix of multipliers of $\mathcal{H}$ with these same properties.

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