COMMUTANTS OF TOEPLITZ OPERATORS WITH SEPARATELY RADIAL POLYNOMIAL SYMBOLS

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Abstract. We study the commuting problem of Toeplitz operators on the Fock space over $\mathbb{C}^n$. Given a separately radial polynomial φ in z and $\bar{z}$, we characterize polynomially bounded functions ψ such that the operators T_ψ and T_φ commute. Several examples and consequences are discussed.

1. Introduction

Let $L^2(\mathbb{C}^n, d\mu)$ be the Hilbert space of all square integrable functions in $\mathbb{C}^n$ with respect to the Gaussian measure $d\mu(z) = \pi^{-n}e^{-|z|^2}dV(z)$. Here dV is the Lebesgue volume measure on $\mathbb{C}^n$. For any two functions $f, g \in L^2(\mathbb{C}^n, d\mu)$, we have the inner product

$$\langle f, g\rangle = \int_{\mathbb{C}^n} f(z)\overline{g(z)}d\mu(z). \tag{1.1}$$

The *Fock space* (also known as the *Segal–Bargmann space*), denoted by $\mathcal{F}_n^2$, consists of all entire functions that belong to $L^2(\mathbb{C}^n, d\mu)$. It is well known that $\mathcal{F}_n^2$ is a closed subspace of $L^2(\mathbb{C}^n, d\mu)$.

Let $\mathbb{N}_0$ denote the set of non-negative integers. For $k = (k_1, \ldots, k_n) \in \mathbb{N}_0^n$ and $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$, we write $k! = k_1! \cdots k_n!$ and $z^k = z_1^{k_1} \cdots z_n^{k_n}$. It is well known that $\{e_k(z) = z^k/\sqrt{k!} : k \in \mathbb{N}_0^n\}$ is an orthonormal basis for $\mathcal{F}_n^2$. For more details on $\mathcal{F}_n^2$ and its variants, see [14].

Let $P : L^2(\mathbb{C}^n, d\mu) \to \mathcal{F}_n^2$ be the orthogonal projection from $L^2(\mathbb{C}^n, d\mu)$ onto $\mathcal{F}_n^2$. For $\varphi \in L^2(\mathbb{C}^n, d\mu)$, the Toeplitz operator $T_\varphi : \mathcal{F}_n^2 \to \mathcal{F}_n^2$ is defined by

$$T_\varphi(f) = P(\varphi f),$$

for all f in $\mathcal{F}_n^2$ for which φf belongs to $L^2(\mathbb{C}^n, d\mu)$. The function φ is called the symbol of T_φ. It is clear that if φ is bounded, then T_φ is a bounded operator with $\|T_\varphi\| \le \|\varphi\|_\infty$.

Recall that a function φ on $\mathbb{C}^n$ is *polynomially bounded* if there exists an integer $d \ge 0$ such that the map $z \mapsto (1 + |z|^d)^{-1}\varphi(z)$ is bounded. It is not difficult to see that for such φ, the domain of T_φ contains all analytic polynomials. In fact, it has been shown by Bauer [3] that if $\varphi_1, \ldots, \varphi_s$ are

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polynomially bounded, then the domain of the product $T_{\varphi_1}\cdots T_{\varphi_s}$ contains all analytic polynomials.

Our main goal in this paper is to study the commuting problem: given a non-constant function φ, find the necessary and sufficient conditions on the function ψ such that $T_\varphi T_\psi = T_\psi T_\varphi$. The commuting problem for Toeplitz operators on the Hardy space of the unit disk was solved by Brown and Halmos in their seminal paper [6] back in the early sixties. Their result has motivated a vast literature on the studies of commuting Toeplitz operators acting on other Hilbert spaces of analytic functions: the Bergman space over the unit disk [1, 2, 9], the Hardy and Bergman spaces over the polydisk or the ball in higher dimensions [7, 10, 11, 12, 13], just to list a few. The interested reader is referred to the above papers for more references.

Recently, Bauer and Lee [5] studied the commuting problem on the Fock space and they obtained the following results in one dimension ($n = 1$).

(A) If φ is a non-constant analytic polynomial and ψ is assumed to be a polynomial in z and $\bar{z}$, then ψ must also be analytic.
(B) If φ is a non-constant, polynomially bounded radial function on $\mathbb{C}$, then ψ must also be radial. Here, a function g is radial if $g(z) = \tilde{g}(|z|)$ for some function $\tilde{g} : [0,\infty) \to \mathbb{C}$. A higher dimensional version of this result [4, Proposition 5.6] was also obtained under the assumption that both φ and ψ are polynomials in z and $\bar{z}$.

The proof of (A) in [5] makes use of a composition formula for Toeplitz operators with polynomial symbols. For any polynomials φ and ψ in z and $\bar{z}$, we have $T_\varphi T_\psi = T_{\varphi\sharp\psi}$, where $\varphi\sharp\psi$ is a polynomial given by

$$\varphi\sharp\psi = \sum_{\gamma\in\mathbb{N}_0^n} \frac{(-1)^{|\gamma|}}{\gamma!}\,\frac{\partial^{|\gamma|}\varphi}{\partial z^\gamma}\,\frac{\partial^{|\gamma|}\psi}{\partial \bar{z}^\gamma}.$$

See [8, Theorem 2] for a proof. From the composition formula, we see that $T_\varphi T_\psi = T_\psi T_\varphi$ if and only if $\varphi\sharp\psi = \psi\sharp\varphi$. In one dimensional case, it follows from this equation (see the proof of [4, Proposition 5.4]) that ψ must be analytic whenever φ is analytic. On the other hand, the situation becomes much more complicated in higher dimensions. If φ is a non-constant analytic polynomial in z_j only, then it can be shown that ψ must be analytic in the variable z_j. One might guess that for each $1 \le j \le n$, if $\partial_{z_j}\varphi \not\equiv 0$, then ψ must be analytic in the variable z_j. It turns out that this naive guess is incorrect. In fact, if $\varphi(z) = z_1 + z_2$ and $\psi(z) = \overline{z}_1 - \overline{z}_2$, then one can check that $\varphi\sharp\psi = \varphi\psi = \psi\sharp\varphi$. Consequently, statement (A) is false when $n \ge 2$. It would be interesting to obtain a correct version of (A) in this case but we have not been aware of such a result.

In [4], Bauer and Le generalized (B) to higher dimensions. They showed that if both φ and ψ are polynomially bounded functions in $\mathbb{C}^n$ and φ is radial such that T_ψ and T_φ commute, then $\psi(\gamma z_1,\ldots,\gamma z_n) = \psi(z_1,\ldots,z_n)$ for a.e. $z = (z_1,\ldots,z_n) \in \mathbb{C}^n$ and a.e. γ on the unit circle $\mathbb{T}$.

The main goal of our paper is to generalize (B) to higher dimensions in a different direction. Let $\mathbb{C}[z,\bar{z}]$ denote the space of all polynomials in z and $\bar{z}$. We assume that $\varphi \in \mathbb{C}[z,\bar{z}]$ is *separately radial* (see Section 2) and investigate polynomially bounded functions ψ for which T_ψ and T_φ commute. We give a description of the functions ψ and discuss several examples. An interesting consequence of our result is that the set of all such functions ψ forms an algebra. We shall also provide an example which illustrates the difference between the commuting problem on the Fock space and the same problem on the Bergman space over the unit ball.

2. Preliminaries and examples

Recall that a function $g : \mathbb{C}^n \to \mathbb{C}$ is called *separately radial* if

$$g(z_1,\ldots,z_n) = \tilde{g}(|z_1|,\ldots,|z_n|)$$

for some function $\tilde{g} : [0,\infty)^n \to \mathbb{C}$. In this section, we investigate Toeplitz operators on $\mathcal{F}^2_n$ whose symbols are separately radial polynomials. These operators are unbounded but they are diagonal with respect to the standard orthonormal basis of monomials. We shall explore their eigenvalues in details.

The following result is well known but for the reader's convenience, we sketch a proof.

Lemma 2.1. *Let g be a polynomially bounded function which is separately radial in the form $g(z) = \tilde{g}(|z_1|,\ldots,|z_n|)$ for $z \in \mathbb{C}^n$. Then T_g is a diagonal operator with respect to the standard orthonormal basis of monomials: $T_g e_m = \omega(g,m)e_m$ for all $m \in \mathbb{N}_0^n$. The eigenvalue $\omega(g,m)$ is given by*

$$\omega(g,m) = \frac{1}{m!}\int_{[0,\infty)^n} \tilde{g}(\sqrt{t_1},\ldots,\sqrt{t_n})t_1^{m_1}\cdots t_n^{m_n}e^{-(t_1+\cdots+t_n)}dt_1\cdots dt_n.$$

Proof. For multi-indices m and k, we compute

$$\langle T_g e_m, e_k\rangle = \langle g e_m, e_k\rangle = \pi^{-n}\int_{\mathbb{C}^n} g(z)\frac{z^m}{\sqrt{m!}}\frac{\overline{z}^k}{\sqrt{k!}}e^{-|z|^2}dV(z).$$

Integration in polar coordinates shows that the integral is zero if $m \neq k$. For $m = k$, the integral becomes

$$\begin{aligned}\langle T_\varphi e_m, e_m\rangle &= \frac{2^n}{\pi^n\, m!}\int_{[0,\infty)^n} \tilde{g}(r)\, r_1^{2m_1+1}\cdots r_n^{2m_n+1}\, e^{-(r_1^2+\cdots+r_n^2)}\, dr_1\cdots dr_n\\ &= \frac{1}{m!}\int_{[0,\infty)^n} \tilde{g}(\sqrt{t_1},\ldots,\sqrt{t_n})t_1^{m_1}\cdots t_n^{m_n}e^{-(t_1+\cdots+t_n)}dt_1\cdots dt_n.\end{aligned}$$

In the second equality, we have used the change of variables $t_j = r_j^2$ for $1 \le j \le n$. The conclusion of the lemma follows. □

When the function φ is a separately radial polynomial, the eigenvalues of T_φ can be computed explicitly. The following result shows that these eigenvalues are polynomials.

Lemma 2.2. *Let φ be a polynomial in $\mathbb{C}[z,\bar{z}]$. If φ is separately radial, then $\omega(\varphi,m)$ is a polynomial in m.*

Proof. By the hypothesis, φ can be written as a linear combination of monomials of the form $|z_1|^{2d_1}\cdots|z_n|^{2d_n}$ for nonnegative integers $d_1,\ldots,d_n$. Hence, it suffices to prove the lemma for φ being such a monomial. By Lemma 2.1, we have

$$\begin{aligned}\omega(\varphi,m) &= \frac{1}{m!}\int_{[0,\infty)^n} t_1^{d_1+m_1}\cdots t_n^{d_n+m_n}e^{-(t_1+\cdots+t_n)}dt_1\cdots dt_n\\ &= \frac{(m_1+d_1)!\cdots(m_n+d_n)!}{m_1!\cdots m_n!}\\ &= [(m_1+d_1)\cdots(m_1+1)]\cdots[(m_n+d_n)\cdots(m_n+1)].\end{aligned}$$

This shows that $\omega(\varphi,m)$ is a polynomial in m. The term with highest degree is $m_1^{d_1}\cdots m_n^{d_n}$. □

Example 2.3. If $\varphi(z)=c_0+c_1|z_1|^2+\cdots+c_n|z_n|^2$, where $c_0,c_1,\ldots,c_n$ are complex numbers, then

$$\begin{aligned}\omega(\varphi,m) &= c_0+c_1(m_1+1)+\cdots+c_n(m_n+1)\\ &= c_1m_1+\cdots+c_nm_n+(c_0+\cdots+c_n).\end{aligned}$$

Remark 2.4. Example 2.3 shows that if $q(m)$ is a linear polynomial in m, then there exists a unique function φ of the form $\varphi(z)=c_0+c_1|z_1|^2+\cdots+c_n|z_n|^2$ such that $\omega(\varphi,m)=q(m)$ for all $m\in\mathbb{N}_0^n$. Furthermore, if $q(-1,\ldots,-1)=0$, then $c_0=0$.

Now given a separately radial polynomial $\varphi\in\mathbb{C}[z,\bar{z}]$, we would like to identify all polynomially bounded functions ψ such that T_φ and T_ψ commute on analytic polynomials, that is, $T_\varphi T_\psi h=T_\psi T_\varphi h$ for all analytic polynomials h. This is equivalent to

$$\langle T_\varphi T_\psi e_m, e_k\rangle = \langle T_\psi T_\varphi e_m, e_k\rangle, \tag{2.1}$$

for all multi-indices m,k in $\mathbb{N}_0^n$.

Since $T_{\overline{\varphi}}e_k=\omega(\overline{\varphi},k)\,e_k=\overline{\omega(\varphi,k)}\,e_k$ by Lemma 2.1, the left hand side becomes

$$\langle T_\varphi T_\psi e_m, e_k\rangle = \langle T_\psi e_m, T_{\overline{\varphi}}e_k\rangle = \omega(\varphi,k)\langle T_\psi e_m, e_k\rangle.$$

Combining with the identity $T_\varphi e_m=\omega(\varphi,m)e_m$, we rewrite (2.1) as

$$\begin{aligned}&\big(\omega(\varphi,k)-\omega(\varphi,m)\big)\langle T_\psi e_m,e_k\rangle=0\\ \Longleftrightarrow\qquad &\big(\omega(\varphi,k)-\omega(\varphi,m)\big)\int_{\mathbb{C}^n}\psi(z)z^m\overline{z}^k\,d\mu(z)=0. \qquad (2.2)\end{aligned}$$

We have shown that T_ψ commutes with T_φ on analytic polynomials if and only if (2.2) holds for all multi-indices m,k in $\mathbb{N}_0^n$. This shows that our commuting problem reduces to the study of functions ψ that satisfies (2.2). As we shall see in the examples below, the following result plays an important role in our analysis.

Definition 2.5. Let $l = (l_1, \dots, l_n)$ be an n-tuple of integers. A function ψ is said to be *l-invariant* if $\psi(\gamma^{l_1} z_1, \dots, \gamma^{l_n} z_n) = \psi(z)$ for a.e. $z \in \mathbb{C}^n$ and a.e. $\gamma \in \mathbb{T}$.

Lemma 2.6 (Lemma 3.2 in [4])**.** *Let $l \in \mathbb{Z}^n$ be given. Suppose ψ is a polynomially bounded function on $\mathbb{C}^n$. Then the following statements are equivalent.*

(a) ψ is l-invariant.

(b) $\int_{\mathbb{C}^n} \psi(w) w^m \overline{w}^k \, d\mu(w) = 0$ for all multi-indices $m, k \in \mathbb{N}_0^n$ satisfying the condition $(k - m) \cdot l = (k_1 - m_1) l_1 + \cdots + (k_n - m_n) l_n \neq 0$.

Let us now consider two examples. In each example, a separately radial polynomial φ is given and we need to identify all polynomially bounded functions ψ such that T_ψ commutes with T_φ. While the first example can be analyzed with the help of Equation (2.2) and Lemma 2.6, the second example is more subtle and can only be fully explained after we present our main result.

Example 2.7. Let $\varphi_1(z) = |z_1|^2 + 2|z_2|^2$ on $\mathbb{C}^2$. Characterize all polynomially bounded functions ψ such that T_{φ_1} and T_ψ commute.

The calculation in Example 2.3 gives $\omega(\varphi_1, m) = m_1 + 2m_2 + 3$. Now equation (2.2) shows that T_ψ commutes with T_{φ_1} if and only if

$$\Big(k_1 + 2k_2 - m_1 - 2m_2\Big) \int_{\mathbb{C}^2} \psi(z) z^m \overline{z}^k \, d\mu(z) = 0, \tag{2.3}$$

for all multi-indicies $m = (m_1, m_2)$ and $k = (k_1, k_2)$ in $\mathbb{N}_0^2$.

The necessary and sufficient condition for (2.3) to hold for all m, k is that whenever $(k - m) \cdot (1, 2) = k_1 + 2k_2 - m_1 - 2m_2 \neq 0$, we have $\int_{\mathbb{C}^2} \psi(z) z^m \overline{z}^k \, d\mu(z) = 0$. By Lemma 2.6, this is equivalent to ψ being $(1, 2)$-invariant, that is,

$$\psi(\gamma z_1, \gamma^2 z_2) = \psi(z_1, z_2)$$

for a.e. $z \in \mathbb{C}^2$ and a.e. $\gamma \in \mathbb{T}$. Separately radial functions as well as the function $z_1^2 \overline{z}_2$ are examples of such ψ.

Example 2.8. Let $\varphi_2(z) = |z_1|^2 |z_2|^2$ on $\mathbb{C}^2$. By Lemma 2.1, we have

$$\omega(\varphi_2, m) = (m_1 + 1)(m_2 + 1).$$

Therefore, T_ψ commutes with T_{φ_2} if and only if for all $m = (m_1, m_2)$ and $k = (k_1, k_2)$ in $\mathbb{N}_0^2$, we have

$$\Big((k_1 + 1)(k_2 + 1) - (m_1 + 1)(m_2 + 1)\Big) \int_{\mathbb{C}^2} \psi(z) z^m \overline{z}^k \, d\mu(z) = 0. \tag{2.4}$$

The set of all m, k for which the first factor vanishes is more complicated than that in Example 2.7. It includes the cases $(m_1, m_2) = (k_1, k_2)$ and $(m_1, m_2) = (k_2, k_1)$. A sufficient condition for (2.4) to hold is that ψ be separately radial. It turns out, even though not obvious, that this condition is also necessary. We shall provide a complete argument after discussing the general result.

In analyzing Equation (2.2), we need another result which shows that for a special class of analytic functions on the half space $\mathbb{K}^n \subset \mathbb{C}^n$, the values of the functions on the lattice $\mathbb{N}^n$ uniquely determine the functions. Here, $\mathbb{K}$ denotes the right half-plane in $\mathbb{C}$.

We recall a definition from [4]. For a polynomially bounded function f on $\mathbb{C}^n$, we define

$$\mathcal{M}f(\zeta) = \int_{\mathbb{C}^n} f(z)|z_1|^{2\zeta_1} \cdots |z_n|^{2\zeta_n}\, d\nu(z),$$

for any $\zeta = (\zeta_1, \ldots, \zeta_n) \in \overline{\mathbb{K}}^n$. It can be checked that $\mathcal{M}f$ is analytic on $\mathbb{K}^n$ and continuous on $\overline{\mathbb{K}}^n$. The following result is a special case of [4, Proposition 3.1].

Lemma 2.9. *Let f be a polynomially bounded function and p be an analytic polynomial on $\mathbb{C}^n$. Let $G(\zeta) = p(\zeta) \cdot (\mathcal{M}f)(\zeta)$ for $\zeta \in \overline{\mathbb{K}}^n$. If $G(k) = 0$ for all $k \in \mathbb{N}^n$, then $G(\zeta) = 0$ for all $\zeta \in \overline{\mathbb{K}}^n$.*

As a consequence of this result, we show that a much stronger version of Equation (2.2) holds.

Proposition 2.10. *Let $\varphi \in \mathbb{C}[z, \bar{z}]$ be separately radial and ψ be polynomially bounded on $\mathbb{C}^n$ such that T_φ and T_ψ commute on analytic polynomials. Then for any $m, k \in \mathbb{N}_0^n$,*

$$\Big(\omega(\varphi, m + \zeta) - \omega(\varphi, k + \zeta)\Big) \int_{\mathbb{C}^n} \psi(z) z^m \bar{z}^k |z^\zeta|^2\, d\mu(z) = 0 \text{ for all } \zeta \in \overline{\mathbb{K}}^n.$$

Consequently, if $\omega(\varphi, m+\zeta) - \omega(\varphi, k+\zeta)$ is not identically zero for $\zeta \in \overline{\mathbb{K}}^n$, then

$$\int_{\mathbb{C}^n} \psi(z) z^m \bar{z}^k\, d\mu(z) = 0.$$

Proof. Since T_ψ commutes with T_φ, Equation (2.2) holds for all multi-indices $m, k \in \mathbb{N}_0^n$. Using the substitution $m \mapsto m + \zeta$ and $k \mapsto k + \zeta$ with $\zeta \in \mathbb{N}_0^n$, we obtain

$$\Big(\omega(\varphi, k + \zeta) - \omega(\varphi, m + \zeta)\Big) \cdot \int_{\mathbb{C}^n} \psi(z) z^m \bar{z}^k |z^\zeta|^2 d\mu(z) = 0.$$

By Lemma 2.9, the above identity also holds for all $\zeta \in \overline{\mathbb{K}}^n$. Since both factors are analytic in ζ, it follows that one of them must be identically zero on $\overline{\mathbb{K}}^n$. As a result, if $\omega(\varphi, m+\zeta) - \omega(\varphi, k+\zeta)$ is not identically zero, then

$$\int_{\mathbb{C}^n} \psi(z) z^m \bar{z}^k |z^\zeta|^2\, d\mu(z) = 0$$

for all $\zeta \in \overline{\mathbb{K}}^n$. Setting $\zeta = 0$, we obtain the conclusion of the proposition. □

Proposition 2.10 suggests that we consider multi-indices m and k for which the polynomials $\omega(\varphi, m+\zeta)$ and $\omega(\varphi, k+\zeta)$ are identically equal for $\zeta \in \overline{\mathbb{K}}^n$. This leads to a definition for general analytic polynomials in several complex variables.

We shall let $\mathbb{C}[\zeta]$ denote the space of all analytic polynomials in $\zeta = (\zeta_1, \ldots, \zeta_n)$. Let q be a polynomial in $\mathbb{C}[\zeta]$. Define

$$\mathrm{Per}(q) = \Big\{a \in \mathbb{C}^n : q(a+\zeta) = q(\zeta) \text{ for all } \zeta \in \mathbb{C}^n\Big\}.$$

Then 0 is always an element of $\mathrm{Per}(q)$ and any $a \neq 0$ belonging to $\mathrm{Per}(q)$ is a period of q. Since q is a polynomial, it follows that if $q(a+\zeta) = q(\zeta)$ for all ζ in a half-plane of the form $u + \mathbb{K}^n$ for some $u \in \mathbb{C}^n$, then a belongs to $\mathrm{Per}(q)$. We also define $\mathrm{Per}_{\mathbb{Z}}(q) = \mathrm{Per}(q) \cap \mathbb{Z}^n$.

Remark 2.11. With the above notation, the conclusion of Proposition 2.10 can be restated as: for any $m, k \in \mathbb{N}_0^n$, if $m - k$ does not belong to $\mathrm{Per}_{\mathbb{Z}}(\omega(\varphi, \zeta))$, then

$$\int_{\mathbb{C}^n} \psi(z) z^m \bar{z}^k \, d\mu(z) = 0.$$

In the following result, we establish a linear structure of $\mathrm{Per}(q)$ and $\mathrm{Per}_{\mathbb{Z}}(q)$. This structure will play an important roll in our main result.

Proposition 2.12. *Let q be a polynomial in $\mathbb{C}[\zeta]$. Then the following statements hold.*

(a) *The set $\mathrm{Per}(q)$ is a vector subspace of $\mathbb{C}^n$.*
(b) *There exist d vectors $u_1, \ldots, u_d \in \mathbb{Z}^n$ $(1 \leq d \leq n)$ such that*

$$\mathrm{Per}_{\mathbb{Z}}(q) = \big\{u_1, \ldots, u_d\big\}^{\perp} = \Big\{u \in \mathbb{Z}^n : u \cdot u_j = 0 \textit{ for all } j = 1, \ldots, d\Big\}.$$

Proof. The proof of (a) is straightforward and we leave it for the interested reader. We only sketch a proof of (b). Let $\mathrm{Per}_{\mathbb{Q}}(q) = \mathrm{Per}(q) \cap \mathbb{Q}^n$. It follows from (a) that $\mathrm{Per}_{\mathbb{Q}}(q)$ is a linear subspace of $\mathbb{Q}^n$. Now consider the standard Euclidean inner product defined on $\mathbb{Q}^n$ and take $\{u_1, \ldots, u_d\}$ to be a basis of the orthogonal complement of $\mathrm{Per}_{\mathbb{Q}}(q)$. Multiplying each vector by a sufficiently large integer if necessary, we may assume that $u_1, \ldots, u_d$ belong to $\mathbb{Z}^n$. Since $\mathrm{Per}_{\mathbb{Z}}(q) = \mathrm{Per}_{\mathbb{Q}}(q) \cap \mathbb{Z}^n$, the conclusion of the lemma follows. □

We conclude this section with some examples.

Example 2.13. Let $q_1(\zeta) = \zeta_1 + 2\zeta_2 + 3$ for $\zeta = (\zeta_1, \zeta_2) \in \mathbb{C}^2$. Recall from Example 2.7 that $q_1(\zeta) = \omega(\varphi_1, \zeta)$, where $\varphi_1(z) = |z_1|^2 + 2|z_2|^2$. We have

$$\begin{aligned}\mathrm{Per}_{\mathbb{Z}}(q_1) &= \big\{(a_1, a_2) \in \mathbb{Z}^2 : q_1(a_1+\zeta_1, a_2+\zeta_2) = q_1(\zeta_1, \zeta_2) \text{ for all } \zeta \in \mathbb{C}^2\big\}\\ &= \big\{(a_1, a_2) \in \mathbb{Z}^2 : a_1 + 2a_2 = 0\big\}\\ &= \big\{(1, 2)\big\}^{\perp}.\end{aligned}$$

Example 2.14. Let $q_2(\zeta) = (\zeta_1 + 1)(\zeta_2 + 1)$. This polynomial appeared in Example 2.8 in which $q_2(\zeta) = \omega(\varphi_2, \zeta)$, where $\varphi_2(z) = |z_1|^2|z_2|^2$. We have

$$\begin{aligned}\mathrm{Per}_{\mathbb{Z}}(q_2) &= \big\{(a_1, a_2) \in \mathbb{Z}^2 : \ q_2(a + \zeta) = q_2(\zeta) \text{ for all } \zeta \in \mathbb{C}^2\big\}\\ &= \big\{(a_1, a_2) \in \mathbb{Z}^2 : \ (\zeta_1 + 1)a_2 + (\zeta_2 + 1)a_1 + a_1a_2 = 0 \text{ for all } \zeta \in \mathbb{C}^2\big\}\\ &= \{(0, 0)\}\\ &= \{(1, 0), (0, 1)\}^{\perp}.\end{aligned}$$

Example 2.15. Let $\varphi_3(z) = |z_1|^2 + \sqrt{3}|z_2|^2$ on $\mathbb{C}^2$. Lemma 2.1 gives

$$q_3(\zeta) = \omega(\varphi_3, \zeta) = \zeta_1 + \sqrt{3}\,\zeta_2 + (1 + \sqrt{3}) \quad \text{ for } \zeta \in \mathbb{C}^2.$$

It then follows that

$$\begin{aligned}\mathrm{Per}_{\mathbb{Z}}(q_3) &= \big\{(a_1, a_2) \in \mathbb{Z}^2 : \ (a_1 + \zeta_1) + \sqrt{3}(a_2 + \zeta_2) = \zeta_1 + \sqrt{3}\,\zeta_2,\\ &\qquad\qquad \text{for all } \zeta \in \mathbb{C}^2\big\}\\ &= \big\{(a_1, a_2) \in \mathbb{Z}^2 : a_1 + \sqrt{3}\,a_2 = 0\big\}\\ &= \{(0, 0)\}\\ &= \{(1, 0), (0, 1)\}^{\perp}.\end{aligned}$$

3. Main results and examples

We are now in a position to state and prove our main result. We shall offer several examples and interesting consequences.

Theorem 3.1. *Let $\varphi \in \mathbb{C}[z, \bar{z}]$ be a separately radial polynomial. Then there are vectors $u_1, \ldots, u_d$ ($1 \le d \le n$) in $\mathbb{Z}^n$ such that for any polynomially bounded function ψ, the following statements are equivalent.*
(a) ψ is u_j-invariant for all $j = 1, \ldots, d$.
(b) $T_\psi T_\varphi = T_\varphi T_\psi$ on analytic polynomials.
Furthermore, the vectors $u_1, \ldots, u_d$ are determined by the relation

$$\{u_1, \ldots, u_d\}^{\perp} = \mathrm{Per}_{\mathbb{Z}}(\omega(\varphi, \zeta)). \tag{3.1}$$

Proof. Lemma 2.2 shows that the function $\omega(\varphi, m)$ is a polynomial in m. By Proposition 2.12, we may choose vectors $u_1, \ldots, u_d$ that satisfies (3.1). By the definition of $\mathrm{Per}_{\mathbb{Z}}(\omega(\varphi, \zeta))$, for each $u \in \mathbb{Z}^n$, the identity $\omega(\varphi, u + \zeta) = \omega(\varphi, \zeta)$ holds for all $\zeta \in \mathbb{C}^n$ if and only if $u \perp \{u_1, \ldots, u_d\}$.

We first prove the implication $(a) \Longrightarrow (b)$. Assume that ψ is u_j-invariant for all $j = 1, \ldots, d$. Let m and k be in $\mathbb{N}_0^n$. If $(m - k) \cdot u_j \neq 0$ for some j, then since ψ is u_j-invariant, Lemma 2.6 shows that

$$\int_{\mathbb{C}^n} \psi(z) z^m \bar{z}^k \, d\mu(z) = 0.$$

If $(m - k) \cdot u_j = 0$ for all $1 \le j \le d$, then $m - k$ belongs to $\mathrm{Per}_{\mathbb{Z}}(\omega(\varphi, \zeta))$. This implies

$$\omega(\varphi, k) = \omega(\varphi, (k - m) + m) = \omega(\varphi, m).$$

In either case, we have

$$\Big(\omega(\varphi,k)-\omega(\varphi,m)\Big)\int_{\mathbb{C}^n}\psi(z)z^m\overline{z}^k\,d\mu(z)=0. \tag{3.2}$$

Consequently, T_φ and T_ψ commute on analytic polynomials.

We now prove $(b)\Longrightarrow(a)$. Assume that T_ψ commutes with T_φ on analytic polynomials. Proposition 2.10 and Remark 2.11 show that for any m,k in $\mathbb{N}_0^n$,

$$\int_{\mathbb{C}^n}\psi(z)z^m\overline{z}^k d\mu(z)=0 \tag{3.3}$$

whenever $m-k$ does not belong to $\mathrm{Per}_{\mathbb{Z}}(\omega(\varphi,\zeta))=\{u_1,\ldots,u_d\}^{\perp}$. Let j be any integer between 1 and d. Suppose m and k are multi-indices such that $(m-k)\cdot u_j\neq 0$. Then equation (3.3) holds because $m-k$ does not belong to $\mathrm{Per}_{\mathbb{Z}}(\omega(\varphi,\zeta))$. Lemma 2.6 now shows that ψ is u_j-invariant. Since j was arbitrary, statement (a) holds. This completes the proof of the theorem. □

Using Theorem 3.1, we re-examine the examples given in Section 2.

Example 3.2. Let $\varphi_1(z)=|z_1|^2+2|z_2|^2$ on $\mathbb{C}^2$ as in Example 2.7. We showed in Example 2.13 that $\mathrm{Per}_{\mathbb{Z}}(\omega(\varphi_1,\zeta))=\{(1,2)\}^{\perp}$. Let ψ be any polynomially bounded function. It follows from Theorem 3.1 that T_ψ commutes with T_φ if and only if ψ is $(1,2)$-invariant. This is consistent with the conclusion obtained in Example 2.7.

Example 3.3. Let $\varphi_2(z)=|z_1|^2|z_2|^2$ on $\mathbb{C}^2$ as in Example 2.8. We showed in Example 2.14 that $\mathrm{Per}_{\mathbb{Z}}(\omega(\varphi_2,\zeta))=\{(1,0),(0,1)\}^{\perp}$. Let ψ be any polynomially bounded function on $\mathbb{C}^2$. It follows from Theorem 3.1 that T_ψ commutes with T_φ if and only if ψ is both $(1,0)$-invariant and $(0,1)$-invariant. Any such function must be separately radial.

Example 3.4. Let $\varphi_3(z)=|z_1|^2+\sqrt{3}|z_2|^2$ on $\mathbb{C}^2$ as in Example 2.15. We have seen that $\mathrm{Per}_{\mathbb{Z}}(\omega(\varphi_3,\zeta))=\{(1,0),(0,1)\}^{\perp}$. The exact same argument as in the previous example shows that T_ψ commutes with T_φ if and only if ψ is separately radial.

Example 3.5. Let $\varphi_4(z)=|z_1|^2-|z_2|^2$ on $\mathbb{C}^2$. Example 2.3 shows

$$q_4(m)=\omega(\varphi_4,m)=m_1-m_2 \text{ for all } m=(m_1,m_2)\in\mathbb{N}_0^2.$$

This implies that

$$\begin{aligned}\mathrm{Per}_{\mathbb{Z}}(q_4)&=\big\{a\in\mathbb{Z}^2: q_4(a+\zeta)=q_4(\zeta)\text{ for all }\zeta\in\mathbb{C}^2\big\}\\&=\big\{a\in\mathbb{Z}^2: a_1-a_2=0\big\}\\&=\{(1,-1)\}^{\perp}.\end{aligned}$$

By Theorem 3.1, for any polynomially bounded function ψ on $\mathbb{C}^2$, the operator T_ψ commutes with T_{φ_4} if and only if ψ is $(1,-1)$-invariant, which means $\psi(\gamma z_1,\bar{\gamma}z_2)=\psi(z_1,z_2)$ for a.e. $\gamma\in\mathbb{T}$ and a.e. $z\in\mathbb{C}^2$. Examples of these functions include separately radial functions as well as analytic functions such as powers of z_1z_2.

Remark 3.6. Example 3.5 illustrates a difference between Toeplitz operators on $\mathcal{F}_n^2$ and on the Bergman space of the unit ball. In fact, a result in [11] shows that on the Bergman space of the unit ball in $\mathbb{C}^2$, T_g commutes with $T_{z_1 z_2}$ if and only if g is analytic.

We now describe several interesting consequences of Theorem 3.1.

Theorem 3.7. *Let φ be a separately radial polynomial. Suppose that ψ_1, ψ_2 are polynomially bounded functions such that both T_{ψ_1} and T_{ψ_2} commute with T_φ. Then $T_{\psi_1\psi_2}$ commutes with T_φ as well. Consequently, the set*

$$\mathcal{X}(\varphi) = \{\psi : \mathbb{C}^n \to \mathbb{C} \text{ is polynomially bounded such that } T_\psi T_\varphi = T_\varphi T_\psi\} \tag{3.4}$$

is an algebra.

Remark 3.8. It is clear that the product $T_{\psi_1} T_{\psi_2}$ commutes with T_φ. On the other hand, since $T_{\psi_1\psi_2}$ is not the same as $T_{\psi_1} T_{\psi_2}$ in general, the conclusion of the theorem is quite nontrivial.

Proof. Assume $T_\varphi T_{\psi_1} = T_{\psi_1} T_\varphi$ and $T_\varphi T_{\psi_2} = T_{\psi_2} T_\varphi$. Then by Theorem 3.1 there exist vectors $u_1, \ldots, u_d \in \mathbb{Z}^n$ such that both ψ_1 and ψ_2 are u_j-invariant for all $1 \le j \le d$. This implies that the product $\psi_1\psi_2$ is also u_j-invariant for all such j. By Theorem 3.1 again, $T_{\psi_1\psi_2}$ commutes with T_φ. This shows that $\mathcal{X}(\varphi)$ is closed under multiplication. On the other hand, it is clear that $\mathcal{X}(\varphi)$ is closed under addition and scalar multiplication. Therefore, $\mathcal{X}(\varphi)$ is an algebra. □

Examples 3.3 and 3.4 illustrate an interesting fact. Even though the polynomials φ_2 and φ_3 are different, the identity $\mathcal{X}(\varphi_2) = \mathcal{X}(\varphi_3)$ holds. It turns out that such phenomenon happens in a more general setting.

Lemma 3.9. *Let $u_1, \ldots, u_d$ be in $\mathbb{Z}^n$. Then there is a linear polynomial q such that $\mathrm{Per}_{\mathbb{Z}}(q) = \{u_1, \ldots, u_d\}^\perp$. As a consequence, there is a separately radial polynomial $\varphi(z) = c_1|z_1|^2 + \cdots + c_n|z_n|^2$ on $\mathbb{C}^n$ so that*

$$\mathrm{Per}_{\mathbb{Z}}(\omega(\varphi, \zeta)) = \{u_1, \ldots, u_d\}^\perp.$$

Proof. Choose d real numbers that are linearly independent over $\mathbb{Q}$, for example, $\sqrt{p_1}, \ldots, \sqrt{p_d}$, where $p_1, \ldots, p_d$ are distinct primes. Define

$$q(\zeta) = \sqrt{p_1}(u_1 \cdot \zeta) + \cdots + \sqrt{p_d}(u_d \cdot \zeta).$$

Then we have

$$\begin{aligned}
\mathrm{Per}_{\mathbb{Z}}(q) &= \big\{a \in \mathbb{Z}^n : q(a + \zeta) = q(\zeta) \quad \text{for all } \zeta \in \mathbb{C}^n\big\} \\
&= \big\{a \in \mathbb{Z}^n : \sqrt{p_1}(u_1 \cdot a) + \cdots + \sqrt{p_d}(u_d \cdot a) = 0\big\} \\
&= \big\{a \in \mathbb{Z}^n : u_1 \cdot a = \cdots = u_d \cdot a = 0\big\} \\
&= \big\{u_1, \ldots, u_d\big\}^\perp.
\end{aligned}$$

The existence of the polynomial φ follows from Remark 2.4. □

Theorem 3.10. *For any separately radial polynomial $\varphi \in \mathbb{C}[z, \bar{z}]$, there exists a separately radial polynomial $\tilde{\varphi}$ of the form*

$$\tilde{\varphi}(z) = \alpha_1|z_1|^2 + \cdots + \alpha_n|z_n|^2$$

such that $\mathcal{X}(\tilde{\varphi}) = \mathcal{X}(\varphi)$.

Proof. Write $\mathrm{Per}_{\mathbb{Z}}\big(\omega(\varphi,\zeta)\big) = \{u_1,\dots,u_d\}^{\perp}$. By Lemma 3.9, there exists $\tilde{\varphi}$ in the required form such that

$$\mathrm{Per}_{\mathbb{Z}}(\omega(\tilde{\varphi},\zeta)) = \{u_1,\dots,u_d\}^{\perp} = \mathrm{Per}_{\mathbb{Z}}(\omega(\varphi,\zeta)).$$

Theorem 3.1 then completes the proof. □

4. Concluding remarks

Throughout the paper, we have restricted our attention to symbols φ being separately radial polynomials. For these symbols, the Toeplitz operator T_φ is diagonal with respect to the standard orthonormal basis. The eigenvalue $\omega(\varphi, m)$ is a polynomial in m. Our description of Toeplitz operators commuting with T_φ relies on the space of periods of this polynomial. Our result raises the question: what is the situation when φ is not a polynomial (but still is separately radial)? In this case, the eigenvalue $\omega(\varphi, m)$ may not be a polynomial in m. It is still possible to consider the function $\omega(\varphi,\zeta)$ with ζ in the right half-space but the periods of this function may not form a vector space over $\mathbb{Q}$, even though they are still closed under addition. Consequently, $\mathrm{Per}_{\mathbb{Z}}(\omega(\varphi,\zeta))$ may not have a convenient description as in Proposition 2.12 (b). Due to this obstacle, difficulties arise in the proof of Theorem 3.1 and we have not been able to resolve. We leave this problem for future investigation.

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