COMMUTANTS OF SEPARATELY RADIAL TOEPLITZ OPERATORS IN SEVERAL VARIABLES

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Abstract. If φ is a bounded separately radial function on the unit ball, the Toeplitz operator T_φ is diagonalizable with respect to the standard orthogonal basis of monomials on the Bergman space. Given such a φ, we characterize bounded functions ψ for which T_ψ commutes with T_φ. Several examples are given to illustrate our results.

1. Introduction

Let $d \geq 1$ be a fixed integer. For $\mathbf{z} = (z_1, \ldots, z_d) \in \mathbb{C}^d$, we denote its Euclidean norm by $|\mathbf{z}| = \sqrt{|z_1|^2 + \cdots + |z_d|^2}$. We write $\mathbb{B}$ for the open unit ball consisting of all $\mathbf{z} \in \mathbb{C}^d$ with $|\mathbf{z}| < 1$. Let ν denote the Lebesgue measure on $\mathbb{B}$ normalized so that $\nu(\mathbb{B}) = 1$. The Bergman space A^2 consists of all holomorphic functions on $\mathbb{B}$ which are square integrable with respect to ν. Since A^2 is a closed subspace of the Hilbert space $L^2 = L^2(\mathbb{B}, \nu)$, there is an orthogonal projection P from L^2 onto A^2. For any bounded measurable function φ on the ball, the Toeplitz operator T_φ is defined by $T_\varphi h = P(\varphi h)$ for $h \in A^2$. It is immediate that T_φ is a bounded linear operator on A^2 with $\|T_\varphi\| \leq \|\varphi\|_\infty$. If φ is holomorphic on $\mathbb{B}$, then T_φ is the multiplication operator on A^2 with symbol φ.

The main goal of this paper is to study the *commuting problem* of Toeplitz operators on A^2: given a non-constant function φ, find the necessary and sufficient conditions on the function ψ such that $T_\varphi T_\psi = T_\psi T_\varphi$. The commuting problem for Toeplitz operators on the Hardy space of the unit disk was solved completely by Brown and Halmos in their seminal paper [8] back in the early sixties. Their result has motivated a vast literature on the studies of commuting Toeplitz operators acting on other Hilbert spaces of analytic functions: the Bergman space over the unit disk [2, 3, 10], the Hardy and Bergman spaces over the polydisk or the ball in higher dimensions [9, 11, 16, 17, 22] and the Fock spaces [6, 5, 1], just to list a few. The interested reader is referred to the above papers for more references. Quite often, an additional assumption on the function φ is imposed. In fact, even on the Bergman space over the unit disk, the general commuting problem remains open.

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The motivation of this paper comes from the following results. Recall that a function φ on $\mathbb{B}$ is called *radial* if there exists a function a on $[0,1)$ such that $\varphi(\mathbf{z}) = a(|\mathbf{z}|)$ for a.e. $\mathbf{z} \in \mathbb{B}$. Čučković and Rao [10] showed that if φ is a non-constant radial function on the unit disk and T_ψ commutes with T_φ on the Bergman space, then ψ must be a radial function as well. This result was later generalized to higher dimensions. In [16], among other things, we showed that for φ a bounded radial function on the unit ball, the operators T_φ and T_ψ commute on the Bergman space if and only if ψ is invariant under the natural diagonal action of the unit circle on $\mathbb{B}$. That is, for each complex number λ on the unit circle, we have $\psi(\lambda\mathbf{z}) = \psi(\mathbf{z})$ for a.e. $\mathbf{z} \in \mathbb{B}$. In this paper we consider a wider class of symbols. We shall assume that φ is *separately radial*, that is, $\varphi(\mathbf{z}) = a(|z_1|, \ldots, |z_d|)$ a.e., for some bounded measurable function a on $\Delta = \{(t_1, \ldots, t_d) \in \mathbb{R}_+^d : t_1^2 + \cdots + t_d^2 < 1\}$. We investigate necessary and sufficient conditions on ψ for which T_ψ commutes with T_φ. We obtain a complete characterization of such ψ when φ is assumed to be a separately radial polynomial. It turns out that the characterization depends heavily on the behavior of the function φ. We shall see several examples that illustrate this dependency. We state here our main results.

Theorem A. *Let φ be a separately radial polynomial in $\mathbf{z}$ and $\bar{\mathbf{z}}$. There then exist tuples of integers $\mathbf{u}_1, \ldots, \mathbf{u}_k$ with $\mathbf{u}_1 = (1, \ldots, 1)$ such that for any bounded functions ψ, the following two statements are equivalent.*

(a) T_ψ commutes with T_φ on the Bergman space A^2.
(b) For all $1 \le j \le k$ and all complex numbers λ with absolute value one, we have

$$\psi(\lambda^{u_{j,1}} z_1, \ldots, \lambda^{u_{j,d}} z_d) = \psi(\mathbf{z}) \ \textit{for a.e.}\ \mathbf{z} \in \mathbb{B},$$

where $\mathbf{u}_j = (u_{j,1}, \ldots, u_{j,d})$.

A converse of Theorem A also holds, as in the following result.

Theorem B. *Let $\mathbf{u}_1 = (1, \ldots, 1)$ and $\mathbf{u}_2, \ldots, \mathbf{u}_k$ be given tuples of integers. Then there exists a separately radial polynomial φ in $\mathbf{z}$ and $\bar{\mathbf{z}}$ such that for any bounded functions ψ, the two statements in Theorem A are equivalent.*

The paper is organized as follows. In Section 2 we investigate the eigenvalues of Toeplitz operators with separately radial symbols. Explicit formulas are given in the case of polynomials. A preliminary study of the commuting problem is considered. In Section 3, we obtain a characterization of Toeplitz operators commuting with a given separately radial Toeplitz operator. The characterization makes use of the notation of periods of functions holomorphic on the right-half space. Section 4 is specially devoted to the case of polynomial symbols. We offer proofs of the main results in this section. Various examples are discussed throughout the paper.

2. Toeplitz operators with separately radial symbols

We first recall several facts about the Bergman space A^2. Denote by $\mathbb{Z}_+$ the set of all non-negative integers. For $\mathbf{z} \in \mathbb{C}^d$ and $\mathbf{m} \in \mathbb{Z}_+^d$, we use the standard multiindex notation:

$$\mathbf{z}^{\mathbf{m}} = z_1^{m_1} \cdots z_d^{m_d}, \;\; \mathbf{m}! = m_1! \cdots m_d!, \;\; \text{and } |\mathbf{m}| = m_1 + \cdots + m_d$$

with the convention that $0^0 = 1$. We write $\mathbb{1}$ to denote the tuple $(1, \ldots, 1)$. Throughout the paper, we use boldface letters $\mathbf{z}, \mathbf{w}$, etc. to denote variables in $\mathbb{C}^d$ and $\mathbf{m}, \mathbf{k}$, etc. to denote multiindices in $\mathbb{Z}_+^d$. On the other hand, z, w, etc. (respectively, m, k, etc.) denote single complex variables (respectively, integers).

The inner product and norm on A^2 is given by

$$\langle f, g\rangle = \int_{\mathbb{B}} f(\mathbf{z})\, \overline{g(\mathbf{z})}\, d\nu(\mathbf{z});$$

and

$$\|f\|^2 = \int_{\mathbb{B}} |f(\mathbf{z})|^2\, d\nu(\mathbf{z}).$$

It is well known (see [21, Propositions 1.4.8 and 1.4.9] and [23, Lemma 1.11]) that monomials of different multi-degrees are orthogonal on the Bergman space and

$$\|\mathbf{z}^{\mathbf{m}}\|^2 = \frac{\mathbf{m}!\, d!}{(|\mathbf{m}| + d)!}.$$

It follows that the collection

$$\left\{ e_{\mathbf{m}}(z) = \sqrt{\frac{(|\mathbf{m}| + d)!}{\mathbf{m}!\, d!}}\, \mathbf{z}^{\mathbf{m}} : \mathbf{m} \in \mathbb{Z}_+^d \right\}$$

form an orthonormal basis for A^2. The reader is referred to [23, Chapter 2] for more details.

A function $\varphi : \mathbb{B} \to \mathbb{C}$ is called *separately radial* if

$$\varphi(z_1, \ldots, z_d) = a(|z_1|, \ldots, |z_d|)$$

for some $a : \Delta \to \mathbb{C}$, where $\Delta = \{(t_1, \ldots, t_d) \in \mathbb{R}_+^d : t_1^2 + \cdots + t_d^2 < 1\}$. Equivalently, φ is invariant under the natural action of the torus $\mathbb{T}^d$ on $\mathbb{B}$.

2.1. Eigenvalues.

We begin with the following result that records the well-known fact that Toeplitz operators with separately radial symbols are diagonalizable with respect to the standard orthonormal basis of monomials.

Proposition 2.1. *Let φ be a bounded separately radial function on $\mathbb{B}$. Then $T_\varphi e_{\mathbf{m}} = \widehat{\varphi}(\mathbf{m})\, e_{\mathbf{m}}$ for all $\mathbf{m} \in \mathbb{Z}_+^d$. The eigenvalue $\widehat{\varphi}(\mathbf{m})$ is given by*

$$\widehat{\varphi}(\mathbf{m}) = \frac{\int_{\mathbb{B}} \varphi(\mathbf{z})\, |\mathbf{z}^{\mathbf{m}}|^2\, d\nu(\mathbf{z})}{\int_{\mathbb{B}} |\mathbf{z}^{\mathbf{m}}|^2\, d\nu(\mathbf{z})} = \frac{(|\mathbf{m}| + d)!}{\mathbf{m}!\, d!} \int_{\mathbb{B}} \varphi(\mathbf{z})\, |\mathbf{z}^{\mathbf{m}}|^2\, d\nu(\mathbf{z}).$$

Remark 2.2. Let $\mathbb{K}$ denote the open right-half plane consisting of all complex numbers whose real parts are positive and $\overline{\mathbb{K}}$ be the close right-half plane (the closure of $\mathbb{K}$). The formula in Proposition 2.1 shows that $\widehat{\varphi}$ extends to a function defined on $\mathbb{K}^d$ as

$$\widehat{\varphi}(\boldsymbol{\zeta}) = \frac{\int_{\mathbb{B}} \varphi(\mathbf{z})\,|z_1|^{2\zeta_1}\cdots|z_d|^{2\zeta_d}\,d\nu(\mathbf{z})}{\int_{\mathbb{B}} |z_1|^{2\zeta_1}\cdots|z_d|^{2\zeta_d}\,d\nu(\mathbf{z})}, \tag{2.1}$$

for $\boldsymbol{\zeta} = (\zeta_1,\ldots,\zeta_d) \in \overline{\mathbb{K}}^d$. An application of Fubini's and Morera's Theorems show that $\widehat{\varphi}$ is holomorphic on $\mathbb{K}^d$ and is continuous on $\overline{\mathbb{K}}^d$. In addition, $\widehat{\varphi}$ is the quotient of two bounded holomorphic functions on $\mathbb{K}^d$.

In the case φ is a radial function, the function $\widehat{\varphi}(\mathbf{m})$ depends only on $|\mathbf{m}|$. Toeplitz operators induced by radial symbols and the C^*-algebra generated by them have been investigated intensively in the literature. See [4, 12, 13, 14], just to name a few. Toeplitz operators induced by separately radial symbols have also appeared in several papers. In particular, Proposition 2.1 and its more general version on bounded Reinhardt domains can be found in [18, 19].

In the case φ is a *polynomial* in $\mathbf{z}$ and $\bar{\mathbf{z}}$ that is separately radial, the eigenvalues of T_φ can be computed more explicitly. Let s be a non-negative integer. We denote by $\mathrm{Rad}_s[\mathbf{z},\bar{\mathbf{z}}]$ the vector space of all separately radial polynomials in $\mathbf{z}$ and $\bar{\mathbf{z}}$ of the form

$$\varphi(\mathbf{z}) = \sum_{|\boldsymbol{\alpha}|\leq s} c_{\boldsymbol{\alpha}}\,|\mathbf{z}^{\boldsymbol{\alpha}}|^2,$$

where the coefficients $c_{\boldsymbol{\alpha}}$'s are complex values. It is evident that the set $\{|\mathbf{z}^{\boldsymbol{\alpha}}|^2 : |\boldsymbol{\alpha}| \leq s\}$ is a linear basis for $\mathrm{Rad}_s[\mathbf{z},\bar{\mathbf{z}}]$.

In the lemma below, we show that when the defining symbol is a separately radial polynomial, the eigenvalues $\widehat{\varphi}(\mathbf{m})$ of the corresponding Toeplitz operator is a rational function of the multiindex $\mathbf{m}$.

Lemma 2.3. *Let φ belong to $\mathrm{Rad}_s[\mathbf{z},\bar{\mathbf{z}}]$. Then $\widehat{\varphi}(\mathbf{m})$ is a rational function of $\mathbf{m}$ in the form*

$$\widehat{\varphi}(\mathbf{m}) = \frac{p(\mathbf{m})}{(|\mathbf{m}|+d+1)\cdots(|\mathbf{m}|+d+s)},$$

where p is a holomorphic polynomial of total degree at most s. Consequently, we may extend $\widehat{\varphi}(\mathbf{m})$ from being defined only on $\mathbb{Z}_+^d$ to $\widehat{\varphi}(\boldsymbol{\zeta})$ defined on $\mathbb{C}^d$ except when the denominator is zero.

Proof. We first compute $\widehat{\varphi}(\mathbf{m})$ when φ is a separately radial monomial in the form $\varphi(\mathbf{z}) = |\mathbf{z}^{\boldsymbol{\alpha}}|^2$ for some $\boldsymbol{\alpha} \in \mathbb{Z}_+^d$. We have

$$\begin{aligned}
\widehat{\varphi}(\mathbf{m}) &= \frac{(|\mathbf{m}|+d)!}{\mathbf{m}!\,d!}\int_{\mathbb{B}} |\mathbf{z}^{\boldsymbol{\alpha}}|^2\,|\mathbf{z}^{\mathbf{m}}|^2\,d\nu(\mathbf{z}) \text{ (by Proposition 2.1)}\\
&= \frac{(|\mathbf{m}|+d)!}{\mathbf{m}!\,d!}\cdot\frac{(\mathbf{m}+\boldsymbol{\alpha})!\,d!}{(|\mathbf{m}|+d+|\boldsymbol{\alpha}|)!} \text{ (see [21, Proposition 1.4.9])}\\
&= \frac{(\mathbf{m}+\boldsymbol{\alpha})!}{\mathbf{m}!}\cdot\frac{1}{(|\mathbf{m}|+d+1)\cdots(|\mathbf{m}|+d+|\boldsymbol{\alpha}|)}\\
&= \frac{[(m_1+1)\cdots(m_1+\alpha_1)]\cdots[(m_d+1)\cdots(m_d+\alpha_d)]}{(|\mathbf{m}|+d+1)\cdots(|\mathbf{m}|+d+|\alpha|)}.
\end{aligned}$$

In general, a polynomial $\varphi \in \mathrm{Rad}_s[\mathbf{z},\bar{\mathbf{z}}]$ can be written as

$$\varphi(\mathbf{z}) = \sum_{|\boldsymbol{\alpha}|\le s} c_{\boldsymbol{\alpha}}|\mathbf{z}^{\boldsymbol{\alpha}}|^2.$$

It follows from Proposition 2.1 and the above computation that

$$\begin{aligned}
\widehat{\varphi}(\mathbf{m}) &= \sum_{|\boldsymbol{\alpha}|\le s} c_{\boldsymbol{\alpha}}\frac{[(m_1+1)\cdots(m_1+\alpha_1)]\cdots[(m_d+1)\cdots(m_d+\alpha_d)]}{(|\mathbf{m}|+d+1)\cdots(|\mathbf{m}|+d+|\boldsymbol{\alpha}|)}\\
&= \frac{p(\mathbf{m})}{(|\mathbf{m}|+d+1)\cdots(|\mathbf{m}|+d+s)},
\end{aligned}$$

where p is a polynomial with total degree at most s. Note that the last ratio may not be in reduced form. $\square$

Example 2.4. Consider $\varphi(\mathbf{z}) = c_1|z_1|^2+\cdots+c_d|z_d|^2$ for complex constants $c_1,\ldots,c_d$. The eigenvalues of T_φ are given by

$$\widehat{\varphi}(\mathbf{m}) = \sum_{j=1}^{d}\frac{c_j\,(m_j+1)}{m_1+\cdots+m_d+d+1} = \frac{c_1m_1+\cdots+c_dm_d+(c_1+\cdots+c_d)}{m_1+\cdots+m_d+d+1}. \tag{2.2}$$

Example 2.5. Consider φ_1 defined on $\mathbb{C}^d$ by $\varphi_1(\mathbf{z}) = |\mathbf{z}|^2 = |z_1|^2+\cdots+|z_d|^2$. The eigenvalues of T_{φ_1} are given by

$$\widehat{\varphi_1}(\mathbf{m}) = \frac{|\mathbf{m}|+d}{|\mathbf{m}|+d+1}.$$

Example 2.6. Consider φ_2 defined on $\mathbb{C}^d$ $(d\ge 2)$ by $\varphi_2(\mathbf{z}) = |z_1z_2|^2$. The eigenvalues of T_{φ_2} are given by

$$\widehat{\varphi_2}(\mathbf{m}) = \frac{(m_1+1)(m_2+1)}{(|\mathbf{m}|+d+1)(|\mathbf{m}|+d+2)}.$$

Example 2.7. To illustrate the difference between the polynomial and the general cases, we consider in this example a Toeplitz operator with a radial rational symbol. Let φ_3 be defined on $\mathbb{C}^2$ by $\varphi_3(\mathbf{z}) = \frac{|z_2|^2}{1-|z_1|^2}$. The eigenvalues of T_{φ_3} can be computed via Proposition 2.1 as

$$\widehat{\varphi_3}(\mathbf{m}) = \frac{(m_1+m_2+2)!}{2!\,m_1!\,m_2!}\int_{\mathbb{B}} \frac{|z_1|^{2m_1}\,|z_2|^{2m_2+2}}{1-|z_1|^2}\,d\nu(z_1,z_2).$$

Using polar coordinates $z_j = r_j e^{i\theta_j}$ $(j=1,2)$ and the fact that

$$d\nu(z_1,z_2) = \frac{2!}{\pi^2} r_1 r_2\,dr_1\,dr_2\,d\theta_1\,d\theta_2,$$

we compute

$$\begin{aligned}
\frac{1}{2!}\int_{\mathbb{B}} \frac{|z_1|^{2m_1}\,|z_2|^{2m_2+2}}{1-|z_1|^2}\,d\nu(z_1,z_2) &= \int_{r_1^2+r_2^2<1} \frac{4r_1^{2m_1+1}r_2^{2m_2+3}}{1-r_1^2}dr_1\,dr_2\\
&= \int_{t_1+t_2<1} \frac{t_1^{m_1}t_2^{m_2+1}}{1-t_1}dt_1\,dt_2\\
&\quad \text{(by the change of variables } t_j = r_j^2\text{)}\\
&= \frac{1}{m_2+2}\int_0^1 t_1^{m_1}(1-t_1)^{m_2+1}dt_1\\
&= \frac{m_1!\,(m_2+1)!}{(m_2+2)\,(m_1+m_2+2)!}.
\end{aligned}$$

The last equality follows from the well-known identity for the Beta function. We thus have

$$\widehat{\varphi_3}(\mathbf{m}) = \frac{m_2+1}{m_2+2}.$$

2.2. **Preliminaries on the commuting problem.** For φ a bounded separately radial function on $\mathbb{B}$, we would like to characterize bounded functions ψ for which T_ψ commutes with T_φ on the Bergman space A^2. For any $\mathbf{m},\mathbf{k}\in\mathbb{Z}_+^d$, since $T_\varphi\mathbf{z}^{\mathbf{m}} = \widehat{\varphi}(\mathbf{m})z^{\mathbf{m}}$ and $T_\varphi^*\mathbf{z}^{\mathbf{k}} = \overline{\widehat{\varphi}(\mathbf{k})}\mathbf{z}^{\mathbf{k}}$, we have

$$\begin{aligned}
\Big\langle [T_\psi, T_\varphi]\mathbf{z}^{\mathbf{m}}, \mathbf{z}^{\mathbf{k}}\Big\rangle &= \langle T_\psi T_\varphi \mathbf{z}^{\mathbf{m}}, \mathbf{z}^{\mathbf{k}}\rangle - \langle T_\varphi T_\psi \mathbf{z}^{\mathbf{m}}, \mathbf{z}^{\mathbf{k}}\rangle\\
&= \big(\widehat{\varphi}(\mathbf{m}) - \widehat{\varphi}(\mathbf{k})\big)\langle T_\psi \mathbf{z}^{\mathbf{m}}, \mathbf{z}^{\mathbf{k}}\rangle.
\end{aligned}$$

This shows that $[T_\psi, T_\varphi] = 0$ on A^2 if and only if

$$\big(\widehat{\varphi}(\mathbf{m}) - \widehat{\varphi}(\mathbf{k})\big)\langle T_\psi \mathbf{z}^{\mathbf{m}}, \mathbf{z}^{\mathbf{k}}\rangle = 0 \quad \text{for all} \quad \mathbf{m},\mathbf{k}\in\mathbb{Z}_+^d. \tag{2.3}$$

We shall need the following result in order to characterize the functions ψ satisfying (2.3). Let $\mathbf{v} = (v_1,\ldots,v_d)$ be a tuple of integers. Then $\mathbf{v}$ gives rise to a diagonal action of the unit circle $\mathbb{T}$ on the unit ball $\mathbb{B}$ via $\gamma\cdot\mathbf{z} = (\gamma^{v_1}z_1,\ldots,\gamma^{v_d}z_d)$ for $\gamma\in\mathbb{T}$ and $\mathbf{z} = (z_1,\ldots,z_d)\in\mathbb{B}$.

Lemma 2.8. *Let $\mathbf{v} = (v_1,\ldots,v_d)$ be an element in $\mathbb{Z}^d$ and s be an integer. Let ψ be a bounded measurable function on $\mathbb{B}$. Then the following two statements are equivalent.*

(a) For each $\gamma \in \mathbb{T}$, we have

$$\psi(\gamma^{v_1} z_1, \ldots, \gamma^{v_d} z_d) = \gamma^s \, \psi(\mathbf{z}) \text{ for a.e. } \mathbf{z} \in \mathbb{B}.$$

We say that ψ is $\mathbf{v}$-homogeneous of degree s. If $s = 0$, we say that ψ is $\mathbf{v}$-invariant.

(b) For all multiindices $\mathbf{m}, \mathbf{k} \in \mathbb{Z}_+^d$ such that $(\mathbf{m} - \mathbf{k}) \cdot \mathbf{v} + s \neq 0$, we have

$$\int_{\mathbb{B}} \psi(\mathbf{z}) \, \mathbf{z}^{\mathbf{m}} \, \bar{\mathbf{z}}^{\mathbf{k}} \, d\nu(\mathbf{z}) = 0.$$

Proof. By the rotation-invariant property of Lebesgue measure, for any unimodular complex γ, we have

$$\int_{\mathbb{B}} \psi(\gamma^{v_1} z_1, \ldots, \gamma^{v_d} z_d) \, \mathbf{z}^{\mathbf{m}} \, \bar{\mathbf{z}}^{\mathbf{k}} \, d\nu(\mathbf{z}) = \gamma^{-(\mathbf{m}-\mathbf{k})\cdot\mathbf{v}} \int_{\mathbb{B}} \psi(\mathbf{z}) \, \mathbf{z}^{\mathbf{m}} \, \bar{\mathbf{z}}^{\mathbf{k}} \, d\nu(\mathbf{z}). \quad (2.4)$$

Assume first (a) holds. Equation (2.4) shows

$$(\gamma^s - \gamma^{-(\mathbf{m}-\mathbf{k})\cdot\mathbf{v}}) \int_{\mathbb{B}} \psi(\mathbf{z}) \, \mathbf{z}^{\mathbf{m}} \, \bar{\mathbf{z}}^{\mathbf{k}} \, d\nu(\mathbf{z}) = 0.$$

If $(\mathbf{m} - \mathbf{k}) \cdot \mathbf{v} + s \neq 0$, by choosing γ so that the first factor is nonzero, we conclude that the integral must be zero. This shows that (b) holds and we have the implication (a)$\longrightarrow$(b).

Now assume that (b) holds. Let γ be a complex number of modulus one. Let $\mathbf{m}$ and $\mathbf{k}$ be multiindices. If $(\mathbf{m} - \mathbf{k}) \cdot \mathbf{v} + s \neq 0$, then (b) together with (2.4) shows that

$$\int_{\mathbb{B}} \psi(\gamma^{v_1} z_1, \ldots, \gamma^{v_d} z_d) \, \mathbf{z}^{\mathbf{m}} \, \bar{\mathbf{z}}^{\mathbf{k}} \, d\nu(\mathbf{z}) = \int_{\mathbb{B}} \gamma^s \, \psi(\mathbf{z}) \, \mathbf{z}^{\mathbf{m}} \, \bar{\mathbf{z}}^{\mathbf{k}} \, d\nu(\mathbf{z}) \quad (2.5)$$

since both sides are zero. On the other hand, if $(\mathbf{m}-\mathbf{k})\cdot\mathbf{v}+s = 0$, then (2.5) is still true by (2.4). We have shown that (2.5) holds for all multiindices $\mathbf{m}$ and $\mathbf{k}$. This forces $\psi(\gamma^{v_1} z_1, \ldots, \gamma^{v_d} z_d) = \gamma^s \, \psi(\mathbf{z})$ for a.e. $z \in \mathbb{B}$. Since γ was arbitrary, we see that (a) holds. $\square$

Remark 2.9. The notion of $\mathbf{v}$-homogeneous polynomials was introduced in an old paper by E. Fisher. If ψ is a $\mathbf{v}$-homogeneous function of degree 0, we say that ψ is $\mathbf{v}$-invariant. Such functions were called $\mathbf{v}$-radially symmetric and were studied in [15], even in the general case where the components of $\mathbf{v}$ are non-integer. We do not need this general notion in our work here.

Using (2.3) and Lemma 2.8, we now analyze several examples.

Example 2.10. Consider the function $\varphi_1(\mathbf{z}) = |\mathbf{z}|^2$ in Example 2.5. By (2.3), the operator T_ψ commutes with T_{φ_1} if and only if for all $\mathbf{m}, \mathbf{k} \in \mathbb{Z}_+^d$,

$$\Big(\frac{|\mathbf{m}| + d}{|\mathbf{m}| + d + 1} - \frac{|\mathbf{k}| + d}{|\mathbf{k}| + d + 1}\Big) \langle T_\psi \mathbf{z}^{\mathbf{m}}, \mathbf{z}^{\mathbf{k}} \rangle = 0.$$

Since the first factor is zero if and only if $|\mathbf{m}| = |\mathbf{k}|$, we conclude that $\langle T_\psi \mathbf{z}^{\mathbf{m}}, \mathbf{z}^{\mathbf{k}} \rangle = 0$ whenever $|\mathbf{m}| \neq |\mathbf{k}|$, that is, $(\mathbf{m} - \mathbf{k}) \cdot (1, \ldots, 1) \neq 0$. By Lemma 2.8, this forces ψ to be $(1, \ldots, 1)$-invariant, that is, for any γ on the unit circle, we have $\psi(\gamma z_1,\dots,\gamma z_d)=\psi(\mathbf{z})$ for a.e. $\mathbf{z}\in\mathbb{B}$. Of course, this example is just a very special case of a more general result in [16].

Example 2.11. Consider $\varphi_2(\mathbf{z})=|z_1z_2|^2$ as in Example 2.6. By (2.3) again, T_ψ commutes with T_{φ_2} if and only if for all $\mathbf{m},\mathbf{k}\in\mathbb{Z}_+^d$,

$$\Big\{\frac{(m_1+1)(m_2+1)}{(|\mathbf{m}|+d+1)(|\mathbf{m}|+d+2)}-\frac{(k_1+1)(k_2+1)}{(|\mathbf{k}|+d+1)(|\mathbf{k}|+d+2)}\Big\}\langle T_\psi\mathbf{z}^{\mathbf{m}},\mathbf{z}^{\mathbf{k}}\rangle=0.$$

The set of pairs $(\mathbf{m},\mathbf{k})$ for which the first factor vanishes is now more complicated. It contains such pairs with $m_1=k_1$ and $m_2=k_2$ but also pairs with $m_1=k_2$ and $m_2=k_1$. As a consequence, the simple approach as in Example 2.10 does not seem to produce an answer. It turns out, as we shall see later from our general result, that ψ must be $(1,\dots,1)$-invariant and radial in both z_1 and z_2. This is equivalent to the condition that for any γ on the unit circle,

$$\psi(\mathbf{z})=\psi(|z_1|,|z_2|,\gamma z_3,\dots,\gamma z_d)\quad\text{ for a.e. }\mathbf{z}\in\mathbb{B}.$$

In particular, in the case $d=2$, the function ψ must be separately radial.

Example 2.12. Consider $\varphi_3(\mathbf{z})=|z_2|^2/(1-|z_1|^2)$ as in Example 2.7. Equation 2.3 shows that T_ψ commutes with T_{φ_3} if and only if for all $\mathbf{m},\mathbf{k}\in\mathbb{Z}_+^d$,

$$\Big(\frac{m_2+1}{m_2+2}-\frac{k_2+1}{k_2+2}\Big)\langle T_\psi\mathbf{z}^{\mathbf{m}},\mathbf{z}^{\mathbf{k}}\rangle=0.$$

The first factor vanishes if and only if $m_2=k_2$, that is, $(\mathbf{m}-\mathbf{k})\cdot(0,1)=0$. We conclude that $\langle T_\psi\mathbf{z}^{\mathbf{m}},\mathbf{z}^{\mathbf{k}}\rangle=0$ whenever $(\mathbf{m}-\mathbf{k})\cdot(0,1)\neq 0$. Lemma 2.8 shows that ψ must be $(0,1)$-invariant. That is, $\psi(z_1,z_2)=\psi(z_1,|z_2|)$ a.e. on $\mathbb{B}$. Note that this class of functions includes all holomorphic functions dependent only on z_1. In particular, the analytic Toeplitz operator T_{z_1} commutes with the non-analytic T_{φ_3}. This shows that the result of Axler-Čučković-Rao [3] does *not* extend to several variables.

Remark 2.13. A commuting property of Toeplitz operators with symbols depending only on z_1 or $|z_2|/\sqrt{1-|z_1|^2}$, related to Example 2.12, is considered in a recent paper [7, Lemma 2.2].

3. The commuting problem for separately radial Toeplitz operators

In order to study the commuting problem with more general separately radial functions, we need to analyze (2.3) more closely. We require the following notation and classical results.

Recall that $\mathbb{K}$ denotes the open right-half complex plane: $\mathbb{K}=\{z\in\mathbb{C}:\Re(z)>0\}$. The space $\mathcal{H}^\infty(\mathbb{K}^d)$ consists of all bounded holomorphic functions on $\mathbb{K}^d$. We use $\mathcal{A}(\overline{\mathbb{K}}^d)$ to denote the space of holomorphic functions that extend continuously to $\overline{\mathbb{K}}^d$. The Nevanlinna class $\mathcal{N}(\overline{\mathbb{K}}^d)$ consists of functions in $\mathcal{A}(\overline{\mathbb{K}}^d)$ that can be written as a quotient of two members of $\mathcal{H}^\infty(\mathbb{K}^d)$. It is clear that $\mathcal{N}(\overline{\mathbb{K}}^d)$ is a subalgebra of $\mathcal{A}(\overline{\mathbb{K}}^d)$.

We have the following classical result [20, p.102] on the zero set of bounded holomorphic functions on $\mathbb{K}$.

Proposition 3.1. *Let f be a bounded holomorphic function on $\mathbb{K}$ that vanish at the pairwise distinct points $a_1, a_2, \ldots$. Suppose that $\inf_{j\geq 1}\{\Re(a_j)\} > 0$ and $\sum_{j=1}^\infty \Re(1/a_j) = \infty$. Then f is identically zero.*

A direct application of Proposition 3.1 together with induction on the dimension d shows that $\mathbb{Z}_+^d$ is the uniqueness set for functions in $\mathcal{N}(\overline{\mathbb{K}}^d)$.

Proposition 3.2. *Let F be a function in $\mathcal{N}(\overline{\mathbb{K}}^d)$ such that $F(\mathbf{m}) = 0$ for all $\mathbf{m} \in \mathbb{Z}_+^d$. Then F is identically zero on $\overline{\mathbb{K}}^d$.*

We introduce the following notion, which plays an important role in our investigation of commuting Toeplitz operators. Let F belong to $\mathcal{A}(\overline{\mathbb{K}}^d)$. We say that $\mathbf{a} \in \mathbb{C}^d$ is a *period* of F if

$$F(\boldsymbol{\zeta} + \mathbf{a}) = F(\boldsymbol{\zeta}) \tag{3.1}$$

for all $\boldsymbol{\zeta} \in \overline{\mathbb{K}}^d \cap (-\mathbf{a} + \overline{\mathbb{K}}^d)$. The identity theorem for holomorphic functions shows that $\mathbf{a}$ is a period for F if (3.1) holds for $\boldsymbol{\zeta}$ belonging to a non-empty open subset of $\mathbb{K}^d \cap (-\mathbf{a} + \mathbb{K}^d)$. We denote by $\mathrm{Per}(F)$ the set of all periods of F. It is clear that $\mathrm{Per}(F)$ is closed under addition and multiplication by -1. Note that in the case F is entire on $\mathbb{C}^d$, we have the usual notion of a period of F.

Now back to condition (2.3) for the commutativity of T_φ and T_ψ:

$$\big(\widehat{\varphi}(\mathbf{m}) - \widehat{\varphi}(\mathbf{k})\big)\langle T_\psi \mathbf{z}^{\mathbf{m}}, \mathbf{z}^{\mathbf{k}}\rangle = 0 \quad \text{for all} \quad \mathbf{m}, \mathbf{k} \in \mathbb{Z}_+^d. \tag{3.2}$$

Replacing $\mathbf{m}$ by $\mathbf{m} + \boldsymbol{\ell}$ and $\mathbf{k}$ by $\mathbf{k} + \boldsymbol{\ell}$ for $\mathbf{m}, \mathbf{k}, \boldsymbol{\ell} \in \mathbb{Z}_+^d$, we have

$$\Big(\widehat{\varphi}(\mathbf{m} + \boldsymbol{\ell}) - \widehat{\varphi}(\mathbf{k} + \boldsymbol{\ell})\Big)\int_{\mathbb{B}} \psi(\mathbf{z})\mathbf{z}^{\mathbf{m}}\bar{\mathbf{z}}^{\mathbf{k}}|\mathbf{z}^{\boldsymbol{\ell}}|^2\, d\nu(\mathbf{z}) = 0 \tag{3.3}$$

Recall that $\widehat{\varphi}$ extends to a function in $\mathcal{N}(\overline{\mathbb{K}}^d)$. On the other hand, define for $\boldsymbol{\zeta} \in \overline{\mathbb{K}}^d$,

$$\Psi_{\mathbf{m},\mathbf{k}}(\boldsymbol{\zeta}) = \int_{\mathbb{B}} \psi(\mathbf{z})\mathbf{z}^{\mathbf{m}}\bar{\mathbf{z}}^{\mathbf{k}}|z_1|^{2\zeta_1}\cdots|z_d|^{2\zeta_d}\, d\nu(\mathbf{z}).$$

Since ψ is bounded, it follows that $\Psi_{\mathbf{m},\mathbf{k}}$ is a bounded function belonging to $\mathcal{A}(\overline{\mathbb{K}}^d)$. Equation (3.3) says that for fixed $\mathbf{m}, \mathbf{k} \in \mathbb{Z}_+^d$, $(\widehat{\varphi}(\mathbf{m} + \boldsymbol{\zeta}) - \widehat{\varphi}(\mathbf{k} + \boldsymbol{\zeta}))\Psi_{\mathbf{m},\mathbf{k}}(\boldsymbol{\zeta}) = 0$ for $\boldsymbol{\zeta} = \boldsymbol{\ell} \in \mathbb{Z}_+^d$. Proposition 3.1 then implies

$$\Big(\widehat{\varphi}(\mathbf{m} + \boldsymbol{\zeta}) - \widehat{\varphi}(\mathbf{k} + \boldsymbol{\zeta})\Big)\Psi_{\mathbf{m},\mathbf{k}}(\boldsymbol{\zeta}) = 0 \quad \text{for all } \boldsymbol{\zeta} \in \overline{\mathbb{K}}^d. \tag{3.4}$$

Since each factor on the left-hand side is holomorphic, one of them must be identically zero on $\overline{\mathbb{K}}^d$. Note that the first factor is identically zero if and only if $\mathbf{m}-\mathbf{k}$ is a period of $\widehat{\varphi}$. Therefore, if $\mathbf{m}-\mathbf{k}$ is not a period of $\widehat{\varphi}$, then $\Psi_{\mathbf{m},\mathbf{k}}$ vanishes identically on $\overline{\mathbb{K}}^d$. In particular, $\Psi_{\mathbf{m},\mathbf{k}}(0)=0$, which gives

$$\langle T_\psi \mathbf{z}^{\mathbf{m}}, \bar{\mathbf{z}}^{\mathbf{k}}\rangle = \int_{\mathbb{B}} \psi(\mathbf{z})\mathbf{z}^{\mathbf{m}}\bar{\mathbf{z}}^{\mathbf{k}}\, d\nu(\mathbf{z}) = 0. \tag{3.5}$$

Conversely, if for any $\mathbf{m},\mathbf{k}\in\mathbb{Z}_+^d$, either $\mathbf{m}-\mathbf{k}$ is a period of $\widehat{\varphi}$ or (3.5) holds, then clearly (3.2) holds. This implies that T_ψ and T_φ commute.

We summarize what we have obtained so far in the following proposition.

Proposition 3.3. *Let φ be a bounded separately radial function and ψ be a bounded function on $\mathbb{B}$. Then the operators T_ψ and T_φ commute on the Bergman space A^2 if and only if for any $\mathbf{m},\mathbf{k}\in\mathbb{Z}_+^d$, either $\mathbf{m}-\mathbf{k}$ is a period of $\widehat{\varphi}$ or $\int_{\mathbb{B}} \psi(\mathbf{z})\mathbf{z}^{\mathbf{m}}\bar{\mathbf{z}}^{\mathbf{k}}\, d\nu(\mathbf{z})=0$.*

Example 3.4. Recall $\varphi_2(\mathbf{z})=|z_1z_2|^2$ in Example 2.11. The eigenvalue function is given by

$$\widehat{\varphi_2}(\boldsymbol{\zeta}) = \frac{(\zeta_1+1)(\zeta_2+1)}{(\Sigma\boldsymbol{\zeta}+d+1)(\Sigma\boldsymbol{\zeta}+d+2)}$$

for all $\boldsymbol{\zeta}\in\overline{\mathbb{K}}^d$, where $\Sigma\boldsymbol{\zeta}=\zeta_1+\cdots+\zeta_d$. It can be verified directly that $\mathrm{Per}(\widehat{\varphi_2})$ consists of vectors $\mathbf{a}=(a_1,\ldots,a_d)$ with $a_1=a_2=0$ and $a_1+\cdots+a_d=0$. Consequently, for any multiindices $\mathbf{m},\mathbf{k}$, we see that $\mathbf{m}-\mathbf{k}$ belongs to $\mathrm{Per}(\widehat{\varphi_2})$ if and only if $m_1=k_1$, $m_2=k_2$ and $|\mathbf{m}|=|\mathbf{k}|$. Proposition 3.3 implies that T_ψ commutes with T_{φ_2} if and only if $\int_{\mathbb{B}} \psi(\mathbf{z})\mathbf{z}^{\mathbf{m}}\bar{\mathbf{z}}^{\mathbf{k}}\, d\nu(\mathbf{z})=0$ whenever $m_1\neq k_1$, or $m_2\neq k_2$, or $|\mathbf{m}|\neq|\mathbf{k}|$. By Lemma 2.8, this is equivalent to the requirement that ψ be $(1,0,\ldots,0)$-, $(0,1,0,\ldots,0)$-, and $(1,\ldots,1)$-invariant, which means that for any complex number τ with $|\tau|=1$, we have

$$\psi(z_1,z_2,\tau z_3,\ldots,\tau z_d)=\psi(|z_1|,|z_2|,z_3,\ldots,z_d) \quad \text{for a.e. } \mathbf{z}\in\mathbb{B}.$$

In the case $d=2$, the function ψ is actually separately radial.

3.1. **Periods of holomorphic functions.** Proposition 3.3 shows that in order to characterize Toeplitz operators commuting with T_φ, we need to understand the set of periods of $\widehat{\varphi}$. We first present an important property of the real periods of general holomorphic functions belonging to $\mathcal{N}(\overline{\mathbb{K}}^d)$.

Proposition 3.5. *Let F be in $\mathcal{N}(\overline{\mathbb{K}}^d)$. Let $\mathbf{a}\in\mathbb{R}^d$ be a period of F. Then $\gamma\cdot\mathbf{a}$ is also a period of F for any complex number γ. As a consequence, if we define $\mathrm{Per}_{\mathbb{Z}}(F)=\mathrm{Per}(F)\cap\mathbb{Z}^d$, then there exist vectors $\mathbf{u}_1,\ldots,\mathbf{u}_k\in\mathbb{Z}^d$ such that*

$$\mathrm{Per}_{\mathbb{Z}}(F)=\Big\{\mathbf{a}\in\mathbb{Z}^d:\ \mathbf{a}\cdot\mathbf{u}_j=0 \text{ for all } j=1,\ldots,k\Big\}. \tag{3.6}$$

We shall denote the right hand side of (3.6) by $\big\{\mathbf{u}_1,\ldots,\mathbf{u}_k\big\}^{\perp}$.

Proof. Write $\mathbf{a} = (a_1, \dots, a_d)$. Put $\mathbf{a}^+ = (a_1^+, \dots, a_d^+)$ and $\mathbf{a}^- = (a_1^-, \dots, a_d^-)$. Recall that for x a real number, $x^+ = \max\{x, 0\}$ and $x^- = \max\{-x, 0\}$. Note that $x = x^+ - x^-$ and $x^+x^- = 0$. Since $\mathbf{a}$ is a period of F, the equation (3.1) holds for all $\boldsymbol{\zeta} \in \mathbb{K}^d \cap (-\mathbf{a} + \mathbb{K}^d)$. In particular, it holds for $\boldsymbol{\zeta} = \mathbf{z} + \mathbf{a}^-$ with $\mathbf{z} \in \mathbb{K}^d$. This implies

$$F(\mathbf{z} + \mathbf{a}^+) - F(\mathbf{z} + \mathbf{a}^-) = 0 \text{ for all } \mathbf{z} \in \mathbb{K}^d.$$

Consequently, for any integer $t \geq 1$, we have

$$F(\mathbf{z} + t\mathbf{a}^+) - F(\mathbf{z} + t\mathbf{a}^-) = 0 \text{ for all } \mathbf{z} \in \mathbb{K}^d.$$

Now fix $\mathbf{z} \in \mathbb{K}^d$. Define a holomorphic function of one complex variable $G(\gamma) = F(\mathbf{z} + \gamma\,\mathbf{a}^+) - F(\mathbf{z} + \gamma\,\mathbf{a}^-)$ for $\gamma \in \overline{\mathbb{K}}$. Note that G belongs to $\mathcal{N}(\overline{\mathbb{K}})$ due to our assumption on F. Because $G(t) = 0$ for all integers $t \geq 1$, Proposition 3.2 implies that G must be identically zero on $\overline{\mathbb{K}}$. Consequently,

$$F(\mathbf{z} + \gamma\,\mathbf{a}^+) = F(\mathbf{z} + \gamma\,\mathbf{a}^-)$$

for all $\gamma \in \overline{\mathbb{K}}$. Since $\mathbf{z} \in \mathbb{K}^d$ was arbitrary, we conclude that $\gamma\,\mathbf{a} = \gamma\,\mathbf{a}^+ - \gamma\,\mathbf{a}^-$ and $-\gamma\,\mathbf{a}$ are periods of F for any $\gamma \in \overline{\mathbb{K}}$.

Let $\mathrm{Per}_{\mathbb{Q}}(F) = \mathrm{Per}(F) \cap \mathbb{Q}$ denote the rational periods of F. Then $\mathrm{Per}_{\mathbb{Q}}(F)$ is a vector space over $\mathbb{Q}^d$. Consider the standard Euclidean inner product on $\mathbb{Q}^d$ and let $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ be a basis for the orthogonal complement of $\mathrm{Per}_{\mathbb{Q}}(F)$. If this orthogonal complement consists of only the zero vector, we take $k = 1$ and $\mathbf{u}_1$ the zero vector. Multiplying each vector by a sufficiently large integer if necessary, we may assume that $\mathbf{u}_1, \dots, \mathbf{u}_k$ belong to $\mathbb{Z}^d$. Since $\mathrm{Per}_{\mathbb{Z}}(F) = \mathrm{Per}_{\mathbb{Q}}(F) \cap \mathbb{Z}^d$, we obtain (3.6). $\square$

Example 3.6. Let $\mathbf{c} = (c_1, \dots, c_d) \in \overline{\mathbb{K}}^d$. Let

$$F(\mathbf{z}) = \langle \mathbf{z}, \mathbf{c} \rangle = \bar{c}_1 z_1 + \cdots + \bar{c}_d z_d \text{ for } \mathbf{z} = (z_1, \dots, z_d) \in \overline{\mathbb{K}}^d.$$

The periods of F are exactly those $\mathbf{a} = (a_1, \dots, a_d) \in \mathbb{C}^d$ with $\langle \mathbf{a}, \mathbf{c} \rangle = 0$, that is, $a_1\bar{c}_1 + \cdots + a_d\bar{c}_d = 0$.

In the simple example below, we see that the conclusion of Proposition 3.5 may not hold if the components of $\mathbf{a}$ are not all real.

Example 3.7. Let $F(\mathbf{z}) = e^{-2z_1 - z_2}$ for $\mathbf{z} = (z_1, z_2) \in \overline{\mathbb{K}}^2$. Then F is a bounded holomorphic function on $\overline{\mathbb{K}}^2$. The periods of F are exactly those $\mathbf{a} = (a_1, a_2) \in \mathbb{C}^2$ with $2a_1 + a_2 = 2k\pi i$ for some integer k. This shows that while $\mathbf{a} = (0, 2\pi i)$ is a period, $\gamma \cdot \mathbf{a} = (0, 2\gamma\pi i)$ is a period if and only if γ is an integer.

3.2. **The main result.** Combining Propositions 3.3 and 3.5, we obtain the main result of this section. This will be used in the proof of Theorem A stated in the Introduction.

Theorem 3.8. *Let φ be a bounded separately radial function on $\mathbb{B}$. There then exist tuples of integers $\mathbf{u}_1, \dots, \mathbf{u}_k$ such that for any bounded functions ψ, the following two statements are equivalent.*

(a) T_ψ *commutes with* T_φ *on the Bergman space* A^2.
(b) ψ *is* $\mathbf{u}_j$*-invariant for all* $1 \leq j \leq k$.

Proof. Since $\widehat{\varphi}$ belongs to $\mathcal{N}(\overline{\mathbb{K}}^d)$, Proposition 3.5 shows the existence of vectors $\mathbf{u}_1, \ldots, \mathbf{u}_k$ in $\mathbb{Z}^d$ such that

$$\mathrm{Per}_{\mathbb{Z}}(\widehat{\varphi}) = \{\mathbf{u}_1, \ldots, \mathbf{u}_k\}^{\perp}.$$

By Lemma 2.8, we see that statement (b) is equivalent to the statement that $\int_{\mathbb{B}} \psi(\mathbf{z})\mathbf{z}^{\mathbf{m}}\mathbf{z}^{\mathbf{k}}\, d\nu(\mathbf{z}) = 0$ whenever $\mathbf{m} - \mathbf{k} \notin \mathrm{Per}_{\mathbb{Z}}(\widehat{\varphi})$.

Now assume that (a) holds. Proposition 3.3 shows that for $\mathbf{m}, \mathbf{k} \in \mathbb{Z}_+^d$ such that $\mathbf{m}-\mathbf{k}$ does not belong to $\mathrm{Per}_{\mathbb{Z}}(\widehat{\varphi})$, we have $\int_{\mathbb{B}} \psi(\mathbf{z})\mathbf{z}^{\mathbf{m}}\mathbf{z}^{\mathbf{k}}\, d\nu(\mathbf{z}) = 0$, hence (b) holds.

Assume that (b) holds. Let $\mathbf{m}, \mathbf{k}$ be any multiindices. If $\mathbf{m} - \mathbf{k}$ belongs to $\mathrm{Per}_{\mathbb{Z}}(\widehat{\varphi})$ then $\widehat{\varphi}(\mathbf{m}) - \widehat{\varphi}(\mathbf{k}) = 0$. If $\mathbf{m} - \mathbf{k}$ does not belong to $\mathrm{Per}_{\mathbb{Z}}(\widehat{\varphi})$ then $\int_{\mathbb{B}} \psi(\mathbf{z})\mathbf{z}^{\mathbf{m}}\mathbf{z}^{\mathbf{k}}\, d\nu(\mathbf{z}) = 0$. It follows that in any case condition (2.3) is satisfied. Hence, T_φ and T_ψ commute. $\square$

4. Toeplitz operators with separately radial polynomial symbols

In this section we investigate in detail separately radial polynomial symbols. We obtain a characterization of the eigenvalue functions of the corresponding Toeplitz operators. We then offer proofs of Theorems A and B.

Let $\mathrm{Pol}_s[\boldsymbol{\zeta}]$ be the vector space of all holomorphic polynomials in $\boldsymbol{\zeta} \in \mathbb{C}^d$ (note that powers of $\bar{\boldsymbol{\zeta}}$ are not allowed) of total degree at most s. It is clear that $\{\boldsymbol{\zeta}^{\boldsymbol{\alpha}} : |\boldsymbol{\alpha}| \leq s\}$ forms a linear basis for $\mathrm{Pol}_s[\boldsymbol{\zeta}]$. We see that the vector spaces $\mathrm{Rad}_s[\mathbf{z}, \bar{\mathbf{z}}]$ and $\mathrm{Pol}_s[\boldsymbol{\zeta}]$ have the same dimension, which is equal to the number of multiindices $\boldsymbol{\alpha}$'s for which $|\boldsymbol{\alpha}| \leq s$. Consequently, the map $|\mathbf{z}^{\boldsymbol{\alpha}}|^2 \mapsto \zeta^{\boldsymbol{\alpha}}$ extends naturally to a vector space isomorphism from $\mathrm{Rad}_s[\mathbf{z}, \bar{\mathbf{z}}]$ onto $\mathrm{Pol}_s[\boldsymbol{\zeta}]$. Via the eigenvalues of Toeplitz operators with separately radial polynomial symbols, we exhibit another isomorphism from $\mathrm{Rad}_s[\mathbf{z}, \bar{\mathbf{z}}]$ onto $\mathrm{Pol}_s[\boldsymbol{\zeta}]$.

For $\boldsymbol{\zeta} \in \mathbb{C}^d$, write $\Sigma\boldsymbol{\zeta} = \zeta_1 + \cdots + \zeta_d$. Note that $\Sigma\mathbf{m} = |\mathbf{m}|$ for $\mathbf{m} \in \mathbb{Z}_+^d$.

Proposition 4.1. *Define the map* $\mathcal{L} : \mathrm{Rad}_s[\mathbf{z}, \bar{\mathbf{z}}] \to \mathrm{Pol}_s[\boldsymbol{\zeta}]$ *by the formula*

$$\mathcal{L}(\varphi) = (\Sigma\boldsymbol{\zeta} + d + 1) \cdots (\Sigma\boldsymbol{\zeta} + d + s)\, \widehat{\varphi}(\boldsymbol{\zeta}).$$

Then $\mathcal{L}$ *is a vector space isomorphism. Consequently, for any polynomial* p *in* $\mathrm{Pol}_s[\boldsymbol{\zeta}]$*, there exists a separately radial polynomial* $\varphi \in \mathrm{Rad}_s[\mathbf{z}, \bar{\mathbf{z}}]$ *such that*

$$\widehat{\varphi}(\mathbf{m}) = \frac{p(\mathbf{m})}{(|\mathbf{m}| + d + 1) \cdots (|\mathbf{m}| + d + s)} \quad \textit{for all} \ \ \mathbf{m} \in \mathbb{Z}_+^d.$$

Proof. For any $\varphi \in \mathrm{Rad}_s[\mathbf{z}, \bar{\mathbf{z}}]$, Lemma 2.3 shows that $\mathcal{L}(\varphi)$ is indeed a polynomial in $\boldsymbol{\zeta}$ of total degree at most s. The linearity of $\mathcal{L}$ comes from the eigenvalue formula in Proposition 2.1. Since the spaces $\mathrm{Rad}_s[\mathbf{z}, \bar{\mathbf{z}}]$ and $\mathrm{Pol}_s[\boldsymbol{\zeta}]$ have the same dimension, to show that $\mathcal{L}$ is an isomorphism, it suffices to show that $\mathcal{L}$ is injective.

Suppose φ belongs to $\ker(\mathcal{L})$. Then $\widehat{\varphi}(\mathbf{m}) = 0$ for all $\mathbf{m} \in \mathbb{Z}_+^d$, which shows that T_φ is the zero operator on A^2. Consequently, $\varphi = 0$. It follows that $\mathcal{L}$ is an injection as required. □

In the following lemma, we characterize rational functions on $\mathbb{C}^d$ that can be represented as eigenvalues of a Toeplitz operator with a separately radial polynomial symbol.

Lemma 4.2. *Let r be a rational function on $\mathbb{C}^d$ given in a reduced form $r(\boldsymbol{\zeta}) = \frac{p(\boldsymbol{\zeta})}{q(\Sigma\boldsymbol{\zeta})}$, where p is a polynomial of $\boldsymbol{\zeta}$ and q is a polynomial of a single variable. Then there exists a separately radial polynomial φ such that $\widehat{\varphi}(\mathbf{m}) = r(\mathbf{m})$ for all multiindices $\mathbf{m}$ if and only if the following two conditions are satisfied:*

(i) total $\deg(p) \leq \deg(q)$*;*

(ii) q has only simple roots, each of which is an integer smaller than or equal to $-d-1$.

Proof. The necessity follows from Lemma 2.3. We only need to prove the sufficiency. Assume that both conditions (i) and (ii) are satisfied. By (ii), there exists a positive integer s such that

$$f(\boldsymbol{\zeta}) = (\Sigma\boldsymbol{\zeta} + d + 1) \cdots (\Sigma\boldsymbol{\zeta} + d + s) \cdot r(\boldsymbol{\zeta})$$

is a polynomial in $\boldsymbol{\zeta}$. By (i), the total degree of f is at most s. Proposition 4.1 shows the existence of a separately radial polynomial $\varphi \in \mathrm{Rad}_s[\mathbf{z}, \bar{\mathbf{z}}]$ such that

$$\widehat{\varphi}(\mathbf{m}) = \frac{f(\mathbf{m})}{(|\mathbf{m}| + d + 1) \cdots (|\mathbf{m}| + d + s)} = r(\mathbf{m})$$

for all multiindices $\mathbf{m}$. This completes the proof of the lemma. □

We are now investigating the integer periods of rational functions arising as eigenvalues of Toeplitz operators with separately radial polynomial symbols. It turns out that except in the trivial case, all periods must always perpendicular to the vector $\mathbb{1}$.

Recall that for F a function defined on the closed right-half space $\overline{\mathbb{K}}^d$, the set $\mathrm{Per}_{\mathbb{Z}}(F)$ consists of all $\mathbf{a} \in \mathbb{Z}^d$ such that

$$F(\boldsymbol{\zeta} + \mathbf{a}) = F(\boldsymbol{\zeta}) \text{ for all } \boldsymbol{\zeta} \in \overline{\mathbb{K}}^d \cap (-\mathbf{a} + \overline{\mathbb{K}}^d).$$

Proposition 4.3. *Suppose F is a non-polynomial rational function of the form*

$$F(\boldsymbol{\zeta}) = \frac{p(\boldsymbol{\zeta})}{q(\Sigma\boldsymbol{\zeta})},$$

where $p \in \mathbb{C}[\boldsymbol{\zeta}]$ *and* $q \in \mathbb{C}[t]$*. Then* $\operatorname{Per}_{\mathbb{Z}}(F) = \{\mathbb{1}\}^{\perp} \cap \operatorname{Per}_{\mathbb{Z}}(p)$ *and there exists* $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\} \subset \mathbb{Z}^d$ *with* $\mathbf{u}_1 = \mathbb{1}$ *such that* $\operatorname{Per}_{\mathbb{Z}}(F) = \{\mathbf{u}_1, \ldots, \mathbf{u}_k\}^{\perp}$*.*

Proof. We first show that any element in $\operatorname{Per}_{\mathbb{Z}}(F)$ is perpendicular to $\mathbb{1}$. Let $\mathbf{v}$ be an element in $\operatorname{Per}_{\mathbb{Z}}(F)$. Proposition 3.5 shows that $s\mathbf{v}$ is also an element in $\operatorname{Per}_{\mathbb{Z}}(F)$ for any positive real number s. Fix $\boldsymbol{\zeta}$ in the domain of F. Then $\boldsymbol{\zeta} + s\mathbf{v}$ also belongs to the domain of F for sufficiently large s. It follows that $F(\boldsymbol{\zeta}) = F(\boldsymbol{\zeta} + s\mathbf{v})$ for all sufficiently large s. Taking limit as $s \to \infty$ gives

$$F(\boldsymbol{\zeta}) = \lim_{s\to\infty} F(\boldsymbol{\zeta} + s\mathbf{v}) = \lim_{s\to\infty} \frac{p(\boldsymbol{\zeta} + s\mathbf{v})}{q(\Sigma\boldsymbol{\zeta} + s\Sigma\mathbf{v})}, \tag{4.1}$$

for each fixed $\boldsymbol{\zeta}$ in the domain of F. Dividing both numerator and denominator by a constant if necessary, we may assume that q is a monic polynomial. Let μ denote the degree of q. If $\Sigma\mathbf{v} \neq 0$, then the leading coefficient of $q(\Sigma\boldsymbol{\zeta} + s\Sigma\mathbf{v})$ as a polynomial of s is $(\Sigma\mathbf{v})^{\mu}$, which is independent of $\boldsymbol{\zeta}$. This would imply that the limit in (4.1) is a polynomial of $\boldsymbol{\zeta}$, which contradicts the assumption that F is not a polynomial. Therefore, we have $\Sigma\mathbf{v} = 0$ as desired.

It is clear that $\{\mathbb{1}\}^{\perp} \cap \operatorname{Per}_{\mathbb{Z}}(p) \subset \operatorname{Per}_{\mathbb{Z}}(F)$. To prove the reverse inclusion, let $\mathbf{v}$ be in $\operatorname{Per}_{\mathbb{Z}}(F)$. We have showed that $\Sigma\mathbf{v} = 0$. It follows that for $\boldsymbol{\zeta} \in \mathbb{K}^d \cap (-\mathbf{v} + \mathbb{K}^d)$,

$$p(\boldsymbol{\zeta} + \mathbf{v}) = q(\Sigma\boldsymbol{\zeta} + \Sigma\mathbf{v})F(\boldsymbol{\zeta} + \mathbf{v}) = q(\Sigma\boldsymbol{\zeta})F(\boldsymbol{\zeta}) = p(\boldsymbol{\zeta}).$$

Therefore, $\mathbf{v}$ belongs to $\operatorname{Per}_{\mathbb{Z}}(p)$.

By Proposition 3.5, there are tuples $\mathbf{u}_2, \ldots, \mathbf{u}_k \in \mathbb{Z}^d$ such that $\operatorname{Per}_{\mathbb{Z}}(p) = \{\mathbf{u}_2, \ldots, \mathbf{u}_k\}^{\perp}$. The conclusion of the proposition then follows. □

We are now ready for the proof of Theorem A stated in Introduction. We restate the result here with a slight modification.

Theorem 4.4. *Let* φ *be a non-constant separately radial polynomial. Write* $\operatorname{Per}_{\mathbb{Z}}(\widehat{\varphi}) = \{\mathbf{u}_1, \ldots, \mathbf{u}_k\}^{\perp}$*, where* $\mathbf{u}_1 = \mathbb{1}$ *and* $\mathbf{u}_2, \ldots, \mathbf{u}_k$ *belong to* $\mathbb{Z}^d$*. Let* ψ *be a bounded function on* $\mathbb{B}$*. Then the following statements are equivalent.*

(a) T_ψ *commutes with* T_φ *on the Bergman space* A^2*.*
(b) ψ *is* $\mathbf{u}_j$*-invariant for all* $1 \leq j \leq k$*.*

Proof. Since φ is a separately radial polynomial, $\widehat{\varphi}$ satisfies the hypothesis of Proposition 4.3. The existence of the tuples $\mathbf{u}_1 = \mathbb{1}, \mathbf{u}_2, \ldots, \mathbf{u}_k$ then follows. Theorem 3.8 now completes the proof. □

Corollary 4.5. *Let* φ *be a non-constant separately radial polynomial and* ψ *be a bounded holomorphic function on* $\mathbb{B}$*. If* T_φ *and* T_ψ *commutes, then* ψ *must be a constant function.*

Proof. By Theorem 4.4, ψ is $\mathbb{1}$-invariant. Since ψ is holomorphic, it must be a constant function. □

Remark 4.6. Example 2.12 shows that the conclusion of Theorem 4.4 may *not* hold if φ is not a polynomial.

We are now in a position to prove Theorem B. We actually provide a more explicit construction.

Theorem 4.7. *Let $\mathbf{u}_1 = \mathbb{1}$ and $\mathbf{u}_2, \ldots, \mathbf{u}_k$ be given tuples of integers. There then exists a separately radial polynomial $\varphi(\mathbf{z}) = c_1|z_1|^2 + \cdots + c_d|z_d|^2$ such that for any bounded function ψ, the two statements in Theorem 4.4 are equivalent.*

Proof. Take $\lambda_1, \ldots, \lambda_k$ to be any k real numbers that are linearly independent over $\mathbb{Q}$ and define $\mathbf{u} = \lambda_1\mathbf{u}_1 + \cdots + \lambda_k\mathbf{u}_k$. Then $\mathbf{u}$ is a non-zero vector. Let φ be the separately radial polynomial given by

$$\varphi(\mathbf{z}) = \langle e_1, \mathbf{u}\rangle|z_1|^2 + \cdots + \langle e_d, \mathbf{u}\rangle|z_d|^2.$$

Here $\{e_1, \ldots, e_d\}$ is the standard basis of $\mathbb{C}^d$. Since $\mathbf{u}$ is not the zero vector, φ is not a constant function. A direct calculation using Example 2.4 shows that

$$\widehat{\varphi}(\mathbf{m}) = \frac{\sum_{j=1}^{d}\langle e_j, \mathbf{u}\rangle(m_j + 1)}{|\mathbf{m}| + d + 1} = \frac{\langle \mathbf{m}, \mathbf{u}\rangle + \langle \mathbb{1}, \mathbf{u}\rangle}{|\mathbf{m}| + d + 1}.$$

This implies $\widehat{\varphi}(\boldsymbol{\zeta}) = (\langle \boldsymbol{\zeta}, \mathbf{u}\rangle + \langle \mathbb{1}, \mathbf{u}\rangle)/(\Sigma\boldsymbol{\zeta} + d + 1)$ for $\boldsymbol{\zeta} \in \mathbb{K}^d$. We compute

$$\begin{aligned}
\operatorname{Per}_{\mathbb{Z}}\big(\langle \boldsymbol{\zeta}, \mathbf{u}\rangle + \langle \mathbb{1}, \mathbf{u}\rangle\big) &= \operatorname{Per}_{\mathbb{Z}}\big(\langle \boldsymbol{\zeta}, \mathbf{u}\rangle\big)\\
&= \{\mathbf{a} \in \mathbb{Z}^d : \langle \mathbf{a}, \mathbf{u}\rangle = 0\} \quad \text{(by Example 3.6)}\\
&= \{\mathbf{a} \in \mathbb{Z}^d : \lambda_1\langle \mathbf{a}, \mathbf{u}_1\rangle + \cdots + \lambda_k\langle \mathbf{a}, \mathbf{u}_k\rangle = 0\}\\
&= \{\mathbf{a} \in \mathbb{Z}^d : \langle \mathbf{a}, \mathbf{u}_1\rangle = \cdots = \langle \mathbf{a}, \mathbf{u}_k\rangle = 0\}\\
&\qquad \text{(since } \lambda_1, \ldots, \lambda_k \text{ are linearly independent over } \mathbb{Q})\\
&= \{\mathbf{u}_1, \ldots, \mathbf{u}_k\}^{\perp}.
\end{aligned}$$

Proposition 4.3 then implies $\operatorname{Per}_{\mathbb{Z}}(\widehat{\varphi}) = \{\mathbf{u}_1, \ldots, \mathbf{u}_k\}^{\perp}$. Theorem 4.4 now completes the proof. $\square$

We illustrate the above construction by the following concrete example.

Example 4.8. Let $\mathbf{u}_1 = (1, 1, 1)$ and $\mathbf{u}_2 = (1, 0, 0)$ in $\mathbb{Z}^3$. Choose $\lambda_1 = 1$ and $\lambda_2 = \sqrt{2}$. Then $\mathbf{u} = \mathbf{u}_1 + \sqrt{2}\,\mathbf{u}_2 = (1 + \sqrt{2}, 1, 1)$. The polynomial φ is

$$\varphi(\mathbf{z}) = (1 + \sqrt{2})|z_1|^2 + |z_2|^2 + |z_3|^2 = |\mathbf{z}|^2 + \sqrt{2}|z_1|^2.$$

Theorem 4.7 shows that for any bounded function ψ on $\mathbb{B}$, the operators T_ψ and T_φ commute if and only if ψ is $(1, 1, 1)$- and $(1, 0, 0)$-invariant, which implies that for any $|\tau| = 1$,

$$\psi(\mathbf{z}) = \psi(|z_1|, \tau z_2, \tau z_3) \text{ for a.e. } \mathbf{z} \in \mathbb{B}.$$

Combining Theorems 4.4 and 4.7 we obtain the following interesting fact about the commuting problem for separately radial polynomials.

Corollary 4.9. *Let φ be a separately radial polynomial in $\mathbf{z}$ and $\bar{\mathbf{z}}$. There then exists a polynomial $\widetilde{\varphi}$ of the form $\widetilde{\varphi}(\mathbf{z}) = c_1|z_1|^2 + \cdots + c_d|z_d|^2$ such that for any function ψ bounded, T_ψ commutes with T_φ if and only if T_ψ commutes with $T_{\widetilde{\varphi}}$.*

5. Remarks on the weighted cases

In this section we briefly discuss our results on weighted Bergman spaces over the unit ball. For $\lambda > -1$, let $d\nu_\lambda(z) = c_\lambda(1-|z|^2)^\lambda\, d\nu(z)$ be the normalized radially weighted Lebesgue measure on $\mathbb{B}$ parametrized by λ. Here, c_λ is the normalizing constant whose exact value is not important to us. Let A^2_λ denote the corresponding weighted Bergman space. The standard orthonormal basis of A^2_λ is given by

$$\Big\{e_{\lambda,\mathbf{m}}(z) = \sqrt{\frac{\Gamma(|\mathbf{m}|+d+\lambda+1)}{\mathbf{m}!\,\Gamma(d+\lambda+1)}}\,\mathbf{z}^{\mathbf{m}} : \mathbf{m}\in\mathbb{Z}^d_+\Big\}.$$

As in the unweighted case, if φ is a bounded separately radial function on $\mathbb{B}$, then the corresponding Toeplitz operator $T_{\lambda,\varphi}$ acting on A^2_λ is also diagonalizable: $T_{\lambda,\varphi}e_{\lambda,\mathbf{m}} = \widehat{\varphi}(\lambda,\mathbf{m})e_{\lambda,\mathbf{m}}$ for all $\mathbf{m}\in\mathbb{Z}^d_+$. In this case, the eigenvalues are given by

$$\widehat{\varphi}(\lambda,\mathbf{m}) = \frac{\Gamma(|\mathbf{m}|+d+\lambda+1)}{\mathbf{m}!\,\Gamma(d+\lambda+1)}\int_{\mathbb{B}}\varphi(\mathbf{z})\,|\mathbf{z}^{\mathbf{m}}|^2\,d\nu_\lambda(\mathbf{z}).$$

A calculation as in Lemma 2.3, making use of [23, Lemma 1.11], shows that for

$$\varphi(\mathbf{z}) = \sum_{|\boldsymbol{\alpha}|\le s} b_{\boldsymbol{\alpha}}|\mathbf{z}^{\boldsymbol{\alpha}}|^2,$$

we have

$$\begin{aligned}\widehat{\varphi}(\lambda,\mathbf{m}) &= \sum_{|\boldsymbol{\alpha}|\le s} b_{\boldsymbol{\alpha}}\frac{[(m_1+1)\cdots(m_1+\alpha_1)]\cdots[(m_d+1)\cdots(m_d+\alpha_d)]}{(|\mathbf{m}|+d+\lambda+1)\cdots(|\mathbf{m}|+d+\lambda+|\boldsymbol{\alpha}|)}\\ &= \frac{p(\lambda,\mathbf{m})}{(|\mathbf{m}|+d+\lambda+1)\cdots(|\mathbf{m}|+d+\lambda+s)},\end{aligned}\tag{5.1}$$

where $p(\lambda,\mathbf{m})$ is a polynomial of total degree at most s in $\mathbf{m}$. This shows that $\widehat{\varphi}(\lambda,\mathbf{m})$ extends to $\mathbb{C}^d$ as a rational function of the form considered in Proposition 4.3. As a result, Theorems A and B in Introduction remain valid on A^2_λ.

Since the spectral sequence of $T_{\lambda,\varphi}$ depends on λ, it is expected that the tuples of integers $\mathbf{u}_1,\ldots,\mathbf{u}_k$ in Theorem A depend on λ. More precisely, the set

$$\mathcal{M}_\lambda(\varphi) = \big\{\psi \text{ bounded} : T_{\lambda,\psi} \text{ commutes with } T_{\lambda,\varphi} \text{ on } A^2_\lambda\big\}$$

depends on λ. It turns out that in the case φ is a radial function (depending only on $|\mathbf{z}|$), [16, Theorem 1.2] shows that $\mathcal{M}_\lambda(\varphi)$ is independent of λ. In addition, this is also the case for all examples of φ that we have considered so far. However, the following example shows that λ plays an important role in the description of $\mathcal{M}_\lambda(\varphi)$.

Example 5.1. Let $\varphi(z) = -2|z_1|^2 + 3|z_1|^4 + 2|z_1|^2|z_2|^2 + |z_3|^4$ on $\mathbb{C}^3$ (so $d = 3$). Formula (5.1) and a direct calculation give

$$\widehat{\varphi}(\lambda, \mathbf{m}) = \frac{(m_1 - m_3)(m_1 - m_3 - 1) - 2\lambda(m_1 + 1)}{(|\mathbf{m}| + \lambda + 4)(|\mathbf{m}| + \lambda + 5)}.$$

We see that $\widehat{\varphi}(\lambda, \mathbf{m})$ extends to $\mathbb{C}^3$ as a rational function of the form

$$\widehat{\varphi}(\lambda, \boldsymbol{\zeta}) = \frac{p(\lambda, \boldsymbol{\zeta})}{(\Sigma\boldsymbol{\zeta} + \lambda + 4)(\Sigma\boldsymbol{\zeta} + \lambda + 5)},$$

where $p(\lambda, \boldsymbol{\zeta}) = (\zeta_1 - \zeta_3)(\zeta_1 - \zeta_3 - 1) - 2\lambda(\zeta_1 + 1)$. Proposition 4.3 shows that $\mathrm{Per}_{\mathbb{Z}}(\widehat{\varphi}(\lambda, \cdot)) = \{\mathbb{1}\}^{\perp} \cap \mathrm{Per}_{\mathbb{Z}}(p(\lambda, \cdot))$. On the other hand, it can be verified that

$$\mathrm{Per}_{\mathbb{Z}}(p(\lambda, \cdot) = \begin{cases} \{(1, 0, -1)\}^{\perp} & \text{if } \lambda = 0, \\ \{(1, 0, 0), (0, 0, 1)\}^{\perp} & \text{if } \lambda \neq 0. \end{cases}$$

Consequently,

$$\mathrm{Per}_{\mathbb{Z}}(\widehat{\varphi}(\lambda, \cdot)) = \begin{cases} \{(1, 1, 1), (1, 0, -1)\}^{\perp} & \text{if } \lambda = 0, \\ \{(1, 1, 1), (1, 0, 0), (0, 0, 1)\}^{\perp} = \{0\} & \text{if } \lambda \neq 0. \end{cases}$$

Theorem 4.4 asserts that for $\lambda \neq 0$, the set $\mathcal{M}_\lambda(\varphi)$ consists of only separately radial functions. On the other hand, $\mathcal{M}_0(\varphi)$ contains functions that are not separately radial, such as $\psi(\mathbf{z}) = z_1 \bar{z}_2^2 z_3$.

For a general separately radial polynomial φ, it may be interesting to describe the dependency of $\mathcal{M}_\lambda(\varphi)$ on λ. However, Example 5.1 shows that this problem may be quite difficult. We leave this open for future research.

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