COMMUTATOR IDEALS OF SUBALGEBRAS OF TOEPLITZ ALGEBRAS ON WEIGHTED BERGMAN SPACES II

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Abstract. For any subset G of L^∞, let $\mathfrak{T}(G)$ denote the algebra generated by all Toeplitz operators T_f with $f \in G$. Let $\mathfrak{CT}(G)$ denote the closed two-sided ideal of $\mathfrak{T}(G)$ generated by all commutators $T_f T_g - T_g T_f$ with $f, g \in G$. In this paper we extend our earlier result in [1]. More specifically, we show that the identity $\mathfrak{CT}(G) = \mathfrak{T}(G)$ holds true for a broader class of G than considered earlier. The main idea is almost the same as that in [1].

We refer the reader to [1] for definitions and basic results which we will use in this paper. As in Section 2 in [1], a function f on $\mathbb{B}_n$ is called a radial function if there is a function $\tilde{f}$ defined on $[0, 1)$ so that $f(z) = \tilde{f}(|z|)$ for all $z \in \mathbb{B}_n$. For such an f and any real number $s \geq 0$, put

$$\omega_\alpha(f, s) = \frac{\Gamma(n+s+\alpha+1)}{\Gamma(\alpha+1)\Gamma(n+s)} \int_0^1 r^{n+s-1}(1-r)^\alpha \tilde{f}(r^{1/2}) \mathrm{d}r.$$

Remark 2.5 in [1] then says that T_f is diagonal with respect to the standard orthonormal basis. In fact we have

$$T_f = \sum_{m \in \mathbb{N}^n} \omega_\alpha(f, |m|) e_m \otimes e_m. \tag{0.1}$$

Here for any $g, h \in A^2_\alpha$, $g \otimes h$ denotes the operator on A^2_α defined by the formula $(g \otimes h)(\varphi) = \langle \varphi, h \rangle_\alpha g$ for all $\varphi \in A^2_\alpha$.

Recall that for $w \in \mathbb{B}_n$ and $0 < r < 1$, $E(w, r)$ denotes the ball centered at w with radius r in the pseudo-hyperbolic metric. If $f(z) = \chi_{E(0,\delta)}(z) =$

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$\chi_{[0,\delta)}(|z|)$ for some $0<\delta<1$ then for any $s\geq 0$,

$$\begin{aligned}\omega_\alpha\big(\chi_{E(0,\delta)},s\big) &= \frac{\Gamma(n+s+\alpha+1)}{\Gamma(\alpha+1)\Gamma(n+s)}\int_0^1 r^{n+s-1}(1-r)^\alpha\chi_{[0,\delta)}(r^{1/2})\mathrm{d}r\\ &= \frac{\Gamma(n+s+\alpha+1)}{\Gamma(\alpha+1)\Gamma(n+s)}\int_0^{\delta^2} r^{n+s-1}(1-r)^\alpha\mathrm{d}r.\end{aligned}$$

Since $\min\{1,(1-\delta^2)^\alpha\}\leq(1-r)^\alpha\leq\max\{1,(1-\delta^2)^\alpha\}$ for all $0\leq r\leq\delta^2$, we have

$$\begin{aligned}\min\{1,(1-\delta^2)^\alpha\} &\leq \omega_\alpha\big(\chi_{E(0,\delta)},s\big)\frac{\Gamma(\alpha+1)\Gamma(n+s)}{\Gamma(n+s+\alpha+1)}\frac{(n+s)}{\delta^{2(n+s)}}\\ &\leq \max\{1,(1-\delta^2)^\alpha\}.\end{aligned}$$

This then implies that for any $0<r<R<1$ and $s\geq 0$,

$$\begin{aligned}0\leq\frac{\omega_\alpha\big(\chi_{E(0,r)},s\big)}{\omega_\alpha\big(\chi_{E(0,R)},s\big)} &\leq \frac{\max\{1,(1-r^2)^\alpha\}}{\min\{1,(1-R^2)^\alpha\}}\Big(\frac{r}{R}\Big)^{2(n+s)}\\ &\leq \frac{\max\{1,(1-r^2)^\alpha\}}{\min\{1,(1-R^2)^\alpha\}}\Big(\frac{r}{R}\Big)^{2n}.\end{aligned}\tag{0.2}$$

Lemma 1. *Suppose* $0<R<1$ *and* $\delta>0$. *Then there exists* $\gamma=\gamma(R,\delta)$ *in* $(0,1)$ *so that for all* $0<r<\gamma$, $T_{\chi_{E(0,r)}}\leq\delta T_{\chi_{E(0,R)}}$.

Proof. From (0.1), we have

$$T_{\chi_{E(0,R)}}=\sum_{m\in\mathbb{N}^n}\omega_\alpha\big(\chi_{E(0,R)},|m|\big)e_m\otimes e_m,$$

and for any $0<r<1$,

$$T_{\chi_{E(0,r)}}=\sum_{m\in\mathbb{N}^n}\omega_\alpha\big(\chi_{E(0,r)},|m|\big)e_m\otimes e_m.$$

From equation (0.2) there is a γ in $(0,1)$ such that for any $0<r<\gamma$ and all $s\geq 0$, we have $0\leq\dfrac{\omega_\alpha\big(\chi_{E(0,r)},s\big)}{\omega_\alpha\big(\chi_{E(0,R)},s\big)}\leq\delta$. This then implies that $T_{\chi_{E(0,r)}}\leq\delta T_{\chi_{E(0,R)}}$. ∎

Theorem 2. *Let* $\{z_j : j\in J\}$ *be a separated sequence in* $\mathbb{B}_n$ *where* J *is either a non-empty finite set or* $\mathbb{N}$. *Let* $0<R<\bar{R}<1$ *and* $\mathcal{M}=\{R_j : j\in J\}\subset(0,1)$ *such that any limit point of* $\mathcal{M}$ *is either* 0 *or is in the open interval* $(R,\bar{R})$. *Suppose* W *is a set with non-empty interior that satisfies the following conditions:*

1. $W\subset\cup_{j\in J}E(z_j,R_j)$,
2. *There exists* $0<r<1$ *such that whenever* $R_j>R$ *(for some* $j\in J$*) we have* $E(z_j,r)\subset W$.

Let G be a linear subspace of $\chi_W L^\infty$ such that $C(\mathbb{B}) \cap \chi_W L^\infty \subset G$ and each function in G is a linear combinations of positive functions in G. Then we have $\mathcal{K} \subset \mathfrak{CT}(G) = \mathfrak{T}(G)$.

Remark 3. *Let W be a subset of $\mathbb{B}_n$ that satisfies the conditions in Theorem 2. Applying the theorem with $G = \chi_W L^\infty$, we see that $\mathfrak{CT}(\chi_W L^\infty) = \mathfrak{T}(\chi_W L^\infty)$. The case where $R_j > R$ for all $j \in J$ is Theorem 1.1 in* [1] *which extends an earlier result of the author in* [2]*. What interesting about Theorem 2 is the case where $R_j \to 0$, which is not covered by* [1]*. In this case the set W is only assumed to have non-empty interior, in addition to the condition that W is a subset of $\cup_{j\in J} E(z_j, R_j)$. Theorem 2 gives a necessary condition on W for the identity $\mathfrak{CT}(\chi_W L^\infty) = \mathfrak{T}(\chi_W L^\infty)$ to hold true.*

Proof of Theorem 2. Since W has an empty interior and $C(\mathbb{B}_n) \cap \chi_W L^\infty \subset G$, Remark 2.9 in [1] shows that $\mathcal{K} \subset \mathfrak{T}(G)$. This implies that $\mathcal{K} \subset \mathfrak{CT}(G)$.

Next, without loss of generality, we may assume that $R_j < \bar{R}$ for all $j \in J$. Choose $\widetilde{R}$ so that $\bar{R} < \widetilde{R} < 1$. By Lemma 3.1 in [1] there is a continuous function η which is supported in $E(0, r)$ such that $[T_\eta, T_{\bar{\eta}}]$ is an injective operator which is diagonal with respect to the standard orthonormal basis of A^2_α.

Let $\delta > 0$ be given. By Lemma 1 there is $0 < \gamma < R$ such that

$$T_{\chi_{E(0,\gamma)}} \leq \delta T_{\chi_{E(0,\tilde{R})}}. \tag{0.3}$$

By Lemma 2.6 in [1] there is a number λ so that

$$T_{\chi_{E(0,\bar{R})}} \leq \lambda [T_\eta, T_{\bar{\eta}}]^2 + \delta T_{\chi_{E(0,\tilde{R})}}. \tag{0.4}$$

Now let

$$N_1 = \{j \in J : R_j < \gamma\} \qquad \text{and} \qquad N_2 = \{j \in J : R_j > R\}.$$

Then by assumption about $\mathcal{M}$, the set $J \backslash (N_1 \cup N_2)$ is a finite set (possibly empty). For any $j \in N_1$, by applying U_{z_j} on both sides of inequality (0.3), we get

$$T_{\chi_{E(z_j,\gamma)}} = U_{z_j} T_{\chi_{E(0,\gamma)}} U_{z_j} \leq \delta U_{z_j} T_{\chi_{E(0,\tilde{R})}} U_{z_j} = \delta T_{\chi_{E(z_j,\tilde{R})}}.$$

This then implies $T_{\chi_{E(z_j,R_j)}} \leq T_{\chi_{E(z_j,\gamma)}} \leq \delta T_{\chi_{E(z_j,\tilde{R})}}$. For any $j \in N_2$, applying U_{z_j} on both sides of inequality (0.4) and arguing as above, we get

$$T_{\chi_{E(z_j,\bar{R})}} \leq \lambda [T_{\eta \circ \varphi_{z_j}}, T_{\bar{\eta} \circ \varphi_{z_j}}]^2 + \delta T_{\chi_{E(z_j,\tilde{R})}}.$$

Hence $T_{\chi_{E(z_j,R_j)}} \le T_{\chi_{E(z_j,\bar{R})}} \le \lambda[T_{\eta\circ\varphi_{z_j}}, T_{\bar{\eta}\circ\varphi_{z_j}}]^2 + \delta T_{\chi_{E(z_j,\widetilde{R})}}$. So

$$\begin{aligned}
T_{\chi_W} &\le \sum_{j\in J} T_{\chi_{E(z_j,R_j)}} \\
&= \sum_{j\in N_1} T_{\chi_{E(z_j,R_j)}} + \sum_{j\in N_2} T_{\chi_{E(z_j,R_j)}} + \sum_{j\in J\setminus N_1\cup N_2} T_{\chi_{E(z_j,R_j)}} \qquad (0.5)\\
&\le \delta \sum_{j\in J} T_{\chi_{E(z_j,\widetilde{R})}} + \lambda \sum_{j\in N_2} [T_{\eta\circ\varphi_{z_j}}, T_{\bar{\eta}\circ\varphi_{z_j}}]^2 + \sum_{j\in J\setminus N_1\cup N_2} T_{\chi_{E(z_j,R_j)}}
\end{aligned}$$

Since $\{z_j : j \in J\}$ is a separated sequence, we can decompose $J = J_1 \cup \cdots \cup J_M$ for some integer M so that $E(z_l,\widetilde{R}) \cap E(z_k,\widetilde{R}) = \emptyset$ for any $l \ne k$ in J_s, where $1 \le s \le M$ (see Lemma 2.3 in [2]). From this we have $\sum_{j\in J} T_{\chi_{E(z_j,\widetilde{R})}} \le M$. For any $j \in J$, the function $\eta\circ\varphi_{z_j}$ is continuous and supported in $E(z_j,r)$, hence it is in G. Proposition 2.3 in [1] shows that $\sum_{j\in N_2\cap J_s} [T_{\eta\circ\varphi_{z_j}}, T_{\bar{\eta}\circ\varphi_{z_j}}]^2$ belongs to $\mathfrak{CT}(G)$ for $1 \le s \le M$. Thus $\sum_{j\in N_2} [T_{\eta\circ\varphi_{z_j}}, T_{\bar{\eta}\circ\varphi_{z_j}}]^2$ belongs to $\mathfrak{CT}(G)$. Also since $T_{\chi_{E(z_j,R_j)}}$ is compact for any j in the finite set $J\setminus N_1 \cup N_2$, $\sum_{j\in J\setminus N_1\cup N_2} T_{\chi_{E(z_j,R_j)}}$ is compact, hence, in $\mathfrak{CT}(G)$. Let π denote the canonical quotient map from $\mathfrak{T}(G)$ onto the quotient algebra $\mathfrak{T}(G)/\mathfrak{CT}(G)$. We then have

$$\pi\Big(\sum_{j\in N_2\cap J_s} [T_{\eta\circ\varphi_{z_j}}, T_{\bar{\eta}\circ\varphi_{z_j}}]^2\Big) = 0 = \pi\Big(\sum_{j\in J\setminus N_1\cup N_2} T_{\chi_{E(z_j,R_j)}}\Big).$$

Let $0 \le f \le 1$ be any function in G. Then since $f \le \chi_W$, (0.5) gives

$$0 \le \pi(T_f) \le \pi(T_{\chi_W}) \le \delta M.$$

But δ was arbitrary, so we conclude that $\pi(T_f) = 0$ for any $f \in G$ with $0 \le f \le 1$. Since any function in G is a linear combination of positive functions in G, we see that $\pi(T_f) = 0$ for all $f \in G$. So $\mathfrak{CT}(G) = \mathfrak{T}(G)$. □

References

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