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ON THE COMMUTATOR IDEAL OF THE TOEPLITZ ALGEBRA ON THE BERGMAN SPACE OF THE UNIT BALL IN $\mathbb{C}^n$

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ABSTRACT. Let L^2_a denote the Bergman space of the open unit ball B_n in $\mathbb{C}^n$, for $n \geq 1$. The Toeplitz algebra $\mathfrak{T}$ is the C$^*-$algebra generated by all Toeplitz operators T_f with $f \in L^\infty$. It was proved by D. Suárez that for $n = 1$, the closed bilateral commutator ideal generated by operators of the form $T_f T_g - T_g T_f$, where $f, g \in L^\infty$, coincides with $\mathfrak{T}$. With a different approach, we can show that for $n \geq 1$, the closed bilateral ideal generated by operators of the above form, where f, g can be required to be continuous on the open unit ball or supported in a nowhere dense set, is also all of $\mathfrak{T}$.

KEYWORDS: *Commutator ideals, Toeplitz operators, Bergman spaces.*

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1. INTRODUCTION

For $n \geq 1$, let $\mathbb{C}^n$ denote the cartesian product of n copies of $\mathbb{C}$. For any two points $z = (z_1, \ldots, z_n)$ and $w = (w_1, \ldots, w_n)$ in $\mathbb{C}^n$, we use the notations $\langle z, w \rangle = z_1\overline{w}_1 + \cdots + z_n\overline{w}_n$ and $|z| = \sqrt{|z_1|^2 + \cdots + |z_n|^2}$ for the inner product and the associated Euclidean norm. Let B_n denote the open unit ball which consists of points $z \in \mathbb{C}^n$ with $|z| < 1$. Let $d\nu$ denote the Lebesgue measure on B_n so normalized that $\nu(B_n) = 1$. Let $\mathrm{d}\mu(z) = (1 - |z|^2)^{-n-1}\,\mathrm{d}\nu(z)$. Then $\mathrm{d}\mu$ is invariant under the action of the group of automorphisms $\mathrm{Aut}(B_n)$ of B_n. Even though $\mathrm{d}\mu$ is an unbounded measure on B_n, it will be very useful for us later.

Let $L^2 = L^2(B_n, \mathrm{d}\nu)$ and $L^\infty = L^\infty(B_n, \mathrm{d}\nu)$. The Bergman space L^2_a is the subspace of L^2 which consists of all holomorphic functions. The orthogonal projection from L^2 onto L^2_a is given by

$$Pf(z) = \int_{B_n} \frac{f(w)}{(1 - \langle z, w \rangle)^{n+1}}\,\mathrm{d}\nu(w),\ f \in L^2,\ z \in B_n.$$

The normalized reproducing kernels for L^2_a are of the form

$$k_z(w) = (1-|z|^2)^{(n+1)/2}(1-\langle w, z\rangle)^{-n-1},\ |z|,|w| < 1.$$

We have $\|k_z\| = 1$ and $\langle g, k_z\rangle = (1-|z|^2)^{(n+1)/2}g(z)$ for all $g \in L^2_a$.

Let $\mathfrak{B}(L^2_a)$ be the C^*-algebra of all bounded linear operators on L^2_a. Let $\mathcal{K}$ denote the ideal of compact operators on L^2_a.

For any $\eta \in L^\infty$ let $M_\eta : L^2 \longrightarrow L^2$ be the operator of multiplication by η and $P_\eta = PM_\eta$. Then $\|P_\eta\| \le \|\eta\|_\infty$. The Toeplitz operator $T_\eta : L^2_a \longrightarrow L^2_a$ is the restriction of P_η to L^2_a. For any subset G of L^∞, let $\mathfrak{T}(G)$ denote the C^*-subalgebra of $\mathfrak{B}(L^2_a)$ generated by $\{T_\eta : \eta \in G\}$. The commutator ideal of this algebra is denoted by $\mathfrak{CT}(G)$. It is well-know that $\mathfrak{CT}(C(\bar{B_n}))$ is contained in $\mathcal{K}$, see [1]. The algebra $\mathfrak{T}(L^\infty)$ which is generated by all Toeplitz operators with bounded symbols is called the full Toeplitz algebra. Its commutator ideal is $\mathfrak{CT}(L^\infty)$.

There have been many results on commutator ideals and abelianizations of Toeplitz algebras acting on Hardy spaces. In contrast with this, there are only few results for Toeplitz algebras on Bergman spaces. Recently, Suárez showed in [5] that the Toeplitz algebra $\mathfrak{T}(L^\infty)$ on the Bergman space of the unit disk coincides with its commutator ideal $\mathfrak{CT}(L^\infty)$. In his paper, Suárez used some explicit computations and identities which are readily available on the unit disk to construct a function $\eta \in L^\infty$ with the property that $\eta > c > 0$ on the disk and T_η is in the commutator ideal $\mathfrak{CT}(L^\infty)$. In higher dimensions, the computations become more complicated and some of the identities which were used by Suárez are not available. We could not find a way to get around these difficulties to construct a function similar to that of Suárez so we tried a different approach. It turns out that our new approach gives more general results about commutator ideals of the Toeplitz algebras. Indeed, we do not need G to be all the functions in L^∞ to get $\mathfrak{CT}(G) = \mathfrak{T}(L^\infty)$. We can take G to be $L^\infty \cap C(B_n)$ - the set of all bounded continuous functions on the open unit ball or we can take G to be all the functions in L^∞ which are supported in a set E where E can be a nowhere dense set with $\nu(E)$ as small as we please.

We next describe a metric on the unit ball which we will mainly use in this paper. For any $z \in B_n$, let φ_z denote the Mobius automorphism of B_n that interchanges 0 and z. For any $z, w \in B_n$, let $\rho(z,w) = |\varphi_z(w)|$. Then ρ is a metric which is invariant under the action of the group of automorphisms Aut(B_n) of B_n. These properties of ρ can be proved by using identities in Theorem 2.2.2 in [4]. Further discussion of this metric will appear later in Section 2.

A collection $\mathcal{W} = \{w_j : j \in J\}$ of points in B_n is said to be separated if $r = \inf\{\rho(w_j, w_k) : j \neq k\} > 0$. It is a consequence of Lemma 2.1 that in this case the index set J is necessarily at most countable. The number r is called the degree of separation of $\mathcal{W}$.

For $z \in B_n$ and $0 < r < 1$, let

$$E(z,r) = \{w \in B_n : \rho(w,z) \le r\}$$

denote the closed $r-$ball centered at z in the ρ metric.

THEOREM 1.1. *Let* $\{w_j : j \in \mathbb{N}\}$ *be a separated sequence of points in* B_n *so that* $B_n = \bigcup_{j\in\mathbb{N}} E(w_j, R)$ *for some* $0 < R < 1$. *Let* η *be a measurable function defined on* $[0,\infty)$ *with* $\eta \geq 0$, $\eta(t) = 0$ *if* $t \geq 1$ *and* $\|\eta\|_\infty = 1$. *For each* $0 < \epsilon < 1$ *put* $\eta_\epsilon(z) = z_1\eta(|z|/\epsilon)$. *Let* G_ϵ *be the set of all functions of the form* $\sum_{j\in F} \eta_\epsilon \circ \varphi_{w_j}$ *or* $\sum_{j\in F} \overline{\eta}_\epsilon \circ \varphi_{w_j}$ *where* F *is a subset of* $\mathbb{N}$. *Then the operator*

$$A_\epsilon = \sum_{j\in\mathbb{N}} [T_{\eta_\epsilon\circ\varphi_{w_j}}, T_{\overline{\eta}_\epsilon\circ\varphi_{w_j}}]^2$$

belongs to the commutator ideal $\mathfrak{CT}(G_\epsilon)$. *Furthermore, for all but countably many* ϵ, *the operator* A_ϵ *is invertible.*

Put $E_\epsilon = \bigcup_{j\in\mathbb{N}} \varphi_{w_j}(\mathrm{supp}(\eta_\epsilon))$.

Then G_ϵ is contained in the subspace $\{\zeta \in L^\infty : \zeta \text{ is supported on } E_\epsilon\}$. If η is supported in a nowhere dense subset of $[0,1]$ then η_ϵ is supported in a nowhere dense subset of B_n, hence E_ϵ - being a countable union of nowhere dense sets, is a nowhere dense subset of B_n, too. Furthermore, we will show that for $\epsilon > 0$, the Lebesgue measure of E_ϵ is $O(\epsilon^{2n})$. We will also show that if η is a continuous function then G_ϵ is a subspace of $C(B_n)$ for all $0 < \epsilon < 1$.

The fact that A_ϵ belongs to the ideal $\mathfrak{CT}(G_\epsilon)$ is proved exactly as in Suárez's paper. The reason is that all the properties of the metric ρ and the kernel functions which were crucial for Suárez's proof hold true in higher dimensions.

The invertibility of A_ϵ follows from a general fact about operators which are diagonalizable with respect to the standard orthonormal basis of L^2_a. In fact, sums of a 'large enough' number of operators which are unitarily equivalent to operators of the above type are invertible. This is the content of Theorem 1.2 which follows.

For any $z \in B_n$, the formula

$$U_z(f) = (f \circ \varphi_z)k_z, \quad f \in L^2$$

defines a bounded operator on L^2. It is well-known that U_z is a unitary self-adjoint operator and $U_z T_\eta U_z^* = T_{\eta\circ\varphi_z}$ for all $z \in B_n$ and all $\eta \in L^\infty$, see, for example, Lemma 7 and 8 in [3].

Also a simple computation reveals that for all $z, w \in B_n$,

$$U_z(k_w) = \left(\frac{|1 - \langle z, w\rangle|}{1 - \langle z, w\rangle}\right)^{n+1} k_{\varphi_z(w)}.$$

This implies

$$U_z(k_w \otimes k_w)U_z^* = k_{\varphi_z(w)} \otimes k_{\varphi_z(w)}.$$

Now for any multi-index $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$, let $|\alpha| = \alpha_1 + \cdots \alpha_n$, $\alpha! = \alpha_1! \cdots \alpha_n!$ and $z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$. Put

$$e_\alpha = \left(\frac{(n+|\alpha|)!}{n!\alpha!} \right)^{1/2} z^\alpha.$$

Then $\{e_\alpha : \alpha \in \mathbb{N}^n\}$ is the standard orthonormal basis for L^2_a, see Proposition 1.4.9 in [4].

Recall that for any two elements f and g in L^2_a, $f \otimes g$ denote the rank one operator $(f \otimes g)u = \langle u, g \rangle f$, for all $u \in L^2_a$.

THEOREM 1.2. *Let $\{s_\alpha : \alpha \in \mathbb{N}^n\}$ be a bounded set of strictly positive real numbers. Let*

$$S = \sum_{\alpha \in \mathbb{N}^n} s_\alpha e_\alpha \otimes e_\alpha.$$

Let $\{w_j : j \in \mathbb{N}\}$ be a separated sequence of points in B_n so that $B_n = \bigcup_{j \in \mathbb{N}} E(w_j, R)$ for some $0 < R < 1$. Then there is a positive constant c so that

$$\sum_{j \in \mathbb{N}} U_{w_j} S U^*_{w_j} \geq c > 0.$$

In the rest of the paper, we will state and prove a couple of lemmas and propositions before giving the proof for Theorem 1.2 in Section 3 and then Theorem 1.1 in Section 4. Some remarks about Theorem 1.1 will be presented in Section 5.

2. BASIC RESULTS

The following inequalities illustrate the fact that the metric ρ in higher dimensions also possesses all the properties used in Suárez's paper. These results are well-known but since we are not aware of an appropriate reference, we sketch here a proof.

LEMMA 2.1. *For any z, w in B_n, the followings hold*

$$\left| \frac{|z| - |w|}{1 - |z||w|} \right| \leq \rho(z, w) \leq \frac{|z - w|}{|1 - \langle z, w \rangle|}.$$

Proof. Using $|\langle z, w \rangle| \leq |z||w|$, we get the inequalities

$$1 - \frac{|z-w|^2}{|1 - \langle z, w \rangle|^2} \leq \frac{(1-|z|^2)(1-|w|^2)}{|1 - \langle z, w \rangle|^2} \leq \frac{(1-|z|^2)(1-|w|^2)}{(1-|z||w|)^2}.$$

Combining the above inequalities with the identity

$$1 - |\varphi_z(w)|^2 = \frac{(1-|z|^2)(1-|w|^2)}{|1 - \langle z, w \rangle|^2} \quad \text{(see Theorem 2.2.2 in [4])}$$

we obtain

$$1 - \frac{|z-w|^2}{|1-\langle z,w\rangle|^2} \le 1 - |\varphi_z(w)|^2 \le 1 - \frac{(|z|-|w|)^2}{(1-|z||w|)^2},$$

from which the stated inequalities follow. ∎

From Lemma 2.1 and the invariance of ρ under the action of $\mathrm{Aut}(B_n)$, we have for any $z, w, u \in B_n$,

$$\begin{aligned}\rho(z,w) &= \rho(\varphi_u(z), \varphi_u(w)) \\ &\ge \Big|\frac{|\varphi_u(z)| - |\varphi_u(w)|}{1-|\varphi_u(z)||\varphi_u(w)|}\Big| \\ &= \frac{|\rho(z,u)-\rho(u,w)|}{1-\rho(z,u)\rho(u,w)}.\end{aligned} \tag{2.1}$$

From the second inequality in Lemma 2.1, we see that if $|z|, |w| \le R < 1$ then

$$\rho(z,w) \le \frac{|z-w|}{|1-\langle z,w\rangle\,|} \le \frac{|z-w|}{1-R^2}. \tag{2.2}$$

For all $0 < r < 1$ and all $0 < R < 1$, from the compactness of $E(0,R)$ in the Euclidean metric, there is an M which depends only on n, r and R so that if $\{w_1, \ldots, w_m\}$ is a subset of $E(0,R)$ and $|w_j - w_k| \ge (1-R^2)r$ for all $j \ne k$ then $m \le M$. Then (2.2) implies that if $\{w_1 \ldots, w_m\}$ is a subset of $E(0,R)$ so that $\rho(w_j, w_k) \ge r$ for all $j \ne k$ then $m \le M$.

The above properties of ρ allow us to prove the following characteristic of a separated collection of points in B_n.

LEMMA 2.2. *Let $\{w_j : j \in J\}$ be a collection of points in B_n so that $\rho(w_j, w_k) > r$ for all $j \ne k$, where $0 < r < 1$. Let $0 < R_1, R_2 < 1$ be given. Then there is an N depending only on n, r, R_1 and R_2 so that for any $u \in B_n$ the set $\{j \in J : E(u, R_1) \cap E(w_j, R_2) \ne \emptyset\}$ has at most N elements.*

Proof. By applying the Mobius automorphism that interchanges 0 and u if necessary, we can assume without loss of generality that $u = 0$. Let $\tilde{R} = \dfrac{R_1+R_2}{1+R_1R_2}$. Suppose $z, w \in B_n$ with $|w| \le R_1$ and $|z| > \tilde{R}$. Then from Lemma 2.1,

$$\rho(z,w) \ge \frac{|z|-|w|}{1-|w||z|} > \frac{\tilde{R}-R_1}{1-\tilde{R}R_1} = R_2.$$

So $E(0,R_1) \cap E(z,R_2) \ne \emptyset$ implies that $|z| \le \tilde{R}$. Hence, $\{j \in J : E(0,R_1) \cap E(z_j, R_2) \ne \emptyset\}$ is a subset of the set $\{j \in J : |w_j| \le \tilde{R}\}$. From the remark preceding the lemma, the second set has at most N elements, where N depends only on n, r, R_1 and R_2. The conclusion of the lemma follows from here. ∎

The following lemma is similar to Lemma 2.1 in [5] but somewhat stronger even though the proof is almost identical. We state here the lemma and give the proof, too.

LEMMA 2.3. *Let $\mathcal{W} = \{w_j : j \in J\}$ be a separated collection of points in B_n and $0 < \sigma < 1$. Then there is a finite decomposition $\mathcal{W} = \mathcal{W}_1 \cup \cdots \cup \mathcal{W}_N$ such that for every $1 \le i \le N$, $E(z,\sigma) \cap E(w,\sigma) = \emptyset$ for all $z \neq w$ in $\mathcal{W}_i$.*

Proof. Let $\mathcal{W}_1 \subset \mathcal{W}$ be a maximal subset so that $E(z,\sigma) \cap E(w,\sigma) = \emptyset$ for all $z \neq w$ in $\mathcal{W}_1$. If $\mathcal{W}_1 = \mathcal{W}$ we are done. Otherwise suppose that $m \ge 2$ and $\mathcal{W}_1, \ldots, \mathcal{W}_{m-1}$ are chosen so that $E(z,\sigma) \cap E(w,\sigma) = \emptyset$ for all $z \neq w$ in $\mathcal{W}_i$, all $1 \le i \le m-1$ and $\mathcal{W} \backslash (\mathcal{W}_1 \cup \cdots \cup \mathcal{W}_{m-1}) \neq \emptyset$. Let $\mathcal{W}_m \subset \mathcal{W} \backslash (\mathcal{W}_1 \cup \cdots \cup \mathcal{W}_{m-1})$ be a maximal subset so that $E(z,\sigma) \cap E(w,\sigma) = \emptyset$ for all $z \neq w$ in $\mathcal{W}_m$. By the maximality at each of the previous steps, if $u \in \mathcal{W}_m$ then for every $1 \le i \le m-1$, there is a $u_i \in \mathcal{W}_i$ so that $E(u_i,\sigma) \cap E(u,\sigma) \neq \emptyset$. Therefore $\{u, u_1, \ldots, u_{m-1}\} \subset \{j \in J : E(u,\sigma) \cap E(w_j,\sigma) \neq \emptyset\}$. From Lemma 2.2, there is an N depending on n, σ and the degree of separation of $\mathcal{W}$ so that $m \le N$. ∎

From now to the end of this section, fix an $r \in (0,1)$ and a sequence of points $\mathcal{W} = \{w_j : j \in \mathbb{N}\}$ in B_n so that $E(w_j, r) \cap E(w_k, r) = \emptyset$ for all $j \neq k$ in $\mathbb{N}$.

Now we state a couple of lemmas which are in Suárez's paper for the case $n = 1$ and for L^p_a with $1 < p < \infty$, see Lemma 2.4-2.6 in [5]. Here we are interested in the case $n \ge 2$ and $p = 2$. The conclusions of those lemmas in our case still hold true with no major changes in the proofs.

LEMMA 2.4. *Let $0 < \beta < 1$ and $r < R < 1$ and let*

$$\Phi(z,w) = \sum_{j \in \mathbb{N}} \chi_{E(w_j,r)}(z) \chi_{B_n \backslash E(w_j,R)}(w) |1 - \langle z, w \rangle|^{-n-1}.$$

Then

$$\int_{B_n} \Phi(z,w)(1-|z|^2)^{-\beta} \, \mathrm{d}\nu(z) \le c_1(\beta)(1-|w|^2)^{-\beta},$$

where $c_1(\beta) > 0$.

LEMMA 2.5. *Let $0 < \beta < 1$ and $r < R < 1$ and $\Phi(z,w)$ as in Lemma 2.4. Then*

$$\int_{B_n} \Phi(z,w)(1-|w|^2)^{-\beta} \, \mathrm{d}\nu(w) \le c_2(\beta, R)(1-|z|^2)^{-\beta},$$

where $c_2(\beta, R) \to 0$ when $R \to 1$.

LEMMA 2.6. *Suppose that $R \in (r, 1)$ and $a_j, A_j \in L^\infty$ are functions of norm ≤ 1 such that*

$$\operatorname{supp} a_j \subset E(w_j, r) \text{ and } \operatorname{supp} A_j \subset B_n \backslash E(w_j, R).$$

Then the operator $\sum_{j \in \mathbb{N}} M_{a_j} P M_{A_j}$ is bounded on L^2, with norm bounded by some constant $k(R) \to 0$ when $R \to 1$.

The following proposition is the case $n \geq 1$ and $p = 2$ of Proposition 2.9 in [5]. Since we have all the needed properties of the metric ρ and all the necessary lemmas, the proof is identical to that of Suárez.

PROPOSITION 2.7. *For each $j \in \mathbb{N}$, let $c_j^1, \ldots, c_j^l, a_j, b_j, d_j^1, \ldots, d_j^m \in L^\infty$ be functions of norm ≤ 1 supported on $E(w_j, r)$. Then*

$$\sum_{j\in\mathbb{N}} T_{c_j^1} \ldots T_{c_j^l}(T_{a_j}T_{b_j} - T_{b_j}T_{a_j})T_{d_j^1} \cdots T_{d_j^m}$$

belongs to the commutator ideal $\mathfrak{CT}(L^\infty)$ of the full Toeplitz algebra.

In the proof of Proposition 2.7, we are dealing only with Toeplitz operators with symbols in the subset G of L^∞ which consists of functions of the form $\sum\limits_{j\in F} f_j$, where F is a subset of $\mathbb{N}$ and f is one of the symbols $c^1, \ldots, c^l, a, b, d^1, \ldots, d^m$. So in the above conclusion, we can replace $\mathfrak{CT}(L^\infty)$ by the smaller ideal $\mathfrak{CT}(G)$.

3. INVERTIBILITY OF SUMS OF RANK ONE PROJECTIONS

From now to the end of this section, fix a bounded set $\{s_\alpha : \alpha \in \mathbb{N}^n\}$ of strictly positive real numbers.

LEMMA 3.1. *Fix $0 < R < 1$ and $\epsilon > 0$ so that $(1+\epsilon)R < 1$. Let $\delta > 0$ be given. Then there is a constant $C(\delta) > 0$ so that for all $|z| \leq R$,*

$$k_z \otimes k_z \leq C(\delta) \sum_{\alpha\in\mathbb{N}^n} s_\alpha\, e_\alpha \otimes e_\alpha + \delta \int\limits_{|w|<(1+\epsilon)R} k_w \otimes k_w \,\mathrm{d}\mu(w). \tag{3.1}$$

Proof. Let f be in L_a^2 and $|z| \leq R$. Let J be a finite subset of $\mathbb{N}^n$.
Put

$$g_J = \sum_{\alpha\in J} \langle f, e_\alpha\rangle e_\alpha \quad \text{and} \quad h_J = \sum_{\alpha\in\mathbb{N}^n\setminus J} \langle f, e_\alpha\rangle e_\alpha.$$

Then

$$\begin{aligned} \langle (k_z \otimes k_z) f, f\rangle &= |\langle f, k_z\rangle|^2 \\ &= |\langle g_J, k_z\rangle + \langle h_J, k_z\rangle|^2 \\ &\leq 2(|\langle g_J, k_z\rangle|^2 + |\langle h_J, k_z\rangle|^2). \end{aligned} \tag{3.2}$$

Now,

$$
\begin{aligned}
\big|\langle h_J, k_z\rangle\big|^2 &= \Big|\sum_{\alpha\in\mathbb{N}^n\setminus J}\langle f, e_\alpha\rangle\langle e_\alpha, k_z\rangle\Big|^2\\
&= (1-|z|^2)^{n+1}\Big|\sum_{\alpha\in\mathbb{N}^n\setminus J}\langle f, e_\alpha\rangle e_\alpha(z)\Big|^2\\
&\le \Big|\sum_{\alpha\in\mathbb{N}^n\setminus J}\langle f, e_\alpha\rangle e_\alpha(z)\Big|^2\\
&\le \Big(\sum_{\alpha\in\mathbb{N}^n\setminus J}|\langle f, e_\alpha\rangle|\Big(\frac{(n+|\alpha|)!}{n!\alpha!}\Big)^{1/2}|z^\alpha|\Big)^2 \qquad (3.3)\\
&\le \Big(\sum_{\alpha\in\mathbb{N}^n\setminus J}|\langle f, e_\alpha\rangle|^2((1+\epsilon)R)^{2|\alpha|}\Big)\\
&\qquad\times\Big(\sum_{\alpha\in\mathbb{N}^n\setminus J}\frac{(n+|\alpha|)!}{n!\alpha!}|z^\alpha|^2((1+\epsilon)R)^{-2|\alpha|}\Big).
\end{aligned}
$$

On the other hand, the homogeneity of the e_α's shows that

$$f((1+\epsilon)R\zeta) = \sum_{\alpha\in\mathbb{N}^n}\langle f, e_\alpha\rangle((1+\epsilon)R)^{|\alpha|}e_\alpha(\zeta)$$

so that the change-of-variable $w = (1+\epsilon)R\zeta$ gives

$$
\begin{aligned}
\int\limits_{|w|<(1+\epsilon)R}\langle(k_w\otimes k_w)f, f\rangle\,\mathrm{d}\mu(w) &= \int\limits_{|w|<(1+\epsilon)R}|f(w)|^2\,\mathrm{d}\nu(w)\\
&= ((1+\epsilon)R)^{2n}\int\limits_{B_n}|f(1+\epsilon)R\zeta|^2\,\mathrm{d}\nu(\zeta)\\
&= ((1+\epsilon)R)^{2n}\sum_{\alpha\in\mathbb{N}^n}|\langle f, e_\alpha\rangle|^2((1+\epsilon)R)^{2|\alpha|} \qquad (3.4)\\
&\ge ((1+\epsilon)R)^{2n}\sum_{\alpha\in\mathbb{N}^n\setminus J}|\langle f, e_\alpha\rangle|^2((1+\epsilon)R)^{2|\alpha|}.
\end{aligned}
$$

This implies

(3.5)
$$\sum_{\alpha\in\mathbb{N}^n\setminus J}|\langle f, e_\alpha\rangle|^2((1+\epsilon)R)^{2|\alpha|} \le (1+\epsilon)R)^{-2n}\int\limits_{|w|<(1+\epsilon)R}\langle(k_w\otimes k_w)f, f\rangle\,\mathrm{d}\mu(w).$$

Inequalities (3.3) and (3.5) imply

$$
\begin{aligned}
|\langle h_J, k_z\rangle|^2 \le \Big(\sum_{\alpha\in\mathbb{N}^n\setminus J}\frac{(n+|\alpha|)!}{n!\alpha!}|z^\alpha|^2((1+\epsilon)R)^{-2|\alpha|}\Big)((1+\epsilon)R)^{-2n}\\
\times\int\limits_{|z|<(1+\epsilon)R}\langle(k_w\otimes k_w)f, f\rangle\,\mathrm{d}\mu(w).
\end{aligned}
$$

Now from the identity

$$K_w(\zeta) = \sum_{\alpha\in\mathbb{N}^n} \overline{e_\alpha(w)} e_\alpha(\zeta),$$

for $w, \zeta \in B_n$, where $K_w(\zeta)$ is the Bergman reproducing kernel, we have

$$\sum_{\alpha\in\mathbb{N}^n} |e_\alpha(w)|^2 = K_w(w) = \frac{1}{(1-|w|^2)^{n+1}}.$$

If we take $w = z/((1+\epsilon)R)$, where $|z| \le R$, we obtain

$$\begin{aligned}\sum_{\alpha\in\mathbb{N}^n} \frac{(n+|\alpha|)!}{n!\alpha!} |z^\alpha|^2((1+\epsilon)R)^{-2|\alpha|} &= \sum_{\alpha\in\mathbb{N}^n} |e_\alpha(z/((1+\epsilon)R))|^2 \\ &\le \frac{1}{(1-1/(1+\epsilon)^2)^{n+1}}.\end{aligned} \tag{3.6}$$

So there is a finite subset J of $\mathbb{N}^n$ which is independent of z so that

$$\sum_{\alpha\in\mathbb{N}^n\setminus J} \frac{(n+|\alpha|)!}{n!\alpha!} |z^\alpha|^2((1+\epsilon)R)^{-2|\alpha|} \le \frac{\delta}{2}((1+\epsilon)R)^{2n}.$$

Hence for this J,

$$|\langle h_J, k_z\rangle|^2 \le \frac{\delta}{2} \int\limits_{|w|<(1+\epsilon)R} \langle (k_w\otimes k_w)f, f\rangle \,\mathrm{d}\mu(w). \tag{3.7}$$

Also,

$$|\langle g_J, k_z\rangle|^2 \le \|g_J\|^2 = \sum_{\alpha\in J} |\langle f, e_\alpha\rangle|^2. \tag{3.8}$$

From inequalities (3.2), (3.7) and (3.8), we conclude that

$$\langle (k_z\otimes k_z)f, f\rangle \le 2\sum_{\alpha\in J} \langle (e_\alpha\otimes e_\alpha)f, f\rangle + \delta \int\limits_{|w|<(1+\epsilon)R} \langle (k_w\otimes k_w)f, f\rangle \,\mathrm{d}\mu(w).$$

Since $s_\alpha > 0$ for all $\alpha \in J$ and J is finite, there is a constant $C(\delta) > 0$ so that $C(\delta)s_\alpha \ge 2$ for all $\alpha \in J$. Then for any $f \in L^2_a$, and any $|z| \le R$,

$$\begin{aligned}\langle (k_z\otimes k_z)f, f\rangle &\le C(\delta)\sum_{\alpha\in J} s_\alpha\langle (e_\alpha\otimes e_\alpha)f, f\rangle + \delta \int\limits_{|w|<(1+\epsilon)R} \langle (k_w\otimes k_w)f, f\rangle \,\mathrm{d}\mu(w) \\ &\le C(\delta)\sum_{\alpha\in\mathbb{N}^n} s_\alpha\langle (e_\alpha\otimes e_\alpha)f, f\rangle + \delta \int\limits_{|w|<(1+\epsilon)R} \langle (k_w\otimes k_w)f, f\rangle \,\mathrm{d}\mu(w).\end{aligned}$$

In other words, for any $|z| \le R$,

$$k_z\otimes k_z \le C(\delta)\sum_{\alpha\in\mathbb{N}^n} s_\alpha e_\alpha\otimes e_\alpha + \delta \int\limits_{|w|<(1+\epsilon)R} k_w\otimes k_w \,\mathrm{d}\mu(w). \quad ∎$$

Proof of Theorem 2. Let $S = \sum_{\alpha\in\mathbb{N}^n} s_\alpha\, e_\alpha \otimes e_\alpha$ and $\mathcal{W} = \{w_j : j \in \mathbb{N}\}$ be as in the hypothesis of Theorem 1.2. Choose an $\epsilon > 0$ so that $(1+\epsilon)R < 1$.

For each $a \in B_n$, apply U_a to the left and U_a^* to the right of both sides of inequality (3.1) in Lemma 3.1, we get

$$\begin{aligned}
U_a(k_z \otimes k_z)U_a^* &\le C(\delta)U_a S U_a^* + \delta \int_{|w|<(1+\epsilon)R} U_a(k_w \otimes k_w)U_a^* \,\mathrm{d}\mu(w)\\
&= C(\delta)U_a S U_a^* + \delta \int_{|w|<(1+\epsilon)R} k_{\varphi_a(w)} \otimes k_{\varphi_a(w)} \,\mathrm{d}\mu(w)\\
&= C(\delta)U_a S U_a^* + \delta \int_{|\varphi_a(\zeta)|<(1+\epsilon)R} k_\zeta \otimes k_\zeta \,\mathrm{d}\mu(\zeta)\\
&\qquad \text{(by the change-of-variable } w = \varphi_a(\zeta))\\
&= C(\delta)U_a S U_a^* + \delta \int_{E(a,(1+\epsilon)R)} k_\zeta \otimes k_\zeta \,\mathrm{d}\mu(\zeta),
\end{aligned}$$

for $|z| \le R$.

Since $U_a(k_z \otimes k_z)U_a^* = k_{\varphi_a(z)} \otimes k_{\varphi_a(z)}$, the above implies

$$k_{\varphi_a(z)} \otimes k_{\varphi_a(z)} \le C(\delta)U_a S U_a^* + \delta \int_{E(a,(1+\epsilon)R)} k_\zeta \otimes k_\zeta \,\mathrm{d}\mu(\zeta). \tag{3.9}$$

For each $|z| \le R$, let

$$T(z) = \sum_{j\in\mathbb{N}} k_{\varphi_{w_j}(z)} \otimes k_{\varphi_{w_j}(z)}.$$

Then (3.9) gives

$$T(z) \le C(\delta) \sum_{j\in\mathbb{N}} U_{w_j} S U_{w_j}^* + \delta \sum_{j\in\mathbb{N}} \int_{E(w_j,(1+\epsilon)R)} k_\zeta \otimes k_\zeta \,\mathrm{d}\mu(\zeta), \tag{3.10}$$

for $|z| \le R$.

Decompose $\mathcal{W} = \mathcal{W}_1 \cup \cdots \cup \mathcal{W}_N$ as in Lemma 2.3, where N depends only on n, $(1+\epsilon)R$ and the degree of separation of $\mathcal{W}$. Then

$$\begin{aligned}
\sum_{j\in\mathbb{N}} \int_{E(w_j,(1+\epsilon)R)} k_\zeta \otimes k_\zeta \,\mathrm{d}\mu(\zeta) &\le \sum_{i=1}^{N} \sum_{w\in\mathcal{W}_i} \int_{E(w,(1+\epsilon)R)} k_\zeta \otimes k_\zeta \,\mathrm{d}\mu(\zeta)\\
&\le \sum_{i=1}^{N} \int_{B_n} k_\zeta \otimes k_\zeta \,\mathrm{d}\mu(\zeta) = N.
\end{aligned}$$

Hence for $|z| \le R$,

$$T(z) \le C(\delta) \sum_{j\in\mathbb{N}} U_{w_j} S U_{w_j}^* + \delta N. \tag{3.11}$$

By integrating $T(z)$ with respect to $\mathrm{d}\nu(z)$ over the ball $|z| < R$, we get

$$
\begin{aligned}
\int\limits_{|z|<R} T(z)\,\mathrm{d}\nu(z) &\geq (1-R^2)^{n+1} \int\limits_{|z|<R} T(z)(1-|z|^2)^{-(n+1)}\,\mathrm{d}\nu(z)\\
&= (1-R^2)^{n+1} \int\limits_{|z|<R} T(z)\,\mathrm{d}\mu(z)\\
&= (1-R^2)^{n+1} \sum_{j\in\mathbb{N}} \int\limits_{|z|<R} k_{\varphi_{w_j}(z)} \otimes k_{\varphi_{w_j}(z)}\,\mathrm{d}\mu(z)\\
&= (1-R^2)^{n+1} \sum_{j\in\mathbb{N}} \int\limits_{E(w_j,R)} k_\zeta \otimes k_\zeta\,\mathrm{d}\mu(\zeta) \qquad (3.12)\\
&\geq (1-R^2)^{n+1} \int\limits_{B_n} k_\zeta \otimes k_\zeta\,\mathrm{d}\mu(\zeta)\\
&\quad \left(\text{since } B_n = \bigcup_{j\in\mathbb{N}} E(w_j,R)\right)\\
&= (1-R^2)^{n+1}.
\end{aligned}
$$

Inequalities (3.11) and (3.12) together imply

$$C(\delta) \sum_{j\in\mathbb{N}} U_{w_j} S U_{w_j}^* + \delta N \geq (1-R^2)^{n+1} R^{-2n}.$$

Now choose δ so small that

$$\delta N \leq 2^{-1}(1-R^2)^{n+1} R^{-2n}.$$

Then we have

$$C(\delta) \sum_{j\in\mathbb{N}} U_{w_j} S U_{w_j}^* \geq 2^{-1}(1-R^2)^{n+1} R^{-2n} > 0. \qquad (3.13) \quad \blacksquare$$

4. PROOF OF THE MAIN THEOREM

Suppose $\eta_\epsilon(z) = z_1\eta(|z|/\epsilon)$ for all $z = (z_1, \ldots, z_n) \in B_n$ as in the hypothesis of Theorem 1.1. We will compute directly $[T_{\eta_\epsilon}, T_{\overline{\eta}_\epsilon}]$ to see that it is a diagonal operator with respect to the standard orthonormal basis.

For any multi-indices α and β in $\mathbb{N}^n$, we have

$$
\begin{aligned}
\langle T_{\eta_\epsilon} e_\alpha, e_\beta\rangle &= \int\limits_{B_n} \eta_\epsilon(z) e_\alpha(z) \overline{e}_\beta(z)\,\mathrm{d}\nu(z)\\
&= \int\limits_{|z|<\epsilon} \eta(|z|/\epsilon) z_1 e_\alpha(z) \overline{e}_\beta(z)\,\mathrm{d}\nu(z).
\end{aligned}
$$

Now,

$$\begin{aligned} z_1 e_\alpha(z) &= \left(\frac{(n+|\alpha|)!}{n!\alpha!}\right)^{1/2} z_1 z^\alpha \\ &= \left(\frac{(n+|\alpha|)!}{n!\alpha!}\ \frac{n!(\alpha+(1,0,\dots,0))!}{(n+|\alpha|+1)!}\right)^{1/2} e_{\alpha+(1,0,\dots,0)}(z) \\ &= \left(\frac{\alpha_1+1}{n+|\alpha|+1}\right)^{1/2} e_{\alpha+(1,0,\dots,0)}(z). \end{aligned}$$

So

$$\begin{aligned} &\langle T_{\eta_\epsilon} e_\alpha, e_\beta\rangle \\ &= \left(\frac{\alpha_1+1}{n+|\alpha|+1}\right)^{1/2} \int\limits_{|z|<\epsilon} \eta(|z|/\epsilon) e_{\alpha+(1,0,\dots,0)}(z)\bar{e}_\beta(z)\,\mathrm{d}\nu(z) \\ &= \left(\frac{\alpha_1+1}{n+|\alpha|+1}\right)^{1/2} \int\limits_0^\epsilon (2n) r^{2n-1}\eta(r/\epsilon) \int\limits_{S^n} e_{\alpha+(1,0,\dots,0))}(r\zeta)\bar{e}_\beta(r\zeta)\,d\sigma(\zeta)\,\mathrm{d}r \\ &= \left(\frac{\alpha_1+1}{n+|\alpha|+1}\right)^{1/2} \\ &\quad\times \begin{cases} 0 & \text{if } \beta \neq \alpha+(1,0,\dots,0) \\ \int\limits_0^\epsilon (2n) r^{2n-1}\eta(r/\epsilon)\left(\frac{n+|\alpha|+1}{n}\right) r^{2|\alpha|+2}\,\mathrm{d}r & \text{if } \beta = \alpha+(1,0,\dots,0) \end{cases} \end{aligned}$$

(see Proposition 1.4.9 in [4])

We have

$$\begin{aligned} &\int\limits_0^\epsilon (2n) r^{2n-1}\eta(r/\epsilon)\left(\frac{n+|\alpha|+1}{n}\right) r^{2|\alpha|+2}\,\mathrm{d}r \\ &\qquad = \int\limits_0^\epsilon 2(n+|\alpha|+1) r^{2n+2|\alpha|+1}\eta(r/\epsilon)\,\mathrm{d}r \\ &\qquad = \epsilon^{2n+2|\alpha|+2} \int\limits_0^1 (n+|\alpha|+1) t^{n+|\alpha|}\eta(t^{1/2})\,\mathrm{d}t \end{aligned}$$

(by the change-of-variable $r = \epsilon t^{1/2}$.)

For $m \geq 0$, put $\gamma_m = \int\limits_0^1 (m+1)t^m\eta(t^{1/2})\,\mathrm{d}t > 0$. Note that γ_m depends only on m and the function η. We then have

$$T_{\eta_\epsilon} e_\alpha = \left(\frac{\alpha_1+1}{n+|\alpha|+1}\right)^{1/2} \epsilon^{2n+2|\alpha|+2} \gamma_{n+|\alpha|}\, e_{\alpha+(1,0,\dots,0)}.$$

From this we see that for any multi-index α,

$$T_{\overline{\eta}_\epsilon} e_\alpha = \left(\frac{\alpha_1}{n+|\alpha|}\right)^{1/2} \epsilon^{2n+2|\alpha|} \gamma_{n+|\alpha|-1}\, e_{\alpha-(1,0,\dots,0)}$$

if $\alpha_1 \geq 1$ and $T_{\overline{\eta}_\epsilon} e_\alpha = 0$ if $\alpha_1 = 0$.

Now for multi-indices α with $\alpha_1 \geq 1$,

$$\begin{aligned} &T_{\eta_\epsilon} T_{\overline{\eta}_\epsilon} e_\alpha \\ &\quad = T_{\eta_\epsilon}\left(\left(\frac{\alpha_1}{n+|\alpha|}\right)^{1/2} \epsilon^{2n+2|\alpha|} \gamma_{n+|\alpha|-1}\, e_{\alpha-(1,0\dots,0)}\right) \\ &\quad = \left(\frac{\alpha_1}{n+|\alpha|}\right)^{1/2} \epsilon^{2n+2|\alpha|} \gamma_{n+|\alpha|-1} \left(\frac{\alpha_1}{n+|\alpha|}\right)^{1/2} \epsilon^{2n+2|\alpha|} \gamma_{n+|\alpha|-1}\, e_\alpha \\ &\quad = \frac{\alpha_1}{n+|\alpha|} \epsilon^{4(n+|\alpha|)} \gamma^2_{n+|\alpha|-1}\, e_\alpha, \end{aligned}$$

and

$$\begin{aligned} &T_{\overline{\eta}_\epsilon} T_{\eta_\epsilon} e_\alpha \\ &\quad = T_{\overline{\eta}_\epsilon}\left(\left(\frac{\alpha_1+1}{n+|\alpha|+1}\right)^{1/2} \epsilon^{2n+2|\alpha|+2} \gamma_{n+|\alpha|}\, e_{\alpha+(1,0,\dots,0)}\right) \\ &\quad = \left(\frac{\alpha_1+1}{n+|\alpha|+1}\right)^{1/2} \epsilon^{2n+2|\alpha|+2} \left(\frac{\alpha_1+1}{n+|\alpha|+1}\right)^{1/2} \epsilon^{2n+2|\alpha|+2} \gamma_{n+|\alpha|}\, e_\alpha \\ &\quad = \frac{\alpha_1+1}{n+|\alpha|+1} \epsilon^{4(n+|\alpha|+1)} \gamma^2_{n+|\alpha|}\, e_\alpha. \end{aligned}$$

Therefore,

$$\begin{aligned} &[T_{\eta_\epsilon}, T_{\overline{\eta}_\epsilon}] e_\alpha \\ &\quad = \left(\frac{\alpha_1}{n+|\alpha|} \epsilon^{4(n+|\alpha|)} \gamma^2_{n+|\alpha|-1} - \frac{\alpha_1+1}{n+|\alpha|+1} \epsilon^{4(n+|\alpha|+1)} \gamma^2_{n+|\alpha|}\right) e_\alpha \\ &\quad = \left(\frac{\alpha_1}{\alpha_1+1}\, \frac{n+|\alpha|+1}{n+|\alpha|} \frac{\gamma^2_{n+|\alpha|-1}}{\gamma^2_{n+|\alpha|}} - \epsilon^4\right) \frac{\alpha_1+1}{n+|\alpha|+1} \epsilon^{4(n+|\alpha|)} \gamma^2_{n+|\alpha|}\, e_\alpha. \end{aligned}$$

This formula also holds for multi-indices α with $\alpha_1 = 0$.

For all $0 < \epsilon < 1$ so that

$$\epsilon^4 \notin \left\{ \frac{\alpha_1}{\alpha_1+1} \frac{n+|\alpha|+1}{n+\alpha|} \frac{\gamma^2_{n+|\alpha|-1}}{\gamma^2_{n+|\alpha|}} : \quad \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n \right\},$$

the operator $T = [T_{\eta_\epsilon}, T_{\overline{\eta}_\epsilon}]^2$ can be written as

$$T = \sum_{\alpha \in \mathbb{N}^n} s_\alpha\, e_\alpha \otimes e_\alpha,$$

where $s_\alpha > 0$ for all α.

Since $\{w_j : j \in \mathbb{N}\}$ is separated and $B_n = \bigcup_{j\in\mathbb{N}} E(w_j, R)$ for some $0 < R < 1$, Theorem 1.2 implies that there is a positive number c so that

$$A_\epsilon = \sum_{j\in\mathbb{N}} U_{w_j} T U_{w_j}^* = \sum_{j\in\mathbb{N}} U_{w_j}[T_{\eta_\epsilon}, T_{\overline{\eta}_\epsilon}]^2 U_{w_j}^* \geq c > 0.$$

Now for each $j \in \mathbb{N}$,

$$\begin{aligned}
U_{w_j}[T_{\eta_\epsilon}, T_{\overline{\eta}_\epsilon}]U_{w_j}^* &= U_{w_j}\Big(T_{\eta_\epsilon}T_{\overline{\eta}_\epsilon} - T_{\overline{\eta}_\epsilon}T_{\eta_\epsilon}\Big)U_{w_j}^* \\
&= U_{w_j}T_{\eta_\epsilon}T_{\overline{\eta}_\epsilon}U_{w_j}^* - U_{w_j}T_{\overline{\eta}_\epsilon}T_{\eta_\epsilon}U_{w_j}^* \\
&= \Big(U_{w_j}T_{\eta_\epsilon}U_{w_j}^*\Big)\Big(U_{w_j}T_{\overline{\eta}_\epsilon}U_{w_j}^*\Big) \\
&\quad - \Big(U_{w_j}T_{\overline{\eta}_\epsilon}U_{w_j}^*\Big)\Big(U_{w_j}T_{\eta_\epsilon}U_{w_j}^*\Big) \\
&= T_{\eta_\epsilon\circ\varphi_{w_j}}T_{\overline{\eta}_\epsilon\circ\varphi_{w_j}} - T_{\overline{\eta}_\epsilon\circ\varphi_{w_j}}T_{\eta_\epsilon\circ\varphi_{w_j}} \\
&= [T_{\eta_\epsilon\circ\varphi_{w_j}}, T_{\overline{\eta}_\epsilon\circ\varphi_{w_j}}].
\end{aligned}$$

Hence $A_\epsilon = \sum_{j\in\mathbb{N}} [T_{\eta_\epsilon\circ\varphi_{w_j}}, T_{\overline{\eta}_\epsilon\circ\varphi_{w_j}}]^2$.

Note that for each j, the function $\eta_\epsilon \circ \varphi_{w_j}$ is supported in the set

$$\{z \in B_n : |\varphi_{w_j}(z)| \leq \epsilon\} = \{z \in B_n : \rho(z, w_j) \leq \epsilon\} = E(w_j, \epsilon).$$

We now decompose $\mathcal{W} = \mathcal{W}_1 \cup \cdots \cup \mathcal{W}_N$ such that $E(z,\epsilon) \cap E(w,\epsilon) = \emptyset$ for all $z \neq w$ in $\mathcal{W}_j$, for all $1 \leq j \leq N$ as in Lemma 2.3. Hence

$$A_\epsilon = \sum_{i=1}^{N} \sum_{w\in\mathcal{W}_i} [T_{\eta_\epsilon\circ\varphi_w}, T_{\overline{\eta}_\epsilon\circ\varphi_w}]^2,$$

where, by Proposition 2.7 and the remark following it, each of the summands is in $\mathfrak{CT}(G_\epsilon)$. Here we remind the reader that G_ϵ is the subset of L^∞ consisting of all functions of the form $\sum_{j\in F} \eta_\epsilon \circ \varphi_{w_j}$ or $\sum_{j\in F} \overline{\eta}_\epsilon \circ \varphi_{w_j}$ where F is a subset of $\mathbb{N}$.

It then follows that A_ϵ itself belongs to $\mathfrak{CT}(G_\epsilon)$.

5. REMARKS

In this section we are discussing some remarks about Theorem 1.1. Our first remark is the existence of a separated sequence as in the hypothesis of Theorem 1.1. This is actually a consequence of Zorn's lemma. In fact, let $0 < r < 1$ and Ω_r be the collection of all sets of points $\{w_j : j \in J\}$ in B_n so that $\rho(w_j, w_k) > r$ for all $j \neq k$. The sets in Ω_r are ordered by inclusion. Apply Zorn's lemma, we get a maximal set in Ω_r. Denote this set by $\{w_j : j \in J\}$. Since J must be infinite and countable, we can assume that J is $\mathbb{N}$. Then for any $z \in B_n$, by maximality there is a $j \in \mathbb{N}$ so that $\rho(z, w_j) \leq r$. Hence $B_n = \bigcup_{j\in\mathbb{N}} E(w_j, r)$.

The second remark is about the set G_ϵ. Note that all functions in G_ϵ vanish on $B_n \backslash E_\epsilon$, where E_ϵ is a subset of $V_\epsilon = \bigcup_{j\in\mathbb{N}} E(w_j, \epsilon)$. The following lemma gives an upper estimate for the Lebesgue measure of V_ϵ for small $\epsilon > 0$.

LEMMA 5.1. *Suppose* $0 < \epsilon_0 < 1$ *so that* $E(w_j, \epsilon_0) \cap E(w_l, \epsilon_0) = \emptyset$ *for all* $j \neq l$. *Then for any* $\epsilon < \epsilon_0$,

$$\nu(V_\epsilon) \leq \Big(\frac{\epsilon}{\epsilon_0}\Big)^{2n} \nu(V_{\epsilon_0}).$$

Proof. For any $0 < \delta < 1$ and $z \in B_n$, we have

$$\begin{aligned}
\nu(E(z,\delta)) &= \int_{E(z,\delta)} \mathrm{d}\nu(w) \\
&= \int_{E(0,\delta)} \frac{(1-|z|^2)^{n+1}}{|1-\langle \zeta, z\rangle|^{2(n+1)}} \, \mathrm{d}\nu(\zeta) \\
&\quad \text{(by the change-of-variable } w = \varphi_z(\zeta)) \\
&= (1-|z|^2)^{n+1} \int_{E(0,1)} \frac{\delta^{2n} \, \mathrm{d}\nu(\zeta)}{|1-\langle \delta\zeta, z\rangle|^{2(n+1)}} \\
&= (1-|z|^2)^{n+1}\delta^{2n} \int_{E(0,1)} (1-(\delta|z|)^2)^{-n-1} |k_{\delta z}(\zeta)|^2 \, \mathrm{d}\nu(\zeta) \\
&= (1-|z|^2)^{n+1}\delta^{2n}(1-(\delta|z|)^2)^{-n-1} \|k_{\delta z}\|^2 \\
&= (1-|z|^2)^{n+1}\delta^{2n}(1-(\delta|z|)^2)^{-n-1}.
\end{aligned}$$

Now for $0 < \epsilon < \epsilon_0 < 1$ as in the hypothesis,

$$\begin{aligned}
\nu(V_\epsilon) &= \sum_{j\in\mathbb{N}} \nu(E(w_j, \epsilon)) \\
&= \sum_{j\in\mathbb{N}} (1-|w_j|^2)^{n+1} \epsilon^{2n} (1-(\epsilon|w_j|)^2)^{-n-1} \\
&= \sum_{j\in\mathbb{N}} (1-|w_j|^2)^{n+1} \epsilon_0^{2n} (1-(\epsilon_0|w_j|)^2)^{-n-1} \Big(\frac{\epsilon}{\epsilon_0}\Big)^{2n} \Big(\frac{1-(\epsilon_0|w_j|)^2}{1-(\epsilon|w_j|)^2}\Big)^{n+1} \\
&\leq \Big(\frac{\epsilon}{\epsilon_0}\Big)^{2n} \sum_{j\in\mathbb{N}} \nu(E(w_j, \epsilon_0)) \\
&\quad \Big(\text{because } \frac{1-(\epsilon_0|w_j|)^2}{1-(\epsilon|w_j|)^2} \leq 1 \text{ for } \epsilon < \epsilon_0\Big) \\
&= \Big(\frac{\epsilon}{\epsilon_0}\Big)^{2n} \nu(V_{\epsilon_0}). \qquad \blacksquare
\end{aligned}$$

This lemma implies that if the separated set is fixed then the Lebesgue measure $\nu(V_\epsilon)$ can be made as small as we please provided that ϵ is small.

To conclude the paper, we will show that if η is a continuous function on $[0,1]$ then G_ϵ is contained in $C(B_n)$ - the space of continuous functions on the open unit ball B_n. This remark together with Theorem 1.1 implies that $\mathfrak{CT}(C(B_n) \cap L^\infty)$ coincides with the full Toeplitz algebra $\mathfrak{T}(L^\infty)$. The reader should compare this with the fact that $\mathfrak{CT}(C(\bar{B}_n))$ is contained in the ideal $\mathcal{K}$ of compact operators.

Suppose η is continuous on $[0,1]$, then for each $j \in \mathbb{N}$ the function $\eta_\epsilon \circ \varphi_{w_i}$ is continuous and supported in the ball $E(w_j, \epsilon)$. Suppose F is a subset of $\mathbb{N}$. Let $f = \sum_{j \in F} \eta_\epsilon \circ \varphi_{w_j}$. Let $0 < R < 1$ be given. By Lemma 2.2, all but a finite number of functions in the series vanish on $E(0,R)$. Thus f - being a finite sum of continuous functions on $E(0,R)$, is continuous on $E(0,R)$ for all $0 < R < 1$. So f is continuous on the open unit ball B_n. Similarly, functions of the form $\sum_{j \in F} \overline{\eta}_\epsilon \circ \varphi_{w_j}$ are also continuous on B_n.

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