COMMUTATOR IDEALS OF SUBALGEBRAS OF TOEPLITZ ALGEBRAS ON WEIGHTED BERGMAN SPACES II

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Abstract. For any subset $G$ of $L^\infty$, let $\mathfrak{T}(G)$ denote the algebra generated by all Toeplitz operators $T_f$ with $f \in G$. Let $\mathfrak{C}\mathfrak{T}(G)$ denote the closed two-sided ideal of $\mathfrak{T}(G)$ generated by all commutators $T_fT_g - T_gT_f$ with $f, g \in G$. In this paper we extend our earlier result in [1]. More specifically, we show that the identity $\mathfrak{C}\mathfrak{T}(G) = \mathfrak{T}(G)$ holds true for a broader class of $G$ than considered earlier. The main idea is almost the same as that in [1].

We refer the reader to [1] for definitions and basic results which we will use in this paper. As in Section 2 in [1], a function $f$ on $\mathbb{B}_n$ is called a radial function if there is a function $\tilde{f}$ defined on $[0, 1)$ so that $f(z) = \tilde{f}(|z|)$ for all $z \in \mathbb{B}_n$. For such an $f$ and any real number $s \geq 0$, put

$$\omega_\alpha(f, s) = \frac{\Gamma(n + s + \alpha + 1)}{\Gamma(\alpha + 1)\Gamma(n + s)} \int_0^1 r^{n+s-1}(1 - r)^\alpha \tilde{f}(r^{1/2})dr.$$  

Remark 2.5 in [1] then says that $T_f$ is diagonal with respect to the standard orthonormal basis. In fact we have

$$T_f = \sum_{m \in \mathbb{N}^n} \omega_\alpha(f, |m|)e_m \otimes e_m. \quad (0.1)$$

Here for any $g, h \in A^2_\alpha$, $g \otimes h$ denotes the operator on $A^2_\alpha$ defined by the formula $(g \otimes h)(\varphi) = \langle \varphi, h \rangle_{\alpha} g$ for all $\varphi \in A^2_\alpha$.

Recall that for $w \in \mathbb{B}_n$ and $0 < r < 1$, $E(w, r)$ denotes the ball centered at $w$ with radius $r$ in the pseudo-hyperbolic metric. If $f(z) = \chi_{E(0, \delta)}(z) = 1$.

\textbf{2000 Mathematics Subject Classification.} Primary 47B35; Secondary 47B47.

\textbf{Key words and phrases.} Commutator ideals; Toeplitz operators; Weighted Bergman spaces.
\( \chi(0, \delta)(|z|) \) for some \( 0 < \delta < 1 \) then for any \( s \geq 0 \),
\[
\omega_\alpha(\chi_{E(0, \delta)}, s) = \frac{\Gamma(n + s + \alpha + 1)}{\Gamma(\alpha + 1)\Gamma(n + s)} \int_0^1 r^{n+s-1}(1-r)^\alpha \chi_{[0, \delta]}(r^{1/2})dr
\]
\[
= \frac{\Gamma(n + s + \alpha + 1)}{\Gamma(\alpha + 1)\Gamma(n + s)} \frac{\delta^2}{\delta^{n+s-1}} \int_0^1 r^{n+s-1}(1-r)^\alpha dr.
\]

Since \( \min\{1, (1 - \delta^2)\alpha\} \leq (1 - r)^\alpha \leq \max\{1, (1 - \delta^2)\alpha\} \) for all \( 0 \leq r \leq \delta^2 \), we have
\[
\min\{1, (1 - \delta^2)\alpha\} \leq \omega_\alpha(\chi_{E(0, \delta)}, s) \frac{\Gamma(\alpha + 1)\Gamma(n + s)}{\Gamma(n + s + \alpha + 1) \delta^{2(n+s)}} \leq \max\{1, (1 - \delta^2)\alpha\}.
\]

This then implies that for any \( 0 < r < R < 1 \) and \( s \geq 0 \),
\[
0 \leq \frac{\omega_\alpha(\chi_{E(0, r)}, s)}{\omega_\alpha(\chi_{E(0, R)}, s)} \leq \frac{\max\{1, (1 - \delta^2)\alpha\}}{\min\{1, (1 - \delta^2)\alpha\}} \left(\frac{r}{R}\right)^{2(n+s)} \leq \frac{\max\{1, (1 - \delta^2)\alpha\}}{\min\{1, (1 - \delta^2)\alpha\}} \left(\frac{r}{R}\right)^{2n}.
\]

**Lemma 1.** Suppose \( 0 < R < 1 \) and \( \delta > 0 \). Then there exists \( \gamma = \gamma(R, \delta) \) in \((0, 1)\) so that for all \( 0 < r < \gamma \), \( T_{\chi_{E(0, r)}} \leq \delta T_{\chi_{E(0, R)}} \).

**Proof.** From (0.1), we have
\[
T_{\chi_{E(0, R)}} = \sum_{m \in \mathbb{N}^n} \omega_\alpha(\chi_{E(0, R)}, |m|) e_m \otimes e_m,
\]
and for any \( 0 < r < 1 \),
\[
T_{\chi_{E(0, r)}} = \sum_{m \in \mathbb{N}^n} \omega_\alpha(\chi_{E(0, r)}, |m|) e_m \otimes e_m.
\]

From equation (0.2) there is a \( \gamma \) in \((0, 1)\) such that for any \( 0 < r < \gamma \) and all \( s \geq 0 \), we have
\[
0 \leq \frac{\omega_\alpha(\chi_{E(0, r)}, s)}{\omega_\alpha(\chi_{E(0, R)}, s)} \leq \delta.
\]

This then implies that \( T_{\chi_{E(0, r)}} \leq \delta T_{\chi_{E(0, R)}} \).

**Theorem 2.** Let \( \{z_j : j \in J\} \) be a separated sequence in \( \mathbb{B}_n \) where \( J \) is either a non-empty finite set or \( \mathbb{N} \). Let \( 0 < R < R' < 1 \) and \( M = \{R_j : j \in J\} \subset (0, 1) \) such that any limit point of \( M \) is either 0 or in the open interval \( (R, R') \). Suppose \( W \) is a set with non-empty interior that satisfies the following conditions:

1. \( W \subset \bigcup_{j \in J} E(z_j, R_j) \).
2. There exists \( 0 < r < 1 \) such that whenever \( R_j > R \) (for some \( j \in J \)) we have \( E(z_j, r) \subset W \).
Let $G$ be a linear subspace of $\chi W L^\infty$ such that $C(\mathbb{B}) \cap \chi W L^\infty \subset G$ and each function in $G$ is a linear combination of positive functions in $G$. Then we have $K \subset \mathcal{C}\mathcal{T}(G) = \mathcal{T}(G)$.

**Remark 3.** Let $W$ be a subset of $\mathbb{B}_n$ that satisfies the conditions in Theorem 2. Applying the theorem with $G = \chi W L^\infty$, we see that $\mathcal{C}\mathcal{T}(\chi W L^\infty) = \mathcal{T}(\chi W L^\infty)$. The case where $R_j > R$ for all $j \in J$ is Theorem 1.1 in [1] which extends an earlier result of the author in [2]. What interesting about Theorem 2 is the case where $R_j \to 0$, which is not covered by [1]. In this case the set $W$ is only assumed to have non-empty interior, in addition to the condition that $W$ is a subset of $\bigcup_{j \in J} E(z_j, R_j)$.

**Proof of Theorem 2.** Since $W$ has an empty interior and $C(\mathbb{B}_n) \cap \chi W L^\infty \subset G$, Remark 2.9 in [1] shows that $K \subset \mathcal{T}(G)$. This implies that $K \subset \mathcal{C}\mathcal{T}(G)$.

Next, without loss of generality, we may assume that $R_j < \bar{R}$ for all $j \in J$. Choose $\bar{R}$ so that $\bar{R} < \tilde{R} < 1$. By Lemma 3.1 in [1] there is a continuous function $\eta$ which is supported in $E(0, r)$ such that $[T_\eta, T_{\bar{\eta}}]$ is an injective operator which is diagonal with respect to the standard orthonormal basis of $A^2_n$.

Let $\delta > 0$ be given. By Lemma 1 there is $0 < \gamma < R$ such that

$$T_{\chi E(0, \gamma)} \leq \delta T_{\chi E(0, \bar{R})}. \tag{0.3}$$

By Lemma 2.6 in [1] there is a number $\lambda$ so that

$$T_{\chi E(0, \bar{R})} \leq \lambda [T_\eta, T_{\bar{\eta}}]^2 + \delta T_{\chi E(0, \bar{R})}. \tag{0.4}$$

Now let

$$N_1 = \{ j \in J : R_j < \gamma \} \quad \text{and} \quad N_2 = \{ j \in J : R_j > R \}.$$

Then by assumption about $M$, the set $J \setminus (N_1 \cup N_2)$ is a finite set (possibly empty). For any $j \in N_1$, by applying $U_{z_j}$ on both sides of inequality [0.3], we get

$$T_{\chi E(z_j, \gamma)} = U_{z_j} T_{\chi E(0, \gamma)} U_{z_j} \leq \delta U_{z_j} T_{\chi E(0, \bar{R})} U_{z_j} = \delta T_{\chi E(z_j, \bar{R})}.$$

This then implies $T_{\chi E(z_j, R_j)} \leq T_{\chi E(z_j, \gamma)} \leq \delta T_{\chi E(z_j, \bar{R})}$. For any $j \in N_2$, applying $U_{z_j}$ on both sides of inequality [0.4] and arguing as above, we get

$$T_{\chi E(z_j, \bar{R})} \leq \lambda [T_{\eta \phi z_j}, T_{\bar{\eta} \phi z_j}]^2 + \delta T_{\chi E(z_j, \bar{R})}.$$
Hence $T_{\chi E(z_j, R_j)} \leq T_{\chi E(z_j, R_j)} \leq \lambda [T_{\eta \circ \varphi_{z_j}}, T_{\eta \circ \varphi_{z_j}}]^2 + \delta T_{\chi E(z_j, R_j)}$. So

$$T_{\chi W} \leq \sum_{j \in J} T_{\chi E(z_j, R_j)} = \sum_{j \in N_1} T_{\chi E(z_j, R_j)} + \sum_{j \in N_2} T_{\chi E(z_j, R_j)} + \sum_{j \in J \setminus N_1 \cup N_2} T_{\chi E(z_j, R_j)} \leq \delta \sum_{j \in J} T_{\chi E(z_j, R_j)} + \lambda \sum_{j \in N_2} [T_{\eta \circ \varphi_{z_j}}, T_{\eta \circ \varphi_{z_j}}]^2 + \sum_{j \in J \setminus N_1 \cup N_2} T_{\chi E(z_j, R_j)} \tag{0.5}
$$

Since $\{z_j : j \in J\}$ is a separated sequence, we can decompose $J = J_1 \cup \cdots \cup J_M$ for some integer $M$ so that $E(z_1, R) \cap E(z_k, R) = \emptyset$ for any $l \neq k$ in $J_s$, where $1 \leq s \leq M$ (see Lemma 2.3 in [2]). From this we have $\sum_{j \in J} T_{\chi E(z_j, R_j)} \leq M$. For any $j \in J$, the function $\eta \circ \varphi_{z_j}$ is continuous and supported in $E(z_j, r)$, hence it is in $G$. Proposition 2.3 in [1] shows that $\sum_{j \in N_2 \cap J_s} [T_{\eta \circ \varphi_{z_j}}, T_{\eta \circ \varphi_{z_j}}]^2$ belongs to $\mathfrak{C}\mathfrak{F}(G)$ for $1 \leq s \leq M$. Thus

$$\sum_{j \in N_2} [T_{\eta \circ \varphi_{z_j}}, T_{\eta \circ \varphi_{z_j}}]^2 \text{ belongs to } \mathfrak{C}\mathfrak{F}(G).$$

Also since $T_{\chi E(z_j, R_j)}$ is compact for any $j$ in the finite set $J \setminus N_1 \cup N_2$, $\sum_{j \in J \setminus N_1 \cup N_2} T_{\chi E(z_j, R_j)}$ is compact, hence, in $\mathfrak{C}\mathfrak{F}(G)$. Let $\pi$ denote the canonical quotient map from $\mathfrak{F}(G)$ onto the quotient algebra $\mathfrak{F}(G)/\mathfrak{C}\mathfrak{F}(G)$. We then have

$$\pi\left(\sum_{j \in N_2 \cap J_s} [T_{\eta \circ \varphi_{z_j}}, T_{\eta \circ \varphi_{z_j}}]^2\right) = 0 = \pi\left(\sum_{j \in J \setminus N_1 \cup N_2} T_{\chi E(z_j, R_j)}\right).
$$

Let $0 \leq f \leq 1$ be any function in $G$. Then since $f \leq \chi W$, (0.5) gives

$$0 \leq \pi(T_f) \leq \pi(T_{\chi W}) \leq \delta M.
$$

But $\delta$ was arbitrary, so we conclude that $\pi(T_f) = 0$ for any $f \in G$ with $0 \leq f \leq 1$. Since any function in $G$ is a linear combination of positive functions in $G$, we see that $\pi(T_f) = 0$ for all $f \in G$. So $\mathfrak{C}\mathfrak{F}(G) = \mathfrak{F}(G)$. □

References