COMPACT TOEPLITZ OPERATORS WITH CONTINUOUS SYMBOLS

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Abstract. For any rotation-invariant positive regular Borel measure ν on the closed unit ball $\overline{\mathbb{B}}_n$ whose support contains the unit sphere $\mathbb{S}_n$, let L^2_a be the closure in $L^2 = L^2(\overline{\mathbb{B}}_n, \mathrm{d}\nu)$ of all analytic polynomials. For a bounded Borel function f on $\overline{\mathbb{B}}_n$, the Toeplitz operator T_f is defined by $T_f(\varphi) = P(f\varphi)$ for $\varphi \in L^2_a$, where P is the orthogonal projection from L^2 onto L^2_a. We show that if f is continuous on $\overline{\mathbb{B}}_n$, then T_f is compact if and only if $f(z) = 0$ for all z on the unit sphere. This is well known when L^2_a is replaced by the classical Bergman or Hardy space.

1. Introduction

As usual, for any integer $n \geq 1$, let $\mathbb{B}_n$ (respectively, $\mathbb{S}_n$) denote the open unit ball (respectively, the unit sphere) in $\mathbb{C}^n$. The closure of $\mathbb{B}_n$ in the Euclidean metric on $\mathbb{C}^n$ is denoted by $\overline{\mathbb{B}}_n$. For $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$, $|z|$ denotes the Euclidean norm of z. For any multi-index $m = (m_1, \ldots, m_n)$ in $\mathbb{N}^n$ (here $\mathbb{N}$ denotes the set of all non-negative integers), $z^m = z_1^{m_1} \cdots z_n^{m_n}$ and $\bar{z}^m = \bar{z}_1^{m_1} \cdots \bar{z}_n^{m_n}$. We also write $|m| = m_1 + \cdots + m_n$ and $m! = m_1! \cdots m_n!$. Let σ denote the rotation-invariant positive Borel measure on $\mathbb{S}_n$ which is normalized so that $\sigma(\mathbb{S}_n) = 1$. Let μ be a positive regular Borel measure on the closed interval $[0,1]$ with $\mu([0,1]) = 1$ and 1 is in the support of μ. Let ν be the product measure of μ and σ. So ν is a regular Borel measure on $\overline{\mathbb{B}}_n$ with unit total mass such that for any $f \in L^1(\overline{\mathbb{B}}_n, \mathrm{d}\nu)$, we have the integration in polar coordinates formula:

$$\int_{\overline{\mathbb{B}}_n} f(z)\mathrm{d}\nu(z) = \int_{[0,1]} \Big(\int_{\mathbb{S}_n} f(r\zeta)\mathrm{d}\sigma(\zeta) \Big)\mathrm{d}r. \tag{1}$$

Let $L^2_a(\overline{\mathbb{B}}_n, \mathrm{d}\nu)$ be the closure of the space of all holomorphic polynomials in $L^2(\overline{\mathbb{B}}_n, \mathrm{d}\nu)$ and let P denote the orthogonal projection from $L^2(\overline{\mathbb{B}}_n, \mathrm{d}\nu)$ onto $L^2_a(\overline{\mathbb{B}}_n, \mathrm{d}\nu)$.

If $\mathrm{d}\mu(r) = \dfrac{2\,\Gamma(n+\alpha+1)}{\Gamma(n)\,\Gamma(\alpha+1)} r^{2n-1}(1-r^2)^{\alpha}\mathrm{d}r$ for some $\alpha > -1$, then ν is a weighted Lebesgue measure on $\mathbb{B}_n$ and $L^2_a(\overline{\mathbb{B}}_n, \mathrm{d}\nu)$ is the familiar weighted Bergman space. If μ is the point mass measure at 1, then $L^2_a(\overline{\mathbb{B}}_n, \mathrm{d}\nu)$ can

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be identified with the Hardy space H^2 on $\mathbb{B}_n$. See [4] for more detail about Bergman and Hardy spaces.

For any bounded Borel function f defined on $\overline{\mathbb{B}}_n$, the Toeplitz operator T_f is the operator on $L^2_a(\overline{\mathbb{B}}_n, \mathrm{d}\nu)$ defined by $T_f\varphi = P(f\varphi)$ for $\varphi \in L^2_a$. The function f is called the symbol of T_f. It is clear that T_f is a bounded operator with $\|T_f\| \leq \|f\|_{L^\infty(\overline{\mathbb{B}}_n, \mathrm{d}\nu)}$. It follows from the density in $C(\overline{\mathbb{B}}_n)$ of polynomials (in z and $\bar{z}$) that if $T_f = 0$ then $f(z) = 0$ for ν-almost all z in $\overline{\mathbb{B}}_n$. So the map $f \mapsto T_f$ from $L^\infty(\overline{\mathbb{B}}_n, \mathrm{d}\nu)$ into the C^*-algebra $\mathfrak{B}(L^2_a(\overline{\mathbb{B}}_n, \mathrm{d}\nu))$ of all bounded linear operators on $L^2_a(\overline{\mathbb{B}}_n, \mathrm{d}\nu)$ is an injective contraction. This map is not an isometry in general. Note that if $\mu(\{1\}) = 0$, then the values of f on the unit sphere do not affect the operator T_f. On the other hand, the values of f on the unit sphere play an important role when $\mu(\{1\}) > 0$.

In this paper we are interested in Toeplitz operators whose symbols behave well near the boundary of $\mathbb{B}_n$. Toeplitz operators (on the classical Hardy and Bergman spaces) whose symbols are continuous functions on $\overline{\mathbb{B}}_n$ and the C^*-algebras generated by them were studied by L. Coburn [1] back in the 1970's. One of many results on this subject is the following theorem.

Theorem 1.1. *Suppose f is in $C(\overline{\mathbb{B}}_n)$. Then T_f is a compact operator if and only if $f(\zeta) = 0$ for all $\zeta \in \mathbb{S}_n$.*

In this paper we will show that Theorem 1.1 still holds true for Toeplitz operators acting on any $L^2_a(\overline{\mathbb{B}}_n, \mathrm{d}\nu)$. That $f|_{\mathbb{S}_n} \equiv 0$ implies T_f is compact is not new. The proof is similar to that of the classical case. On the other hand, the proof of the converse requires a different argument. The usual approach which involves reproducing kernels does not seem to work for general ν. The reason is that for such a ν, even though reproducing kernels exist, there is no useful formula for them. Theorem 1.1 for a general rotation-invariant positive Borel measure ν on the unit disk was showed by T. Nakazi and R. Yoneda in [2]. This paper was in fact inspired by theirs.

2. Toeplitz operators with compactly supported symbols

In this section we show that if f is a bounded Borel function whose support is contained in a compact subset of $\mathbb{B}_n$, then T_f is a Hilbert-Schmidt operator.

For multi-indexes $m, k \in \mathbb{N}^n$, from formula (1) and Propositions 1.4.8 and 1.4.9 in [3], we have

$$\begin{aligned}\int_{\overline{\mathbb{B}}_n} z^m \bar{z}^k \mathrm{d}\nu(z) &= \int_{[0,1]} \Big(\int_{\mathbb{S}_n} \zeta^m \zeta^k \mathrm{d}\sigma(\zeta) \Big) r^{2|m|} \mathrm{d}\mu(r) \\ &= \begin{cases} 0 & \text{if } m \neq k, \\ \dfrac{(n-1)!\; m!}{(n-1+|m|)!} \int_{[0,1]} r^{2|m|} \mathrm{d}\mu(r) & \text{if } m = k. \end{cases}\end{aligned}$$

For $s \in \mathbb{N}$, let $\alpha_s = \displaystyle\int_{[0,1]} r^{2s} \mathrm{d}\mu(r)$. For $m \in \mathbb{N}^n$ and $z \in \mathbb{C}^n$, put

$$e_m(z) = \Big(\frac{(n-1+|m|)!}{(n-1)!\ m!\ \alpha_{|m|}} \Big)^{1/2} z^m.$$

Then from the above computation and the definition of $L^2_a(\overline{\mathbb{B}}_n, \mathrm{d}\nu)$, it follows that the set $\{e_m : m \in \mathbb{N}^n\}$ is an orthonormal basis for $L^2_a(\overline{\mathbb{B}}_n, \mathrm{d}\nu)$.

Proposition 2.1. *Let f be a bounded Borel function on $\overline{\mathbb{B}}_n$ such that for some $0 < \delta < 1$, $f(z) = 0$ whenever $|z| > \delta$. Then T_f is a Hilbert-Schmidt operator.*

Proof. For $z \in \mathbb{B}_n$ with $|z| \leq \delta$, we have

$$\begin{aligned}
\sum_{m\in\mathbb{N}^n} |e_m(z)|^2 &= \sum_{m\in\mathbb{N}^n} \frac{(n-1+|m|)!}{(n-1)!\ m_1!\cdots m_n!} \frac{|z_1|^{2m_1}\cdots|z_n|^{2m_n}}{\alpha_{|m|}} \\
&= \sum_{M=0}^{\infty} \frac{(n-1+M)!}{(n-1)!\ M!\ \alpha_M} \sum_{|m|=M} \frac{M!}{m_1!\cdots m_n!} |z_1|^{2m_1}\cdots|z_n|^{2m_n} \\
&= \sum_{M=0}^{\infty} \frac{(n-1+M)!}{(n-1)!\ M!\ \alpha_M} (|z_1|^2 + \cdots + |z_n|^2)^M \\
&\leq \sum_{M=0}^{\infty} \frac{(n-1+M)!}{(n-1)!\ M!\ \alpha_M} \delta^{2M}. \qquad (2)
\end{aligned}$$

Now $\lim\limits_{M\to\infty} (\alpha_M)^{1/M} = \lim\limits_{M\to\infty} \Big(\int_{[0,1]} r^{2M} \mathrm{d}\mu(r) \Big)^{1/M} = \|r^2\|_{L^\infty([0,1],\mathrm{d}\mu)} = 1$, where the last identity follows from the fact that 1 is in the support of μ. Thus the infinite sum in (2) is convergent. So for each $0 < \delta < 1$, there is a constant $C(\delta) < \infty$ such that $\sum\limits_{m\in\mathbb{N}^n} |e_m(z)|^2 \leq C(\delta)$ for all $|z| \leq \delta$.

Now suppose f satisfies the hypothesis of the proposition. Then

$$\begin{aligned}
\sum_{m,k\in\mathbb{N}^n} |\langle T_f e_m, e_k\rangle|^2 &\leq \sum_{m,k\in\mathbb{N}^n} \Big(\int_{\overline{\mathbb{B}}_n} |f(z) e_m(z) e_k(z)| \mathrm{d}\nu(z) \Big)^2 \\
&\leq \sum_{m,k\in\mathbb{N}^n} \int_{\overline{\mathbb{B}}_n} |f(z)|^2 |e_m(z)|^2 |e_k(z)|^2 \mathrm{d}\nu(z) \\
&\quad \text{(by Holder's inequality)} \\
&= \int_{|z|\leq\delta} |f(z)|^2 \Big(\sum_{m\in\mathbb{N}^n} |e_m(z)|^2 \Big) \Big(\sum_{k\in\mathbb{N}^n} |e_k(z)|^2 \Big) \mathrm{d}\nu(z) \\
&\leq (C(\delta))^2 \int_{|z|\leq\delta} |f(z)|^2 \mathrm{d}\nu(z) < \infty.
\end{aligned}$$

This shows that T_f is a Hilbert-Schmidt operator. $\square$

The following corollary proves the "if" part of Theorem 1.1. The "only if" part will follow from a more general result which will be presented in Section 3.

Corollary 2.2. *If $f \in C(\overline{\mathbb{B}}_n)$ such that $f(\zeta) = 0$ for all $|\zeta| = 1$ then T_f is compact.*

Proof. Since f can be uniformly approximated on $\overline{\mathbb{B}}_n$ by continuous functions with compact supports in $\mathbb{B}_n$, Proposition 2.1 shows that T_f can be approximated in the operator norm by Hilbert-Schmidt operators. Hence T_f is a compact operator. $\square$

3. Compact Toeplitz operators with continuous symbols

We begin this section with a proposition that relates the boundary values of f with $\langle T_f e_m, e_m\rangle$ as $|m| \to \infty$.

Proposition 3.1. *Let f be a bounded Borel function on $\overline{\mathbb{B}}_n$ such that for σ-almost all $\zeta \in \mathbb{S}_n$, we have $f(\zeta) = \lim\limits_{r\uparrow 1} f(r\zeta)$. If $\lim\limits_{|m|\to\infty} \langle T_f e_m, e_m\rangle = \alpha$, then $\int_{\mathbb{S}_n} f(\zeta)\mathrm{d}\sigma(\zeta) = \alpha$.*

Proof. Without loss of generality, we may assume that $\alpha = 0$. For any function g in $L^1(\overline{\mathbb{B}}_n, \mathrm{d}\nu)$ and any positive integer M we have

$$\begin{aligned}
&\sum_{|m|=M} \langle T_g e_m, e_m\rangle \\
&= \sum_{|m|=M} \frac{(n-1+|m|)!}{(n-1)!\ m!\ \alpha_{|m|}} \int_{\mathbb{B}_n} g(z) z^m \bar{z}^m \mathrm{d}\nu(z) \\
&= \frac{(n-1+M)!}{(n-1)!\ M!\ \alpha_M} \int_{\mathbb{B}_n} g(z) \Big\{ \sum_{|m|=M} \frac{M!}{m_1!\cdots m_n!} |z_1|^{2m_1}\cdots|z_n|^{2m_n} \Big\} \mathrm{d}\nu(z) \\
&= \frac{(n-1+M)!}{(n-1)!\ M!\ \alpha_M} \int_{\mathbb{B}_n} g(z)(|z_1|^2+\cdots+|z_n|^2)^M \mathrm{d}\nu(z) \\
&= \frac{(n-1+M)!}{(n-1)!\ M!\ \alpha_M} \int_{[0,1]} \Big(\int_{\mathbb{S}_n} g(r\zeta)\mathrm{d}\sigma(\zeta) \Big) r^{2M} \mathrm{d}\mu(r).
\end{aligned} \tag{3}$$

In particular, if $g(z) = 1$ for all $z \in \overline{\mathbb{B}}_n$, then $\frac{(n-1+M)!}{(n-1)!\ M!} = \sum\limits_{|m|=M} 1$. This shows that the set $\{m = (m_1, \ldots, m_n) \in \mathbb{N}^n : m_1 + \cdots + m_n = M\}$ has $\frac{(n-1+M)!}{(n-1)!\ M!}$ elements. This formula can, of course, be showed directly by an elementary combinatoric argument.

Let $\epsilon > 0$ be given. There is an integer M_ϵ such that for all $m \in \mathbb{N}^n$ with $|m| > M_\epsilon$ we have $|\langle T_f e_m, e_m\rangle| < \epsilon$. Thus for any $M > M_\epsilon$, (3) with f in place of g gives

$$\begin{aligned}\Big|\frac{1}{\alpha_M}\int_{[0,1]}\Big(\int_{\mathbb{S}_n} f(r\zeta)\mathrm{d}\sigma(\zeta)\Big)r^{2M}\mathrm{d}\mu(r)\Big| &\le \frac{(n-1)!\ M!}{(n-1+M)!}\sum_{|m|=M}|\langle T_f e_m, e_m\rangle| \\ &\le \frac{(n-1)!\ M!}{(n-1+M)!}\sum_{|m|=M}\epsilon \\ &= \epsilon.\end{aligned}$$

This shows that

$$\lim_{M\to\infty}\frac{1}{\alpha_M}\int_{[0,1]}\Big(\int_{\mathbb{S}_n} f(r\zeta)\mathrm{d}\sigma(\zeta)\Big)r^{2M}\mathrm{d}\mu(r) = 0. \tag{4}$$

For each $0\le r\le 1$, put $\varphi(r) = \int_{\mathbb{S}_n} f(r\zeta)\mathrm{d}\sigma(\zeta)$. Since f is bounded on $\overline{\mathbb{B}}_n$ and $f(r\zeta)\to f(\zeta)$ as $r\uparrow 1$ for σ-almost all $\zeta\in\mathbb{S}_n$, Lebesgue Dominated Convergence Theorem implies that $\varphi(r)\to\varphi(1)$ as $r\uparrow 1$. We now show that $\lim\limits_{M\to\infty}\frac{1}{\alpha_M}\int_{[0,1]}\varphi(r)r^{2M}\mathrm{d}\mu(r) = \varphi(1)$. Let $\epsilon>0$ be given. There is a δ in $[0,1)$ such that $|\varphi(r)-\varphi(1)|<\epsilon$ for all $a\le r\le 1$. Therefore,

$$\begin{aligned}\Big|\Big(\frac{1}{\alpha_M}\int_{[0,1]}\varphi(r)r^{2M}\mathrm{d}\mu(r)\Big)-\varphi(1)\Big| &= \Big|\frac{1}{\alpha_M}\int_{[0,1]}(\varphi(r)-\varphi(1))r^{2M}\mathrm{d}\mu(r)\Big| \\ &\le \frac{1}{\alpha_M}\int_{[0,a)}|\varphi(r)-\varphi(1)|r^{2M}\mathrm{d}\mu(r) \\ &\quad + \frac{1}{\alpha_M}\int_{[a,1]}|\varphi(r)-\varphi(1)|r^{2M}\mathrm{d}\mu(r) \\ &\le 2\|\varphi\|_\infty\frac{1}{\alpha_M}\int_{[0,a)}r^{2M}\mathrm{d}\mu(r)+\epsilon.\end{aligned}$$

Now since 1 is in the support of μ, an elementary argument shows that $\lim\limits_{M\to\infty}\frac{1}{\alpha_M}\int_{[0,a)}r^{2M}\mathrm{d}\mu(r) = 0$. (See Lemma 2 in [2] for a detailed proof.) By taking $M\to\infty$ in the above inequalities, we conclude that

$$\limsup_{M\to\infty}\Big|\Big(\frac{1}{\alpha_M}\int_{[0,1]}\varphi(r)r^{2M}\mathrm{d}\mu(r)\Big)-\varphi(1)\Big|\le\epsilon.$$

Since ϵ was arbitrary, we get

$$\lim_{M\to\infty}\frac{1}{\alpha_M}\int_{[0,1]}\varphi(r)r^{2M}\mathrm{d}\mu(r) = \varphi(1). \tag{5}$$

Now (4) and (5) imply that $\varphi(1)=0$, which means $\int_{\mathbb{S}_n} f(\zeta)\mathrm{d}\sigma(\zeta) = 0$. $\square$

Corollary 3.2. *Suppose f is a bounded Borel function on $\overline{\mathbb{B}}_n$ such that for σ-almost all $\zeta\in\mathbb{S}_n$, $f(\zeta)=\lim\limits_{r\uparrow 1}f(r\zeta)$ and that T_f is a compact operator on*

$L_a^2(\overline{\mathbb{B}}_n, \mathrm{d}\nu)$. Then $f(\zeta) = 0$ for σ-almost all ζ in $\mathbb{S}_n$. From this, the "only if" part of Theorem 1.1 follows.

Proof. For any multi-indexes $l_1, l_2 \in \mathbb{N}^n$, the operator $T_{fe_{l_1}\bar{e}_{l_2}} = T_{\bar{e}_{l_2}} T_f T_{e_{l_1}}$ is compact. Thus we have $\lim_{|m|\to\infty} \langle T_{fe_{l_1}\bar{e}_{l_2}} e_m, e_m \rangle = 0$. By Proposition 3.1 and the fact that for σ-almost all $\zeta \in \mathbb{S}_n$, $\lim_{r\uparrow 1} f(r\zeta) e_{l_1}(r\zeta)\bar{e}_{l_2}(r\zeta) = f(\zeta)e_{l_1}(\zeta)\bar{e}_{l_2}(\zeta)$, which is a positive multiple of $f(\zeta)\zeta^{l_1}\bar{\zeta}^{l_2}$, we conclude that $\int_{\mathbb{S}_n} f(\zeta)\zeta^{l_1}\bar{\zeta}^{l_2}\mathrm{d}\sigma(\zeta) = 0$. Since this is true for all multi-indexes l_1 and l_2, we have $f(\zeta) = 0$ for σ-almost all $\zeta \in \mathbb{S}_n$. □

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References

[1] Lewis A. Coburn, *Singular integral operators and Toeplitz operators on odd spheres*, Indiana Univ. Math. J. **23** (1973/74), 433–439. MR 0322595 (48 #957)

[2] Takahiko Nakazi and Rikio Yoneda, *Compact Toeplitz operators with continuous symbols on weighted Bergman spaces*, Glasg. Math. J. **42** (2000), no. 1, 31–35. MR 1739694 (2000i:47052)

[3] Walter Rudin, *Function theory in the unit ball of* $\mathbf{C}^n$, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Science], vol. 241, Springer-Verlag, New York, 1980. MR 601594 (82i:32002)

[4] Kehe Zhu, *Spaces of holomorphic functions in the unit ball*, Graduate Texts in Mathematics, vol. 226, Springer-Verlag, New York, 2005. MR 2115155 (2006d:46035)

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