COMPACT TOEPLITZ OPERATORS ON SEGAL-BARGMANN TYPE SPACES

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Abstract. We consider Toeplitz operators with symbols enjoying a uniform radial limit on Segal-Bargmann type spaces. We show that such an operator is compact if and only if the limiting function vanishes on the unit sphere. The structure of the C*-algebra generated by Toeplitz operators whose symbols admit continuous uniform radial limits is also analyzed.

1. Introduction

Let ν be a regular Borel probability measure on $\mathbb{C}^n$ that is rotation-invariant. Then there is a regular Borel probability measure μ on $[0,\infty)$ so that the formula

$$\int_{\mathbb{C}^n} f(z)d\nu(z) = \int_0^\infty \int_{\mathbb{S}} f(r\zeta)d\sigma(\zeta)d\mu(r) \tag{1.1}$$

holds for all functions f in $L^1(\mathbb{C}^n, d\nu)$, where $\mathbb{S}$ is the unit sphere and σ is the normalized surface area measure on $\mathbb{S}$. Throughout the paper, we also require that μ satisfy the following three conditions:

$(C1)$ $\sup\{r : r \in \operatorname{supp} \mu\} = \infty$;

$(C2)$ $\hat{\mu}(m) = \int_0^\infty r^m d\mu(r) < \infty \quad \text{for} \quad \text{all} \quad m \geq 0$;

$(C3)$ $\lim_{m\to\infty} \dfrac{(\hat{\mu}(2m+1))^2}{\hat{\mu}(2m)\hat{\mu}(2m+2)} = 1.$

The first condition means that μ does not have bounded support, while the second condition assures that the function spaces we are interested in contain all holomorphic polynomials at least. The necessity of assuming the third condition will be confirmed by Proposition 2.4 below. While many Gaussian type measures on $\mathbb{C}^n$ satisfy all three conditions above, there are measures that satisfy $(C1)$ and $(C2)$ but not $(C3)$. See [3, Section 3.3] for such examples.

The space $\mathcal{H} = H^2(\mathbb{C}^n, d\nu)$, as a closed subspace of the Hilbert space $L^2(\mathbb{C}^n, d\nu)$ of square integrable functions with respect to ν, is defined to be

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the space of all entire functions f for which

$$\|f\|^2 = \int_{\mathbb{C}^n} |f(z)|^2 d\nu(z) < \infty.$$

For any multi-index $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$ (here $\mathbb{N}_0$ denotes the set of all non-negative integers), we write $\alpha! = \alpha_1! \cdots \alpha_n!$ and $|\alpha| = \alpha_1 + \cdots + \alpha_n$. We also write $z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$ and $\bar{z}^\alpha = \bar{z}_1^{\alpha_1} \cdots \bar{z}_n^{\alpha_n}$ for $z = (z_1, \ldots, z_n)$ in $\mathbb{C}^n$. Put $c_\alpha = \int_{\mathbb{S}} |\zeta^\alpha|^2 d\sigma(\zeta) = \dfrac{(n-1)!\alpha!}{(n-1+|\alpha|)!}$. We then have

$$\int_{\mathbb{C}^n} z^\alpha \bar{z}^\beta d\nu(z) = \int_0^\infty r^{|\alpha|+|\beta|} d\mu(r) \int_{\mathbb{S}} \zeta^\alpha \bar{\zeta}^\beta d\sigma(\zeta) = \begin{cases} 0 & \text{if } \alpha \neq \beta; \\ c_\alpha \hat{\mu}(2|\alpha|) & \text{if } \alpha = \beta \end{cases} \tag{1.2}$$

for all multi-indices α and β. This shows that the space $\mathcal{H}$ has the orthonormal basis $\left\{e_\alpha(z) = \frac{z^\alpha}{\sqrt{c_\alpha \hat{\mu}(2|\alpha|)}} : \alpha \in \mathbb{N}_0^n\right\}$, which is usually referred to as the standard orthonormal basis.

Using Cauchy formula and the assumption about μ, we see that for each compact set Q, there is a constant C_Q such that

$$\sup_{z \in Q} |f(z)| \leq C_Q \|f\| \tag{1.3}$$

for $f \in \mathcal{H}$. This implies that the evaluation functional at each point in $\mathbb{C}^n$ is bounded on $\mathcal{H}$. As a consequence, there is a reproducing kernel $K(w, z) = K_z(w)$ such that $f(z) = \langle f, K_z \rangle$ for $z \in \mathbb{C}^n$. It follows from (1.3) that $\sup_{z \in Q} \|K_z\| \leq C_Q$. It is also standard that $K(z, z) = \sum_\alpha |e_\alpha(z)|^2$. See [2] for a different approach about the existence of the reproducing kernel.

Let P denote the orthogonal projection from $L^2(\mathbb{C}^n, d\nu)$ onto $\mathcal{H}$. For a bounded Borel function f on $\mathbb{C}^n$, the Toeplitz operator $T_f : \mathcal{H} \to \mathcal{H}$ is defined by

$$T_f(u) = PM_f u = P(fu), \quad u \in \mathcal{H}.$$

Here $M_f : L^2(\mathbb{C}^n, d\nu) \to L^2(\mathbb{C}^n, d\nu)$ is the operator of multiplication by f. The function f is called the symbol of T_f. We also define the Hankel operator $H_f : \mathcal{H} \to \mathcal{H}^\perp$ by

$$H_f u = (I - P) M_f u = (I - P)(fu), \quad u \in \mathcal{H},$$

where $\mathcal{H}^\perp$ is the orthogonal complement of $\mathcal{H}$ in $L^2(\mathbb{C}^n, d\nu)$. It is immediate that $\|T_f\| \leq \|f\|_\infty$ and $\|H_f\| \leq \|f\|_\infty$.

For f, g bounded Borel functions on $\mathbb{C}^n$, the following basic identities follow easily from the definition of Toeplitz and Hankel operators:

$$T_{gf} - T_g T_f = H_{\bar{g}}^* H_f$$

and

$$(T_g)^* = T_{\bar{g}}, \quad T_{af+bg} = aT_f + bT_g,$$

where a, b are complex numbers and $\bar{g}$ denotes the complex conjugate of g.

When ν is the standard Gaussian measure $d\nu(z) = (2\pi)^{-n} e^{-\frac{|z|^2}{2}} dV(z)$, it can be verified directly that the associated measure μ satisfies the conditions $(C1)$-$(C3)$ and in this case, $\mathcal{H}$ is the standard Segal-Bargmann space $H^2(\mathbb{C}^n)$ (also known as the Fock space).

It is well known (on $H^2(\mathbb{C}^n)$ but similar argument still works for general $\mathcal{H}$) that if f is a bounded function such that $f(z) \to 0$ as $|z| \to \infty$, then T_f is compact. The converse of this does not hold in general, but it will hold if we put some restrictions on the behavior of f near infinity.

We say that a bounded Borel function f defined on $\mathbb{C}^n$ has a *uniform radial limit* (at infinity) if there is a function f_∞ on $\mathbb{S}$ such that

$$\lim_{r\to\infty} \sup_{\zeta\in\mathbb{S}} |f(r\zeta) - f_\infty(\zeta)| = 0.$$

The function f_∞ will be called the uniform radial limit of f.

We define $\mathcal{S}$ to be the space of all bounded Borel functions on $\mathbb{C}^n$ which have a *continuous* uniform radial limit. Then $\mathcal{S}$, equipped with the supremum norm, is a C*-subalgebra of the algebra of all bounded Borel functions on $\mathbb{C}^n$.

In this note, we will show that for any bounded Borel function f which has a uniform radial limit f_∞, the operator T_f is compact if and only if the limiting function f_∞ vanishes on $\mathbb{S}$. We also study the C*-algebra generated by T_f with $f \in \mathcal{S}$. We show that this algebra is an extension of the compact operators by continuous functions on the unit sphere and this extension is equivalent to a known extension given by Toeplitz operators acting on the Hardy space of the unit sphere.

This paper is organized as follows. In Section 2, we will give some preliminaries. The main results will be provided in Section 3.

2. Preliminaries

The first result of this section, regarding compactness of Toeplitz and Hankel operators whose symbols vanish at infinity, is standard. For the reader's convenience, we provide here a proof.

Lemma 2.1. *Suppose f is bounded on $\mathbb{C}^n$ such that $\lim_{|z|\to\infty} f(z) = 0$. Then the operator $M_f|_{\mathcal{H}}$ is compact. As a result, the operators $T_f = PM_f|_{\mathcal{H}}$ and $H_f = (1-P)M_f|_{\mathcal{H}}$ are both compact on $\mathcal{H}$.*

Proof. For any $0 < r < \infty$, let $\mathbb{B}_r = \{z \in \mathbb{C}^n : |z| \leq r\}$ and let $f_r = f\chi_{\mathbb{B}_r}$ where $\chi_{\mathbb{B}_r}$ is the characteristic function of $\mathbb{B}_r$. Then $\|f - f_r\|_\infty \to 0$ as $r \to \infty$, which gives $\|M_f - M_{f_r}\| \to 0$ as $r \to \infty$. Thus, it reduces to show that $M_{f_r}|_{\mathcal{H}}$ is compact for each r.

We will in fact show that $M_{f_r}|_{\mathcal{H}}$ is a Hilbert-Schmidt operator. We have

$$\sum_{\alpha\in\mathbb{N}_0^n}\|M_{f_r}e_\alpha\|^2 = \int_{\mathbb{C}^n}|f_r(z)|^2\Big(\sum_{\alpha\in\mathbb{N}_0^n}|e_\alpha(z)|^2\Big)d\nu(z)$$
$$= \int_{\mathbb{C}^n}|f_r(z)|^2K(z,z)d\nu(z) = \int_{\mathbb{B}_r}|f(z)|^2K(z,z)d\nu(z) < \infty,$$

because the function $z\mapsto K(z,z)$ is bounded on compact sets. Therefore $M_{f_r}|_{\mathcal{H}}$ is a Hilbert-Schmidt operator. □

Remark 2.2. For f in $\mathcal{S}$ with uniform radial limit f_∞, we define

$$g(z) = \begin{cases} f_\infty(\frac{z}{|z|}) & \text{if } z\neq 0,\\ 0 & \text{if } z = 0.\end{cases} \tag{2.1}$$

Then g is continuous on $\mathbb{C}^n\backslash\{0\}$ by the continuity of f_∞ and we have

$$\lim_{|z|\to\infty}|f(z)-g(z)| = \lim_{r\to\infty}\sup_{\zeta\in\mathbb{S}}|f(r\zeta)-g(r\zeta)| = \lim_{r\to\infty}\sup_{\zeta\in\mathbb{S}}|f(r\zeta)-f_\infty(\zeta)| = 0.$$

Lemma 2.1 implies that both $T_f - T_g$ and $H_f - H_g$ are compact.

For $1\le j\le n$, put $\chi_j(0) = 0$ and $\chi_j(z) = \frac{z_j}{|z|}$ if $z\neq 0$, and $\chi(z) = (\chi_1(z),\ldots,\chi_n(z))$. We also denote by δ_j the multi-index $(\delta_{1j},\ldots,\delta_{nj})$, where δ_{kl} is the usual Kronecker notation.

Lemma 2.3. *Let $S = T_{\bar\chi_1}T_{\chi_1}+\cdots+T_{\bar\chi_n}T_{\chi_n}$ and $T = T_{\chi_1}T_{\bar\chi_1}+\cdots+T_{\chi_n}T_{\bar\chi_n}$. Then for any multi-index α we have*

$$Se_\alpha = \frac{(\hat\mu(2|\alpha|+1))^2}{\hat\mu(2|\alpha|)\hat\mu(2|\alpha|+2)}e_\alpha,$$
$$Te_\alpha = \begin{cases} 0 & \text{if } \alpha = 0;\\ \dfrac{|\alpha|}{n+|\alpha|-1}\times\dfrac{(\hat\mu(2|\alpha|-1))^2}{\hat\mu(2|\alpha|-2)\hat\mu(2|\alpha|)}e_\alpha & \text{if } \alpha\neq 0.\end{cases}$$

In particular, the operators S and T are diagonal with respect to the standard orthonormal basis.

Proof. Let j be an integer between 1 and n and α a multi-index. Using the orthogonality of the set $\{\zeta^\beta : \beta\in\mathbb{N}_0^n\}$ with respect to σ on $\mathbb{S}$, we see that $T_{\chi_j}e_\alpha = P(\chi_j e_\alpha)$ is a scalar multiple of $e_{\alpha+\delta_j}$. To determine the multiple, we compute

$$\begin{aligned}\langle T_{\chi_j} e_\alpha, e_{\alpha+\delta_j}\rangle &= \langle \chi_j e_\alpha, e_{\alpha+\delta_j}\rangle\\ &= \frac{\Big(\int_{[0,\infty)} t^{2|\alpha|+1} d\mu(t)\Big)\Big(\int_{\mathbb{S}} |\zeta^{\alpha+\delta_j}|^2 d\sigma(\zeta)\Big)}{\Big(c_\alpha c_{\alpha+\delta_j} \hat{\mu}(2|\alpha|)\hat{\mu}(2|\alpha|+2)\Big)^{1/2}}\\ &= \frac{\hat{\mu}(2|\alpha|+1) c_{\alpha+\delta_j}}{\Big(c_\alpha c_{\alpha+\delta_j} \hat{\mu}(2|\alpha|)\hat{\mu}(2|\alpha|+2)\Big)^{1/2}}\\ &= \Big(\frac{\alpha_j+1}{n+|\alpha|} \times \frac{(\hat{\mu}(2|\alpha|+1))^2}{\hat{\mu}(2|\alpha|)\hat{\mu}(2|\alpha|+2)}\Big)^{1/2}.\end{aligned}$$

It now follows that, for $\beta \in \mathbb{N}_0^n$,

$$\begin{aligned}\langle T_{\bar{\chi}_j} e_\alpha, e_\beta\rangle &= \langle P(\bar{\chi}_j e_\alpha), e_\beta\rangle = \langle e_\alpha, P(\chi_j e_\beta)\rangle\\ &= \begin{cases} 0 & \text{if } \alpha \neq \beta+\delta_j;\\ \Big(\frac{\beta_j+1}{n+|\beta|} \times \frac{(\hat{\mu}(2|\beta|+1))^2}{\hat{\mu}(2|\beta|)\hat{\mu}(2|\beta|+2)}\Big)^{1/2} & \text{if } \alpha = \beta+\delta_j.\end{cases}\end{aligned}$$

This implies

$$T_{\bar{\chi}_j} e_\alpha = \begin{cases} 0 & \text{if } \alpha_j = 0;\\ \Big(\frac{\alpha_j}{n+|\alpha|-1} \times \frac{(\hat{\mu}(2|\alpha|-1))^2}{\hat{\mu}(2|\alpha|-2)\hat{\mu}(2|\alpha|)}\Big)^{1/2} e_{\alpha-\delta_j} & \text{if } \alpha_j \geq 1.\end{cases}$$

With the above formulas for T_{χ_j} and $T_{\bar{\chi}_j}$, we have

$$\begin{aligned}Se_\alpha &= \sum_{j=1}^n T_{\bar{\chi}_j} T_{\chi_j} e_\alpha = \Big(\sum_{j=1}^n \frac{\alpha_j+1}{n+|\alpha|} \times \frac{(\hat{\mu}(2|\alpha|+1))^2}{\hat{\mu}(2|\alpha|)\hat{\mu}(2|\alpha|+2)}\Big) e_\alpha\\ &= \frac{(\hat{\mu}(2|\alpha|+1))^2}{\hat{\mu}(2|\alpha|)\hat{\mu}(2|\alpha|+2)} e_\alpha,\\ Te_\alpha &= \sum_{j=1}^n T_{\chi_j} T_{\bar{\chi}_j} e_\alpha\\ &= \begin{cases} 0 & \text{if } \alpha = 0;\\ \Big(\sum_{j=1}^n \frac{\alpha_j}{n+|\alpha|-1} \times \frac{(\hat{\mu}(2|\alpha|-1))^2}{\hat{\mu}(2|\alpha|-2)\hat{\mu}(2|\alpha|)}\Big) e_\alpha & \text{if } \alpha \neq 0.\end{cases}\\ &= \begin{cases} 0 & \text{if } \alpha = 0;\\ \frac{|\alpha|}{n+|\alpha|-1} \times \frac{(\hat{\mu}(2|\alpha|-1))^2}{\hat{\mu}(2|\alpha|-2)\hat{\mu}(2|\alpha|)} e_\alpha & \text{if } \alpha \neq 0.\end{cases}\end{aligned}$$ □

Let $\mathcal{G} = \{f \in L^\infty(\mathbb{C}^n, d\nu) : H_f \text{ is compact on } \mathcal{H}\}$. It is immediate that $\mathcal{G}$ is a closed linear subspace of $L^\infty(\mathbb{C}^n, d\nu)$. Using the identity

$$H_{fg} = (I-P)M_{fg}|_{\mathcal{H}} = H_f P M_g|_{\mathcal{H}} + (I-P)M_f H_g,$$

we also see that $\mathcal{G}$ is a subalgebra of $L^\infty(\mathbb{C}^n, d\nu)$.

For standard Gaussian measure $d\nu(z) = (2\pi)^{-n} e^{-\frac{|z|^2}{2}} dV(z)$, a description of $\mathcal{G}$ was given in [1]. It was showed there that $\mathcal{G}$ is self-adjoint (that is, f belongs to $\mathcal{G}$ if and only if $\bar{f}$ belongs to $\mathcal{G}$) and it contains the algebra ESV of functions that are "eventually slowly varying", which in turn contains $\mathcal{S}$. Their approach relied heavily on the explicit form of the measure and it does not seem to work for general μ. It turns out that the inclusion $\mathcal{S} \subset \mathcal{G}$ does not always hold unless μ satisfies condition *(C3)*. But curiously, the condition *(C3)* was not explicitly used anywhere in [1]. For general μ satisfying *(C3)*, we do not know whether $\mathcal{G}$ is self-adjoint.

Proposition 2.4. *The inclusion $\mathcal{S} \subset \mathcal{G}$ holds if and only if*

$$\lim_{m\to\infty} \frac{(\hat{\mu}(2m+1))^2}{\hat{\mu}(2m)\hat{\mu}(2m+2)} = 1. \tag{2.2}$$

Proof. Suppose $\mathcal{S}$ is contained in $\mathcal{G}$. Then in particular, H_{χ_j} is compact for all $j = 1, \ldots, n$. This shows that the operator

$$\sum_{j=1}^{n} H^*_{\chi_j} H_{\chi_j} = \sum_{j=1}^{n} \left(T_{|\chi_j|^2} - T_{\bar{\chi}_j} T_{\chi_j}\right) = I - \sum_{j=1}^{n} T_{\bar{\chi}_j} T_{\chi_j} \tag{2.3}$$

is compact. It now follows from Lemma 2.3 that (2.2) holds.

Now suppose that (2.2) holds. We need to show $\mathcal{S} \subset \mathcal{G}$. It follows from Remark 2.2 that we only need to show that $\mathcal{G}$ contains functions of the form (2.1). It suffices to prove that χ_j and $\bar{\chi}_j$ belong to $\mathcal{G}$ for $j = 1, \ldots, n$ since $\mathcal{G}$ is a closed subalgebra of $L^\infty(\mathbb{C}^n, d\nu)$. Now Lemma 2.3 together with (2.3) and (2.2) implies that $H^*_{\chi_1} H_{\chi_1} + \cdots + H^*_{\chi_n} H_{\chi_n}$ is compact. Therefore $H^*_{\chi_j} H_{\chi_j}$, and hence H_{χ_j}, is compact for all $j = 1, \ldots, n$. Similar argument shows that $H_{\bar{\chi}_1}, \ldots, H_{\bar{\chi}_n}$ are all compact. □

Using the basic identity relating Toeplitz and Hankel operators, our assumption about μ and Proposition 2.4, we obtain

Corollary 2.5. *For any $f \in \mathcal{S}$ and $h \in L^\infty$, the semi-commutators $T_{fh} - T_f T_h$ and $T_{fh} - T_h T_f$ are both compact on $\mathcal{H}$.*

Recall that the Hardy space $H^2(\mathbb{S})$ is the closure of the span of analytic monomials $\{\zeta^\alpha : \alpha \in \mathbb{N}_0^n\}$ in $L^2(\mathbb{S}) = L^2(\mathbb{S}, d\sigma)$. Since analytic monomials of different degrees are orthogonal and $c_\alpha = \int_{\mathbb{S}} |\zeta^\alpha|^2 d\sigma(\zeta)$, the set $\{\tilde{e}_\alpha(\zeta) = \frac{\zeta^\alpha}{\sqrt{c_\alpha}} : \alpha \in \mathbb{N}_0^n\}$ is an orthonormal basis for $H^2(\mathbb{S})$. There is a natural unitary operator $U : \mathcal{H} \to H^2(\mathbb{S})$ given by $Ue_\alpha = \tilde{e}_\alpha$.

Let $\tilde{P}$ be the orthogonal projection from $L^2(\mathbb{S})$ onto $H^2(\mathbb{S})$. For any bounded Borel function g on $\mathbb{S}$, the Toeplitz operator $\widetilde{T}_g$ is defined by $\widetilde{T}_g h = \tilde{P}(gh)$ for $h \in H^2(\mathbb{S})$.

The next theorem shows that condition (2.2) is equivalent to the compactness of certain differences of Toeplitz operators on $\mathcal{H}$ and $H^2(\mathbb{S})$. The case $n = 1$ appeared in [2], where condition (2.2) was first (as far as we know) discussed.

Theorem 2.6. *The operator $T_f - U^*\widetilde{T}_{f_\infty}U$ is compact on $\mathcal{H}$ for any $f \in \mathcal{S}$ with uniform radial limit f_∞ if and only if* (2.2) *holds.*

Proof. Recall that for $1 \le j \le n$, $\chi_j(z) = \frac{z_j}{|z|}$ if $|z| \ne 0$ and $\chi_j(0) = 0$. A calculation as in Lemma 2.3 shows that for any multi-index α,

$$\{T_{\chi_1} - U^*\widetilde{T}_{\zeta_1}U\}e_\alpha = \Big(\frac{\alpha_1+1}{n+|\alpha|}\Big)^{\frac{1}{2}}\Big[\frac{\hat{\mu}(2|\alpha|+1)}{(\hat{\mu}(2|\alpha|)\hat{\mu}(2|\alpha|+2))^{\frac{1}{2}}} - 1\Big]e_{\alpha+\delta_1}.$$

This implies that (2.2) holds if the operator $T_{\chi_1} - U^*\widetilde{T}_{\zeta_1}U$ is compact.

Now suppose (2.2) holds. Similar to the proof of Lemma 3 in [4], we write $T_{\chi_1} - U^*\widetilde{T}_{\zeta_1}U = DS_{\delta_1}$, where $S_{\delta_1}e_\alpha = e_{\alpha+\delta_1}$, $De_\alpha = 0$ if $\alpha_1 = 0$ and

$$De_{\alpha+\delta_1} = \Big(\frac{\alpha_1+1}{n+|\alpha|}\Big)^{\frac{1}{2}}\Big[\frac{\hat{\mu}(2|\alpha|+1)}{(\hat{\mu}(2|\alpha|)\hat{\mu}(2|\alpha|+2))^{\frac{1}{2}}} - 1\Big]e_{\alpha+\delta_1}.$$

Then D is compact, so is $T_{\chi_1} - U^*\widetilde{T}_{\zeta_1}U$. Hence, by symmetry, $T_{\chi_j} - U^*\widetilde{T}_{\zeta_j}U$ is compact for all $1 \le j \le n$. It also follows that $T_{\bar{\chi}_j} - U^*\widetilde{T}_{\bar{\zeta}_j}U$ is compact. Using the identity $\widetilde{T}_{\bar{\zeta}^\beta\zeta^\alpha} = \widetilde{T}^{\beta_1}_{\bar{\zeta}_1}\cdots\widetilde{T}^{\beta_n}_{\bar{\zeta}_n}\widetilde{T}^{\alpha_1}_{\zeta_1}\cdots\widetilde{T}^{\alpha_n}_{\zeta_n}$, we see that $T^{\beta_1}_{\bar{\chi}_1}\cdots T^{\beta_n}_{\bar{\chi}_n}T^{\alpha_1}_{\chi_1}\cdots T^{\alpha_n}_{\chi_n} - U^*\widetilde{T}_{\bar{\zeta}^\beta\zeta^\alpha}U$ is compact for all multi-indices α and β.

On the other hand, $T_{\bar{\chi}^\beta\chi^\alpha} - T^{\beta_1}_{\bar{\chi}_1}\cdots T^{\beta_n}_{\bar{\chi}_n}T^{\alpha_1}_{\chi_1}\cdots T^{\alpha_n}_{\chi_n}$ is also compact by Corollary 2.5. We then conclude that $T_{\bar{\chi}^\beta\chi^\alpha} - U^*\widetilde{T}_{\bar{\zeta}^\beta\zeta^\alpha}U$ is compact for multi-indices α, β. A standard approximation argument using polynomials in ζ and $\bar{\zeta}$ on $\mathbb{S}$ shows that $T_g - U^*\widetilde{T}_{f_\infty}U$ is compact for all g given by (2.1), where f_∞ is continuous on $\mathbb{S}$. Remark 2.2 then implies that $T_f - U^*\widetilde{T}_{f_\infty}U$ is compact for all $f \in \mathcal{S}$ with uniform radial limit f_∞. □

We close the section with the following elementary result, which will be useful in proving one of our main theorems in the next section.

Lemma 2.7. *Suppose φ is a real-valued function which is locally integrable on $[0,\infty)$ with respect to μ. Then we have*

$$\liminf_{k\to\infty}\frac{\int_{[0,\infty)}\varphi(t)t^k d\mu(t)}{\int_{[0,\infty)}t^k d\mu(t)} \ge \liminf_{t\to\infty}\varphi(t). \tag{2.4}$$

As a consequence, if $\lim_{t\to\infty}\varphi(t)$ exists or ∞ or $-\infty$, then

$$\lim_{k\to\infty}\frac{\int_{[0,\infty)}\varphi(t)t^k d\mu(t)}{\int_{[0,\infty)}t^k d\mu(t)} = \lim_{t\to\infty}\varphi(t). \tag{2.5}$$

Proof. There is nothing to prove if the the right hand side of (2.4) is $-\infty$, so we may assume that it is a finite real number or ∞. Let α be any real number such that $\alpha < \liminf_{t\to\infty}\varphi(t)$. Then there is a number $r > 0$ so that

$\varphi(t) - \alpha \geq 0$ for all $t \geq r$. We then have

$$\begin{aligned}\int_{[0,\infty)} \varphi(t) t^k d\mu(t) &= \alpha \int_{[0,\infty)} t^k d\mu(t) + \int_{[0,\infty)} (\varphi(t) - \alpha) t^k d\mu(t) \\ &\geq \alpha \int_{[0,\infty)} t^k d\mu(t) + \int_{[0,r)} (\varphi(t) - \alpha) t^k d\mu(t) \\ &\geq \alpha \int_{[0,\infty)} t^k d\mu(t) - r^k \int_{[0,r)} |\varphi(t) - \alpha| d\mu(t).\end{aligned}$$

On the other hand,

$$\int_{[0,\infty)} t^k d\mu(t) \geq \int_{[2r,\infty)} t^k d\mu(t) \geq (2r)^k \mu([2r,\infty)) > 0.$$

Therefore,

$$\begin{aligned}\frac{\int_{[0,\infty)} \varphi(t) t^k d\mu(t)}{\int_{[0,\infty)} t^k d\mu(t)} &\geq \alpha - \frac{r^k \int_{[0,r)} |\varphi(t) - \alpha| d\mu(t)}{\int_{[0,\infty)} t^k d\mu(t)} \\ &\geq \alpha - \frac{r^k \int_{[0,r)} |\varphi(t) - \alpha| d\mu(t)}{(2r)^k \mu([2r,\infty))}.\end{aligned}$$

Taking liminf as $k \to \infty$, we see that the left hand side of (2.4) is at least α. Since α is arbitrarily smaller than $\liminf_{t\to\infty} \varphi(t)$, the inequality in (2.4) holds.

To obtain (2.5), apply (2.4) to both φ and $-\varphi$. □

3. Main Results

Our first result gives a necessary condition for the compactness of the Toeplitz operator T_f, when the radial limit at infinity of f exists almost everywhere.

Theorem 3.1. *Suppose that f is a bounded function on $\mathbb{C}^n$ so that the radial limit $g(\zeta) = \lim_{r\to\infty} f(r\zeta)$ exists for $\sigma-$almost every $\zeta \in \mathbb{S}$. If T_f is compact on $\mathcal{H}$, then $g(\zeta) = 0$ for $\sigma-$almost every $\zeta \in \mathbb{S}$.*

Proof. Using formula (1.1) as in the proof of Proposition 3.1 of [6], we have

$$\frac{1}{\hat{\mu}(2m)} \int_0^\infty \Big[\int_{\mathbb{S}} f(r\zeta) d\sigma(\zeta) \Big] r^{2m} d\mu(r) = \frac{(n-1)!m!}{(n-1+m)!} \sum_{|\alpha|=m} \langle T_f e_\alpha, e_\alpha \rangle.$$

Let $\psi(r) = \int_{\mathbb{S}} f(r\zeta) d\sigma(\zeta)$. Then by our assumption and Lebesgue Dominated Convergence Theorem, $\lim_{r\to\infty} \psi(r) = \int_{\mathbb{S}} g(\zeta) d\sigma(\zeta)$.

Since T_f is compact, $\lim_{|\alpha|\to\infty} \langle T_f e_\alpha, e_\alpha \rangle = 0$. It follows that

$$\lim_{m\to\infty} \frac{1}{\hat{\mu}(2m)} \int_0^\infty \psi(r) r^{2m} d\mu(r) = \lim_{m\to\infty} \frac{(n-1)!m!}{(n-1+m)!} \sum_{|\alpha|=m} \langle T_f e_\alpha, e_\alpha \rangle = 0.$$

Here we have used the fact that the set $\{\alpha \in \mathbb{N}_0^n : |\alpha| = m\}$ has cardinality $\frac{(n-1+m)!}{(n-1)!m!}$. Now applying Lemma 2.7 to the real part and imaginary part of ψ respectively, we obtain

$$\int_{\mathbb{S}} g(\zeta)d\sigma(\zeta) = \lim_{r\to\infty} \psi(r) = \lim_{m\to\infty} \frac{1}{\hat{\mu}(2m)} \int_0^\infty \psi(r) r^{2m} d\mu(r) = 0.$$

For any multi-indices α, β, we have $T_{f\chi^\alpha\bar{\chi}^\beta} = H_{\bar{f}}^* H_{\chi^\alpha\bar{\chi}^\beta} + T_f T_{\chi^\alpha\bar{\chi}^\beta}$. (Recall that $\chi(z) = (\chi_1(z), \ldots, \chi_n(z)) = \frac{z}{|z|}$ for $z \neq 0$ and $\chi(0) = 0$.) Therefore, $T_{f\chi^\alpha\bar{\chi}^\beta}$ is compact by the compactness of T_f and $H_{\chi^\alpha\bar{\chi}^\beta}$ (by Proposition 2.4). Since

$$\lim_{r\to\infty} f(r\zeta)\chi(r\zeta)^\alpha \bar{\chi}(r\zeta)^\beta = g(\zeta)\zeta^\alpha\bar{\zeta}^\beta$$

for $\sigma-$almost every $\zeta \in \mathbb{S}$, the preceding argument implies $\int_{\mathbb{S}} g(\zeta)\zeta^\alpha\bar{\zeta}^\beta = 0$. Because this holds true for any multi-indices α and β, we conclude that $g(\zeta) = 0$ for $\sigma-$almost every $\zeta \in \mathbb{S}$. □

Corollary 3.2. *Let f be a bounded function on $\mathbb{C}^n$ with uniform radial limit f_∞ on $\mathbb{S}$. Then T_f is compact if and only if f_∞ vanishes on $\mathbb{S}$.*

Proof. The "only if" part is a direct consequence of the above theorem. The "if" part is Lemma 2.1. □

Remark 3.3. If the limiting function f_∞ is assumed to be continuous on $\mathbb{S}$, one may prove the "only if" part of Corollary 3.2 by using Theorem 2.6. In fact, the compactness of T_f on $\mathcal{H}$ implies that $\widetilde{T}_{f_\infty}$ is compact on the Hardy space $H^2(\mathbb{S})$. By [4, Lemma 2], f_∞ vanishes on $\mathbb{S}$.

When f_∞ is not continuous, the "only if" part of Corollary 3.2 seems to be new even for the standard Segal-Bargmann space.

Remark 3.4. The limit in Corollary 3.2 must be uniform in ζ. We will construct a bounded function f such that $\lim_{r\to\infty} f(r\zeta) = 0$ for each $\zeta \in \mathbb{S}$ but T_f is not compact on the standard Segal-Bargmann space $H^2(\mathbb{C}^n)$.

We first observe the following fact. For any $\zeta \in \mathbb{S}$ and any $r > 0$, if $R \geq 1+2/r$ then any ray from the origin passing through a point in $\mathbb{B}(R\zeta, 1)$ intersects the unit sphere $\mathbb{S}$ at a point belonging to $\mathbb{B}(\zeta, r)$. In fact, let z be in $\mathbb{B}(R\zeta, 1)$. Then the ray from the origin passing through z intersects the unit sphere at $z/|z|$. Since $|z - R\zeta| < 1$ and $|\zeta| = 1$, we obtain

$$\Big|\frac{z}{|z|} - \zeta\Big| = \Big|\frac{z - |z|\zeta}{|z|}\Big| \leq \frac{|z - R\zeta| + |R - |z||}{|z|} \leq \frac{2|z - R\zeta|}{|z|} < \frac{2}{R-1} \leq r.$$

Now choose a sequence $\zeta_1, \zeta_2, \ldots$ in $\mathbb{S}$ and a sequence of positive real numbers $r_1, r_2, \ldots$ converging to 0 such that the sets $\mathbb{B}(\zeta_1, r_1), \mathbb{B}(\zeta_2, r_2), \ldots$ are pairwise disjoint. For each j, put $R_j = 1 + 2/r_j$. From the above observation, we see that for each $\zeta \in \mathbb{S}$, the ray $L_\zeta = \{r\zeta : r > 0\}$ intersects the ball $\mathbb{B}(R_j\zeta_j, 1)$ only if ζ belongs to $\mathbb{B}(\zeta_j, r_j)$. Therefore, for such a ζ, there is at most one j so that L_ζ intersects $\mathbb{B}(R_j\zeta_j, 1)$. This shows that the function $f=\sum_{j=1}^{\infty}\chi_{\mathbb{B}(R_j\zeta_j,1)}$ is bounded by 1 (because the sets $\mathbb{B}(R_j\zeta_j,1)$ are pairwise disjoint) and $\lim_{r\to\infty}f(r\zeta)=0$ for all $\zeta\in\mathbb{S}$.

We claim that T_f is not compact on $H^2(\mathbb{C}^n)$. Recall that the normalized kernel functions of $H^2(\mathbb{C}^n)$ have the form $k_a(z)=e^{\langle z,a\rangle/2-|a|^2/4}$ for $a,z\in\mathbb{C}^n$. It is well known that $k_a\to 0$ weakly in $H^2(\mathbb{C}^n)$ as $|a|\to\infty$.

For each j, put $a_j=R_j\zeta_j$. We have

$$\begin{aligned}\langle T_f k_{a_j},k_{a_j}\rangle &= \int_{\mathbb{C}^n} f|k_{a_j}|^2 d\nu \geq \int_{\mathbb{B}(a_j,1)} |k_{a_j}(z)|^2(2\pi)^{-n}e^{-|z|^2/2}dV(z)\\ &= \int_{\mathbb{B}(a_j,1)}(2\pi)^{-n}e^{-|z-a_j|^2/2}dV(z)=\int_{\mathbb{B}(0,1)}d\nu=\nu(\mathbb{B}(0,1)).\end{aligned}$$

This shows that $\|T_f k_{a_j}\|\geq\nu(\mathbb{B}(0,1))$ for all $j\geq 1$. Since $k_{a_j}\to 0$ weakly as $j\to\infty$, we conclude that T_f is not a compact operator.

In the rest of the paper, we study the structure of the C^*-algebra $\mathfrak{T}(\mathcal{S})$ generated by all Toeplitz operators T_f, where f belongs to $\mathcal{S}$.

Recall that a C*-subalgebra of $\mathcal{B}(\mathcal{H})$ is irreducible if its commutant consists of only scalar multiples of the identity operator. Equivalently, the only reducing subspaces of the subalgebra are $\{0\}$ and $\mathcal{H}$.

Proposition 3.5. *$\mathfrak{T}(\mathcal{S})$ is irreducible.*

Proof. Suppose that Q is an operator on $\mathcal{H}$ that commutes with all elements of $\mathfrak{T}(\mathcal{S})$. In particular, $QT_{\bar{\chi}_j}=T_{\bar{\chi}_j}Q$ for $j=1,\ldots,n$. Let $\varphi=Q(e_0)$. Using the computations in the proof of Lemma 2.3 and

$$0=QT_{\bar{\chi}_j}(e_0)=T_{\bar{\chi}_j}Q(e_0)=T_{\bar{\chi}_j}(\varphi)=\sum_{\alpha}\langle\varphi,e_\alpha\rangle T_{\bar{\chi}_j}e_\alpha,$$

we obtain $\langle\varphi,e_\alpha\rangle=0$ whenever $\alpha_j\geq 1$. Since j can be any integer between 1 and n, we conclude that φ is a multiple of e_0. Thus $Q(e_0)=\varphi=\langle\varphi,e_0\rangle e_0$.

It also follows from the proof of Lemma 2.3 that there is a constant d_α such that $d_\alpha\prod_{j=1}^n(T_{\chi_j})^{\alpha_j}(e_0)=e_\alpha$. Then

$$\begin{aligned}Q(e_\alpha) &= d_\alpha Q\prod_{j=1}^n(T_{\chi_j})^{\alpha_j}(e_0)=d_\alpha\prod_{j=1}^n(T_{\chi_j})^{\alpha_j}Q(e_0)\\ &= d_\alpha\prod_{j=1}^n(T_{\chi_j})^{\alpha_j}(\langle\varphi,e_0\rangle e_0)=\langle\varphi,e_0\rangle d_\alpha\prod_{j=1}^n(T_{\chi_j})^{\alpha_j}(e_0)=\langle\varphi,e_0\rangle e_\alpha.\end{aligned}$$

So $Q=\langle\varphi,e_0\rangle I$, which implies that $\mathfrak{T}(\mathcal{S})$ is irreducible. □

We are now ready for the description of $\mathfrak{T}(\mathcal{S})$ as an extension of the compact operators by continuous functions on the unit sphere.

Theorem 3.6. *The following statements hold:*

(a) *The commutator ideal $\mathfrak{CT}$ of $\mathfrak{T}(\mathcal{S})$ is the same as the ideal $\mathcal{K}$ of compact operators on $\mathcal{H}$.*

(b) *We have* $\mathfrak{T}(\mathcal{S}) = \{T_f + K : f \in \mathcal{S} \text{ and } K \in \mathcal{K}\}$. *Moreover, there is a short exact sequence*

$$0 \to \mathcal{K} \xrightarrow{\ \iota\ } \mathfrak{T}(\mathcal{S}) \xrightarrow{\ \rho\ } C(\mathbb{S}) \to 0. \tag{3.1}$$

Here ι *is the inclusion map and* $\rho(T_f + K) = f_\infty$ *for* $K \in \mathcal{K}$ *and* $f \in \mathcal{S}$ *with uniform radial limit* f_∞.

Proof. We have showed that $\mathfrak{T}(\mathcal{S})$ is irreducible. On the other hand, $\mathfrak{T}(\mathcal{S})$ contains a non-zero compact operator (e.g. $T_{|\chi_j|^2} - T_{\bar{\chi}_j}T_{\chi_j}$). It then follows from a well known result in the theory of C^*-algebras ([5, Corollary I.10.4]) that $\mathfrak{T}(\mathcal{S})$ contains the ideal $\mathcal{K}$ of compact operators. Thus the commutator ideal $\mathfrak{CT}$ contains the commutator ideal of $\mathcal{K}$, which is exactly $\mathcal{K}$.

For any $f, h \in \mathcal{S}$, the commutator $T_fT_h - T_hT_f$ is compact by Corollary 2.5. This implies $\mathfrak{CT} \subset \mathcal{K}$, which finishes the proof of (a).

For the proof of (b), we consider the map $\Phi : \mathcal{S} \to \mathfrak{T}(\mathcal{S})/\mathcal{K}$ defined by $\Phi(f) = T_f + \mathcal{K}$. Corollary 2.5 shows that Φ is a $*$-homomorphism of C^*-algebras (recall that $\mathcal{S}$ with the supremum norm is a C*-algebra). By a standard result in the theory of C^*-algebras ([5, Theorem I.5.5]), the range $\Phi(\mathcal{S})$ is a closed C^*-subalgebra. On the other hand, $\Phi(\mathcal{S})$ contains the quotient classes of all generators of $\mathfrak{T}(\mathcal{S})$. So it follows that $\Phi(\mathcal{S}) = \mathfrak{T}(\mathcal{S})/\mathcal{K}$ and $\mathfrak{T}(\mathcal{S}) = \{T_f + K : f \in \mathcal{S} \text{ and } K \in \mathcal{K}\}$.

Furthermore, we know from Corollary 3.2 that the kernel of Φ is the ideal $\ker(\Phi) = \{f \in \mathcal{S} : f_\infty = 0 \text{ on } \mathbb{S}, \text{ where } f_\infty \text{ is the uniform radial limit of } f\}$. On the other hand, it can be showed that $\mathcal{S}/\ker(\Phi)$ is isometrically $*$-isomorphic to $C(\mathbb{S})$ and the map $f + \ker(\Phi) \mapsto f_\infty$ is a $*$-isomorphism. Thus there is a $*$-isomorphism $\widetilde{\Phi} : C(\mathbb{S}) \to \mathfrak{T}(\mathcal{S})/\mathcal{K}$ induced by Φ. This then gives the required short exact sequence, where $\rho = \widetilde{\Phi}^{-1} \circ \pi$ with $\pi : \mathfrak{T}(\mathcal{S}) \to \mathfrak{T}(\mathcal{S})/\mathcal{K}$ the quotient map. Also for $f \in \mathcal{S}$ with radial limit f_∞ and $K \in \mathcal{K}$,

$$\rho(T_f + K) = \widetilde{\Phi}^{-1}(\pi(T_f + K)) = \widetilde{\Phi}^{-1}(T_f + \mathcal{K}) = f_\infty. \quad \square$$

The short exact sequence in Theorem 3.6 says, in the language of Brown-Douglas-Fillmore (BDF) Theory (see [5, Chapter IX]), that $\mathfrak{T}(\mathcal{S})$ is an extension of the compact operators $\mathcal{K}$ by $C(\mathbb{S})$.

Let $\widetilde{\mathfrak{T}}$ be the C*-algebra generated by all Toeplitz operators $\widetilde{T}_g$ acting on the Hardy space $H^2(\mathbb{S})$, where g is continuous on $\mathbb{S}$. Coburn [4] showed that $\widetilde{\mathfrak{T}} = \{\widetilde{T}_g + K : g \in C(\mathbb{S}) \text{ and } K \in \mathcal{K}\}$, the commutator ideal of $\widetilde{\mathfrak{T}}$ coincides with the compact operators $\mathcal{K}$ and there is a short exact sequence

$$0 \to \mathcal{K} \xrightarrow{\ \iota\ } \widetilde{\mathfrak{T}} \xrightarrow{\ \tilde{\rho}\ } C(\mathbb{S}) \to 0, \tag{3.2}$$

where $\tilde{\rho}(\widetilde{T}_g + K) = g$, for $g \in C(\mathbb{S})$ and $K \in \mathcal{K}$.

Out last result shows that the extensions (3.1) and (3.2) are in fact equivalent. This implies that they give rise to the same element of the extension group of $\mathcal{K}$ by $C(\mathbb{S})$.

Theorem 3.7. *There is a* $*$*-isomorphism* $\Psi : \widetilde{\mathfrak{T}} \to \mathfrak{T}(\mathcal{S})$ *such that* $\Psi(\mathcal{K}) = \mathcal{K}$ *and* $\rho \circ \Psi = \widetilde{\rho}$.

Proof. Recall that $U : \mathcal{H} \to H^2(\mathbb{S})$ is the unitary operator defined by $Ue_\alpha = \tilde{e}_\alpha$, where $\{e_\alpha : \alpha \in \mathbb{N}_0^n\}$ (respectively, $\{\tilde{e}_\alpha : \alpha \in \mathbb{N}_0^n\}$) is the standard orthonormal basis for $\mathcal{H}$ (respectively, $H^2(\mathbb{S})$).

For any A in $\widetilde{\mathfrak{T}}$, define $\Psi(A) = U^*AU$. Since $A = \widetilde{T}_{f_\infty} + K$ for some $f_\infty \in C(\mathbb{S})$ and $K \in \mathcal{K}$, using Theorem 2.6 we obtain $\Psi(A) = U^*\widetilde{T}_{f_\infty}U + U^*KU = T_f + K'$, where K' is compact and f is any function whose uniform radial limit is f_∞. This shows that the image of Ψ is contained in $\mathfrak{T}(\mathcal{S})$ and

$$\rho \circ \Psi(A) = \rho(T_f + K') = f_\infty = \tilde{\rho}(\widetilde{T}_{f_\infty} + K) = \tilde{\rho}(A).$$

On the other hand, for any $B \in \mathfrak{T}(\mathcal{S})$, there is a function $f \in \mathcal{S}$ and a compact operator K so that $B = T_f + K$. By Theorem 2.6 again, $K' = U^*\widetilde{T}_{f_\infty}U - T_f$ is compact. This shows that the operator $A = \widetilde{T}_{f_\infty} + U(K - K')U^*$ belongs to $\widetilde{\mathfrak{T}}$ and we have

$$\Psi(A) = U^*\widetilde{T}_{f_\infty}U + K - K' = T_f + K = B.$$

Therefore Ψ is a $*$-isomorphism from $\widetilde{\mathfrak{T}}$ onto $\mathfrak{T}(\mathcal{S})$. This completes the proof of the theorem. ∎

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