Coposinormality of the Cesàro matrices

Trieu Le and H. C. Rhaly Jr.

Abstract. It is already known that the Cesàro matrices of orders one and two are coposinormal operators on ℓ^2. Here it is shown that the Cesàro matrices of all orders are coposinormal; the proof employs posinormality, achieved by means of a diagonal interrupter, and makes use of the Zeilberger's algorithm and computational assistance by Maple™.

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1. Introduction

Let $a := \{a(n)\}_{n\geq 0}$ be a sequence of nonegative numbers with $a(0) > 0$. Put $S(n) = \sum_{j=0}^{n} a(j)$. The Nörlund matrix $M = [m(i,j)]_{i,j\geq 0}$ is an infinite lower triangular matrix defined by

$$m(i,j) = \begin{cases} a(i-j)/S(i) & \text{for } 0 \leq j \leq i \\ 0 & \text{for } j > i. \end{cases}$$

For a real number $\alpha \geq 1$, the Cesàro matrix of order α, denoted by $C(\alpha)$, is generated by $a_\alpha(n) = \binom{n+\alpha-1}{\alpha-1}$. (See [10, p. 442].) In this case,

$$S_\alpha(n) = \sum_{j=0}^{n} \binom{j+\alpha-1}{\alpha-1} = \binom{n+\alpha}{\alpha}.$$

For $j \leq i$, the$(i,j)^{th}$ entry of $C(\alpha)$ is given by

$$m_\alpha(i,j) = \frac{\binom{i-j+\alpha-1}{\alpha-1}}{\binom{i+\alpha}{\alpha}} = \frac{\alpha\,\Gamma(i-j+\alpha)\,\Gamma(i+1)}{\Gamma(i-j+1)\,\Gamma(i+\alpha+1)}. \tag{1.1}$$

We shall consider $C(\alpha)$ as an operator on ℓ^2. Stirling's approximation of the Gamma function shows the existence of a constant K_α independent of i and j such that

$$m_\alpha(i,j) \leq K_\alpha \frac{(i-j+1)^{\alpha-1}}{(i+1)^\alpha} \leq \frac{K_\alpha}{i+1} = K_\alpha\, m_1(i,j).$$

Here $m_1(i,j)$ is the $(i,j)^{th}$ entry of the Cesàro matrix of order one, $C(1)$. It is well known that $C(1)$ gives rise to a bounded operator on ℓ^2. As a consequence, for all $\alpha \geq 1$, the operator $C(\alpha)$ is bounded on ℓ^2 and $\|C(\alpha)\| \leq K_\alpha \|C(1)\|$.

If $\mathcal{B}(H)$ denotes the space of all bounded linear operators on a Hilbert space H, then the operator $A \in \mathcal{B}(H)$ is *hyponormal* if

$$\langle (A^*A - AA^*)f, f\rangle \geq 0$$

for all $f \in H$. The operator $A \in \mathcal{B}(H)$ is said to be *posinormal* (see [2, 6]) if

$$AA^* = A^*PA$$

for some positive operator $P \in \mathcal{B}(H)$, called the *interrupter*. The operator A is *coposinormal* if A^* is posinormal. Hyponormal operators are necessarily posinormal, but they need not be coposinormal, as the unilateral shift illustrates.

First, consider $C(1)$, whose entries m_{ij} are given by

$$m_{ij} = \begin{cases} \frac{1}{i+1} & \text{for} \quad 0 \leq j \leq i \\ 0 & \text{for} \quad j > i. \end{cases}$$

In [6] it was observed that $C(1)$ satisfies

$$C(1)C(1)^* = C(1)^*P_1C(1)$$

where

$$P_1 :\equiv \operatorname{diag}\left\{\frac{n+1}{n+2} : n \geq 0\right\};$$

therefore,

$$\begin{aligned}\big\langle (C(1)^*C(1) - C(1)C(1)^*)f, f\big\rangle &= \big\langle (C(1)^*C(1) - C(1)^*P_1C(1))f, f\big\rangle \\ &= \big\langle (I - P_1)C(1)f, C(1)f\big\rangle \geq 0\end{aligned}$$

for all $f \in \ell^2$, so $C(1)$ is a hyponormal operator on ℓ^2. In this manner posinormality was used to give a proof of hyponormality for $C(1)$ that is different from an earlier one found in [1]. The coposinormality of $C(1)$ was demonstrated in [6].

Next, consider $C(2)$, the Cesàro matrix of order 2, whose entries m_{ij} given by

$$m_{ij} = \begin{cases} \frac{2(i+1-j)}{(i+1)(i+2)} & \text{for} \quad 0 \leq j \leq i \\ 0 & \text{for} \quad j > i. \end{cases}$$

It was recently discovered in [8] that $C(2)$ satisfies

$$C(2)C(2)^* = C(2)^*P_2C(2)$$

where

$$P_2 :\equiv \operatorname{diag}\left\{\frac{(n+1)(n+2)}{(n+3)(n+4)} : n \geq 0\right\}$$

with

$$I - P_2 \geq 0,$$

so $C(2)$ is also hyponormal on ℓ^2. The computations in [8] centered on coposinormality, and the diagonal form of P_2 emerged somewhat serendipitously from those computations.

These observations led the second author to conjecture in [9] that for all integer values of $\alpha \geq 1$ it holds true that

$$C(\alpha)C(\alpha)^* = C^*(\alpha)P_\alpha C(\alpha) \tag{1.2}$$

where

$$P_\alpha :\equiv \operatorname{diag}\left\{\frac{(n+1)\cdots(n+\alpha)}{(n+\alpha+1)\cdots(n+2\alpha)} \,:\, n \geq 0\right\}. \tag{1.3}$$

The first author then set out to prove that conjecture, and this paper is the result.

It follows from (1.1) that $m_\alpha(i,j)$ is well defined whenever α is a complex number which is neither zero nor a negative integer. We may then define $C(\alpha)$ for all such α. With the help of Zeilberger's Algorithm [5, Chapter 6] and Maple™ [4], we are able to show in this note that, for a suitably defined diagonal operator P_α,

$$C(\alpha)C(\alpha)^* = C(\alpha)^* P_\alpha C(\alpha)$$

holds for all complex numbers $\alpha \in \mathbb{C}\backslash\{0,-1,-2,\ldots\}$.

2. Main Results

First the definition of P_α must be modified appropriately when α is not a positive integer. We define $P_\alpha = \operatorname{diag}\{d_\alpha(n) : n \geq 0\}$, where

$$d_\alpha(n) = \frac{\left|\Gamma(n+\alpha+1)\right|^2}{\Gamma(n+1)\,\Gamma(n+\alpha+\bar{\alpha}+1)}. \tag{2.1}$$

In the case $\alpha = 1$ (respectively, $\alpha = 2$), our matrix P_α coincides with P_1 (respectively, P_2) as specified in the introduction.

For any complex number α with a positive real part, we have $d_\alpha(n) > 0$, and an application of Cauchy-Schwarz's inequality shows that $d_\alpha(n) \leq 1$ for all $n \geq 0$. Indeed,

$$\begin{aligned}
\Gamma(n+1)\,\Gamma(n+\alpha+\bar{\alpha}+1) &= \Big(\int_0^\infty t^n e^{-t}\,dt\Big)\Big(\int_0^\infty t^{n+\alpha+\bar{\alpha}}\,e^{-t}\,dt\Big) \\
&\geq \Big(\int_0^\infty t^{n+(\alpha+\bar{\alpha})/2}\,e^{-t}\,dt\Big)^2 \\
&= \Big(\int_0^\infty |t^{n+\alpha}|\,e^{-t}\,dt\Big)^2 \\
&\geq \Big|\int_0^\infty t^{n+\alpha}\,e^{-t}\,dt\Big|^2 = \Big|\Gamma(n+\alpha+1)\Big|^2.
\end{aligned}$$

This shows that P_α is a positive contractive operator on ℓ^2. Furthermore, the well-known asymptotic behavior of the Gamma function shows that $\lim_{n\to\infty} d_\alpha(n) = 1$. As a result, P_α is an invertible operator.

Our main result in this note is the following theorem.

Theorem 2.1. *For any complex number $\alpha \in \mathbb{C}\backslash\{0,-1,-2,\ldots\}$, we have*

$$C(\alpha)C(\alpha)^* = C(\alpha)^* P_\alpha C(\alpha). \tag{2.2}$$

Proof. Since both sides of (2.2) are self-adjoint matrices, it suffices to show that for any integers $v, i \geq 0$, the $(i, i+v)^{th}$ entries of both sides are equal. To this end, we shall fix v. For $i \geq 0$, let $A(i)$ (respectively, $B(i)$) be the $(i, i+v)^{th}$ entry of $C(\alpha)C(\alpha)^*$ (respectively, $C(\alpha)^* P_\alpha C(\alpha)$). Clearly, $A(i)$ and $B(i)$ depend on v and α as well but since only i changes in our argument below, we have dropped v and α for the sake of simplifying the notation.

We compute, using the fact that $\overline{\Gamma}(z) = \Gamma(\bar{z})$ for $z \in \mathbb{C}\backslash\{0,-1,-2,\ldots\}$,

$$\begin{aligned}
A(i) &= \sum_{\ell=0}^{\infty} m_\alpha(i,\ell) \cdot \overline{m}_\alpha(i+v,\ell) \\
&= \sum_{\ell=0}^{i} \frac{\alpha\,\Gamma(i-\ell+\alpha)\,\Gamma(i+1)}{\Gamma(i-\ell+1)\,\Gamma(i+\alpha+1)} \cdot \frac{\bar{\alpha}\,\Gamma(i+v-\ell+\bar{\alpha})\,\Gamma(i+v+1)}{\Gamma(i+v-\ell+1)\,\Gamma(i+v+\bar{\alpha}+1)} \\
&= \frac{|\alpha|^2\,\Gamma(i+1)\,\Gamma(i+v+1)}{\Gamma(i+\alpha+1)\,\Gamma(i+v+\bar{\alpha}+1)} \sum_{\ell=0}^{i} \frac{\Gamma(i-\ell+\alpha)\,\Gamma(i+v-\ell+\bar{\alpha})}{\Gamma(i-\ell+1)\,\Gamma(i+v-\ell+1)} \\
&= \frac{|\alpha|^2\,\Gamma(i+1)\,\Gamma(i+v+1)}{\Gamma(i+\alpha+1)\,\Gamma(i+v+\bar{\alpha}+1)} \sum_{k=0}^{i} \frac{\Gamma(k+\alpha)\,\Gamma(k+v+\bar{\alpha})}{\Gamma(k+1)\,\Gamma(k+v+1)}. \qquad (2.3)
\end{aligned}$$

The last equality follows from the change of indices $k = i - \ell$. Note that $A(0) = \frac{\bar{\alpha}}{v+\bar{\alpha}}$. Using the **ZeilbergerRecurrence** command in Maple™, we find that $A(i)$ satisfies the recurrence relation

$$(i+\alpha+1)(i+v+\bar{\alpha}+1)A(i+1) - (i+1)(i+v+1)A(i) = |\alpha|^2. \tag{2.4}$$

To prove this, we use (2.3) with $i+1$ in place of i and properties of the Gamma function to write

$$\begin{aligned}
&(i+\alpha+1)(i+v+\bar{\alpha}+1)A(i+1) \\
&\qquad = \frac{|\alpha|^2\,\Gamma(i+2)\,\Gamma(i+v+2)}{\Gamma(i+\alpha+1)\,\Gamma(i+v+\bar{\alpha}+1)} \sum_{k=0}^{i+1} \frac{\Gamma(k+\alpha)\,\Gamma(k+v+\bar{\alpha})}{\Gamma(k+1)\,\Gamma(k+v+1)} \\
&\qquad = |\alpha|^2 + \frac{|\alpha|^2\,\Gamma(i+2)\,\Gamma(i+v+2)}{\Gamma(i+\alpha+1)\,\Gamma(i+v+\bar{\alpha}+1)} \sum_{k=0}^{i} \frac{\Gamma(k+\alpha)\,\Gamma(k+v+\bar{\alpha})}{\Gamma(k+1)\,\Gamma(k+v+1)} \\
&\qquad = |\alpha|^2 + (i+1)(i+v+1)A(i).
\end{aligned}$$

The recurrence relation (2.4) then follows.

Let us now compute the $(i, i+v)^{th}$ entry of the matrix $C(\alpha)^* P_\alpha C(\alpha)$, using the formulas (1.1) and (2.1),

$$B(i) = \sum_{\ell=0}^{\infty} \overline{m}_\alpha(\ell,i) \cdot d_\alpha(\ell) \cdot m_\alpha(\ell,i+v) = \sum_{\ell=i+v}^{\infty} \overline{m}_\alpha(\ell,i) \cdot d_\alpha(\ell) \cdot m_\alpha(\ell,i+v)$$

$$= \sum_{\ell=i+v}^{\infty} \frac{|\alpha|^2\,\Gamma(\ell-i+\bar{\alpha})\,\Gamma(\ell+1)\,\Gamma(\ell-i-v+\alpha)}{\Gamma(\ell-i+1)\,\Gamma(\ell-i-v+1)\,\Gamma(\ell+\alpha+\bar{\alpha}+1)}$$

$$= \sum_{k=0}^{\infty} \frac{|\alpha|^2\,\Gamma(k+v+\bar{\alpha})\,\Gamma(k+i+v+1)\,\Gamma(k+\alpha)}{\Gamma(k+v+1)\,\Gamma(k+1)\,\Gamma(k+i+v+\alpha+\bar{\alpha}+1)},$$

by the change of indices $\ell = k+i+v$. The **ZeilbergerRecurrence** command again shows that $B(i)$ satisfies the same recurrence relation as $A(i)$ does. That is,

$$(i+\alpha+1)(i+v+\bar{\alpha}+1)B(i+1) - (i+1)(i+v+1)B(i) = |\alpha|^2. \quad (2.5)$$

This time, verifying (2.5) by a direct calculation does not seem easy. However, by following the Zeilberger's telescoping method and assistance by Maple™, we shall demonstrate that the above recurrence equation indeed holds. To do that, let $F(i,k)$ be the summands in the series of $B(i)$. The command **Zeilberger** in Maple™ provides the following identity

$$\begin{aligned}(i+\alpha+1)(i+v+\bar{\alpha}+1)F(i+1,k) - (i+1)(i+v+1)F(i,k)\\ = (k+1)(k+v+1)F(i,k+1) - k(k+v)F(i,k).\end{aligned} \quad (2.6)$$

Divided by $F(i+1,k)$, equation (2.6) boils down to

$$\begin{aligned}&(i+\alpha+1)(i+v+\bar{\alpha}+1) - (i+1)(i+v+1)\frac{k+i+v+\alpha+\bar{\alpha}+1}{k+i+v+1}\\ &\quad = (k+1)(k+v+1)\frac{(k+v+\bar{\alpha})(k+\alpha)}{(k+v+1)(k+1)} - k(k+v)\frac{k+i+v+\alpha+\bar{\alpha}+1}{k+i+v+1},\end{aligned}$$

which can now be verified by a direct calculation, or by any symbolic calculator. From (2.6), summing in $k=0$ to ∞ and noticing that the right hand side is telescoping, together with the fact that

$$\lim_{k\to\infty} k(k+v)F(i,k) = |\alpha|^2,$$

we obtain (2.5). Now for $i=0$,

$$\begin{aligned}B(0) &= |\alpha|^2 \sum_{k=0}^{\infty} \frac{\Gamma(k+v+\bar{\alpha})\,\Gamma(k+\alpha)}{\Gamma(k+1)\,\Gamma(k+v+\alpha+\bar{\alpha}+1)}\\ &= |\alpha|^2\,\frac{\Gamma(v+\bar{\alpha})\,\Gamma(\alpha)}{\Gamma(v+\alpha+\bar{\alpha}+1)}\ {}_2F_1(v+\bar{\alpha},\ \alpha;\ v+\alpha+\bar{\alpha}+1;\ 1)\\ &= |\alpha|^2\,\frac{\Gamma(v+\bar{\alpha})\,\Gamma(\alpha)}{\Gamma(v+\alpha+\bar{\alpha}+1)} \cdot \frac{\Gamma(v+\alpha+\bar{\alpha}+1)}{\Gamma(\alpha+1)\,\Gamma(v+\bar{\alpha}+1)}\\ &\qquad \text{(by Gauss's Hypergeometric Theorem)}\\ &= \frac{\bar{\alpha}}{v+\bar{\alpha}}.\end{aligned}$$

Here ${}_2F_1(a,b;c;1)$ denotes the hypergeometric series with upper parameters a, b and lower parameter c, evaluated at 1. Gauss's Hypergeometric Theorem says that

$$ {}_2F_1(a,b;c;1) = \frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)}, $$

whenever $c-a-b$ has a positive real part. We would like to mention that the proof of Gauss's Theorem and other identities involving hypergeometric series can be done using the WZ method. See [5] for more details.

Since $A(0) = B(0)$, and $A(i)$ and $B(i)$ satisfy the same recurrence relation, we conclude that $A(i) = B(i)$ for all non-negative integers i, which completes the proof of the theorem. □

Corollary 2.2. *For all $\alpha \geq 1$, $C(\alpha)$ is a posinormal and hyponormal operator on ℓ^2.*

We thank B. E. Rhoades for pointing out that the corollary above also follows from [11, Theorem 2].

Corollary 2.3. *For all $\alpha \geq 1$, $C(\alpha)$ is a coposinormal operator on ℓ^2.*

Proof. Apply [7, Theorem 1(d)], using the fact that the interrupter P_α in the proof of the theorem above is invertible. □

Corollary 2.4. *For all $\alpha \geq 1$, both $C(\alpha)$ and $C(\alpha)^*$ are injective and have dense range with*

$$ \mathrm{Ran}(C(\alpha)) = \mathrm{Ran}(C(\alpha)^*). $$

Proof. Since $C(\alpha)$ is posinormal, it follows from [6, Theorem 2.1 and Corollary 2.3] that

$$ \mathrm{Ran}(C(\alpha)) \subseteq \mathrm{Ran}(C(\alpha)^*) $$

and

$$ \mathrm{Ker}(C(\alpha)) \subseteq \mathrm{Ker}(C(\alpha)^*). $$

Since $C(\alpha)^*$ is also known to be posinormal (by Corollary 2.3), the reverse inclusions must also hold; therefore,

$$ \mathrm{Ker}(C(\alpha)) = \mathrm{Ker}(C(\alpha)^*) $$

and

$$ \mathrm{Ran}(C(\alpha)) = \mathrm{Ran}(C(\alpha)^*). $$

It is easy to see that $\mathrm{Ker}(C(\alpha)) = \{0\}$. Consequently, both $C(\alpha)$ and $C(\alpha)^*$ are one-to-one, and both have dense range. □

Corollary 2.5. *For all $\alpha \geq 1$, $C(\alpha)^k$ is both posinormal and coposinormal for each positive integer k.*

Proof. This follows from [3, Corollary 1(b)]. □

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Trieu Le
University of Toledo
Dept. of Mathematics and Statistics
University of Toledo
Toledo, OH 43606, U.S.A.
e-mail: `Trieu.Le2@utoledo.edu`

H. C. Rhaly Jr.
1081 Buckley Drive
Jackson, MS 39206, U.S.A.
e-mail: `rhaly@member.ams.org`