Decomposing algebraic $m$-isometric tuples

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Abstract

We show that any $m$-isometric tuple of commuting algebraic operators on a Hilbert space can be decomposed as a sum of a spherical isometry and a commuting nilpotent tuple. Our approach applies as well to tuples of algebraic operators that are hereditary roots of polynomials in several variables.

Keywords: $m$-isometry, nilpotent, commuting tuple

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1. Introduction

The notion of $m$-isometries was introduced and studied by Agler [3] back in the eighties. A bounded linear operator $T$ on a complex Hilbert space $\mathcal{H}$ is called $m$-isometric if it satisfies the operator equation

$$\sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} T^*kT^k = 0,$$

where $T^*$ is the adjoint operator of $T$. Equivalently, for all $v \in \mathcal{H}$,

$$\sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} \|T^k v\|^2 = 0.$$

In a series of papers [5] [6] [7], Agler and Stankus gave an extensive study of $m$-isometric operators. It is clear that any 1-isometric operator is an isometry. Multiplication by $z$ on the Dirichlet space over the unit disk is not an isometry but it is a 2-isometry. Richter [30] showed that any
cyclic 2-isometry arises from multiplication by $z$ on certain Dirichlet-type spaces. Very recently, researchers have been interested in algebraic properties, cyclicity and supercyclicity of $m$-isometries, among other things. See [28, 24, 14, 16, 15, 18, 13, 12, 26, 11, 22] and the references therein.

It was showed by Agler, Helton and Stankus [4, Section 1.4] that any $m$-isometry $T$ on a finite dimensional Hilbert space admits a decomposition $T = S + N$, where $S$ is a unitary and $N$ is a nilpotent operator satisfying $SN = NS$. In [12], it was showed that if $S$ is an isometry on any Hilbert space and $N$ is a nilpotent operator of order $n$ commuting with $S$ then the sum $S + N$ is a strict $(2n - 1)$-isometry. This result has been generalized to $m$-isometries by several authors [26, 11, 22].

Let $A$ be a positive operator on $H$. An operator $T$ is called an $(A,m)$-isometry if it is a solution to the operator equation

$$\sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} T^{*k} A T^k = 0.$$ 

Such operators were introduced and studied by Sid Ahmed and Saddi in [8], then by other authors [17, 25, 29, 23, 19, 10]. In the case $m = 1$, we call such operators $A$-isometries. Since $A$ is positive, the map $v \mapsto \|v\|_A := \langle Av, v \rangle$ (where $\langle \cdot, \cdot \rangle$ denotes the inner product on $H$) gives rise to a seminorm. In the case $A$ is injective, $\| \cdot \|_A$ becomes a norm. It follows that an operator $T$ is $(A,m)$-isometric if and only if $T$ is $m$-isometric with respect to $\| \cdot \|_A$. As a result, several algebraic properties of $(A,m)$-isometries follow from the corresponding properties of $m$-isometries with more or less similar proofs (see [8, 10]). However, there are great differences between $(A,m)$-isometries and $m$-isometries, specially when $A$ is not injective. For example, it is known [5] that the spectrum of an $m$-isometry must either be a subset of the unit circle or the entire closed unit disk. On the other hand, [10, Theorem 2.3] shows that for any compact set $K$ on the plane that intersects the unit circle, there exist a non-zero positive operator $A$ and an $(A,1)$-isometry whose spectrum is exactly $K$. The following question was asked in [10].

**Question 1.** Let $T$ be an $(A,m)$-isometry on a finite dimensional Hilbert space. Is it possible to write $T$ as a sum of an $A$-isometry and a commuting nilpotent operator?

In this paper, we shall answer Question 1 in the affirmative. Indeed, we are able to prove a much more general result, in the setting of multivariable operator theory.
Gleason and Richter [20] considered the multivariable setting of \( m \)-isometries and studied their properties. A commuting \( d \)-tuple of operators \( \mathbf{T} = [T_1, \ldots, T_d] \) is said to be an \( m \)-isometry if it satisfies the operator equation

\[
\sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} \sum_{|\alpha|=k} \frac{k!}{\alpha!} (T^\alpha)^* T^\alpha = 0.
\] (1.1)

Here \( \alpha = (\alpha_1, \ldots, \alpha_d) \) denotes a multiindex of non-negative integers. We have also used the standard multiindex notation: \(|\alpha| = \alpha_1 + \cdots + \alpha_d\), \( \alpha! = \alpha_1! \cdots \alpha_d! \) and \( T^\alpha = T_1^{\alpha_1} \cdots T_d^{\alpha_d} \). Note that 1-isometric tuples are called spherical isometries. It was shown in [20] that the \( d \)-shift on the Drury-Arveson space over the unit ball in \( \mathbb{C}^d \) is \( d \)-isometric. This generalizes the single-variable fact that the unilateral shift on the Hardy space \( H^2 \) over the unit disk is an isometry. Gleason and Richter also studied spectral properties of \( m \)-isometric tuples and they constructed a list of examples of such operators, built from single-variable \( m \)-isometries. Many algebraic properties of \( m \)-isometric tuples have been discovered by the author in an unpublished work and independently by Gu [21]. As an application of our main result in this note, we shall answer the following question in the affirmative.

**Question 2.** Let \( \mathbf{T} \) be an \( m \)-isometric tuple acting on a finite dimensional Hilbert space. Is it possible to write \( \mathbf{T} \) as a sum of a 1-isometric \( \mathbf{S} \) (that is, a spherical isometry) and a nilpotent tuple \( \mathbf{N} \) that commutes with \( \mathbf{S} \)?

To state our main result, we first generalize the notion of \((A, m)\)-isometric operators to tuples. Let \( A \) be any bounded operator on \( \mathcal{H} \) (we do not need to assume that \( A \) is positive). A commuting tuple \( \mathbf{T} = [T_1, \ldots, T_d] \) is said to be \((A, m)\)-isometric if

\[
\sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} \sum_{|\alpha|=k} \frac{k!}{\alpha!} (T^\alpha)^* A T^\alpha = 0.
\] (1.2)

It is clear that \((I, m)\)-isometric tuples (here \( I \) stands for the identity operator) are the same as \( m \)-isometric tuples. We shall call \((A, 1)\)-isometric tuples spherical \( A \)-isometric. They are tuples \( \mathbf{T} \) that satisfies

\[
T_1^* A T_1 + \cdots + T_d^* A T_d = A.
\]

A main result in the paper is the following theorem.
Theorem 1.1. Suppose $T$ is an $(A,m)$-isometric tuple on a finite dimensional Hilbert space. Then there exist a spherical $A$-isometric tuple $S$ and a nilpotent tuple $N$ commuting with $S$ such that $T = S + N$.

In the case of a single operator, Theorem 1.1 answers Question 1 in the affirmative. In the case $A = I$, we also obtain an affirmative answer to Question 2.

2. Hereditary calculus and applications

Our approach uses a generalization of the hereditary functional calculus developed by Agler [1, 2]. We begin with some definitions and notation. We use boldface lowercase letters, for example $x, y$, to denote $d$-tuples of complex variables. Let $\mathbb{C}[x,y]$ denote the space of polynomials in commuting variables $x$ and $y$ with complex coefficients. Let $A$ be a bounded linear operator on a Hilbert space $\mathcal{H}$ and $X, Y$ be two $d$-tuples of commuting bounded operators on $\mathcal{H}$. These two tuples may not commute with each other. We denote by $X^*$ the tuple $[X_1^*, \ldots, X_d^*]$. Let $f \in \mathbb{C}[x,y]$. If

$$f(x,y) = \sum_{\alpha,\beta} c_{\alpha,\beta} x^\alpha y^\beta,$$

where the sum is finite, then we define

$$f(A; X, Y) = \sum_{\alpha,\beta} c_{\alpha,\beta} (X^\alpha)^* A Y^\beta. \quad (2.1)$$

It is clear that the map $f \mapsto f(A; X, Y)$ is linear from $\mathbb{C}[x,y]$ into $(\mathcal{B}(\mathcal{H}))^d$. If $g \in \mathbb{C}[x,y]$ depending only on $x$, then $g(A; X, Y) = g(X^*) A$. On the other hand, if $h \in \mathbb{C}[x,y]$ depending only on $y$, then $h(A; X, Y) = A h(Y)$. Furthermore, if $F = g f h$, then

$$F(A; X, Y) = g(X^*) f(A; X, Y) h(Y). \quad (2.2)$$

If $X = Y$, we shall write $f(A; X)$ instead of $f(A; X, X)$. In the case $A = I$, the identity operator, we shall use $f(X, Y)$ to denote $f(I; X, Y)$. Therefore, $f(X)$ denotes $f(I; X, X)$. We say that $X$ is a hereditary root of $f$ if $f(X) = 0$.

Example 2.1. Define $p_m(x,y) = \left( \sum_{j=1}^d x_j y_j - 1 \right)^m \in \mathbb{C}[x,y]$. It is then clear that $T$ is $m$-isometric if and only if $T$ is a hereditary root of $p_m$, that is, $p_m(T) = 0$. Similarly, $T$ is $(A,m)$-isometric if and only if $p_m(A; T) = 0$. 


Even though the map $f \mapsto f(A;X,Y)$ is not multiplicative in general, it turns out that its kernel is an ideal of $\mathbb{C}[x,y]$. This observation will play an important role in our approach.

**Proposition 2.2.** Let $A$ be a bounded linear operator and let $X$ and $Y$ be two $d$-tuples of commuting operators. Define

$$\mathcal{J}(A;X,Y) = \{ f \in \mathbb{C}[x,y] : f(A;X,Y) = 0 \}.$$  

Then $\mathcal{J}(A;X,Y)$ is an ideal of $\mathbb{C}[x,y]$.

**Proof.** For simplicity of the notation, throughout the proof, let us write $\mathcal{J}$ for $\mathcal{J}(A;X,Y)$. It is clear that $\mathcal{J}$ is a vector subspace of $\mathbb{C}[x,y]$. Now let $f$ be in $\mathcal{J}$ and $g$ be in $\mathbb{C}[x,y]$. We need to show that $gf$ belongs to $\mathcal{J}$. By linearity, it suffices to consider the case $g$ is a monomial $g(x,y) = x^\alpha y^\beta$ for some multi-indices $\alpha$ and $\beta$. By 2.2,

$$(fg)(A;X,Y) = (X^\alpha)^* \cdot f(A;X,Y) \cdot Y^\beta = 0,$$

since $f(A;X,Y) = 0$. This shows that $fg$ belongs to $\mathcal{J}$ as desired.

If $f$ is a polynomial of $y$ in the form $f(y) = \sum \alpha c_\alpha y^\alpha$, we define $\bar{f}(x)$ as

$$\bar{f}(x) = \sum \alpha \bar{c}_\alpha x^\alpha.$$  

In the case $A$ is positive and $X = Y$, we obtain an additional property of the ideal $\mathcal{J}(A;Y,Y)$ as follows.

**Proposition 2.3.** Let $A$ be a positive operator and $Y$ be a $d$-tuple of commuting operators. Suppose $f_1, \ldots, f_m$ are polynomials of $y$ such that the sum $\bar{f}_1(x)f_1(y) + \cdots + \bar{f}_m(x)f_m(y)$ belongs to $\mathcal{J}(A;Y,Y)$. Then $f_1(y), \ldots, f_m(y)$ also belong to $\mathcal{J}(A;Y,Y)$.

**Proof.** Note that $\bar{f}_j(Y^*) = (f_j(Y))^*$ for all $j$. By the hypotheses, we have

$$(f_1(Y))^* Af_1(Y) + \cdots + (f_m(Y))^* Af_m(Y) = 0,$$

which implies

$$[(A^{1/2} f_1(Y))^*[A^{1/2} f_1(Y)] + \cdots + [A^{1/2} f_m(Y)]^*[A^{1/2} f_m(Y)] = 0.$$  

It follows that for all $j$, we have $A^{1/2} f_j(Y) = 0$, which implies $Af_j(Y) = 0$. Therefore, $f_j(y) \in \mathcal{J}(A;Y,Y)$ for all $j$.  

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Recall that the radical ideal of an ideal \(\mathcal{I} \subset \mathbb{C}[x, y]\), denoted by \(\text{Rad}(\mathcal{I})\), is the set of all polynomials \(p \in \mathbb{C}[x, y]\) such that \(p^N \in \mathcal{I}\) for some positive integer \(N\). In the following proposition, we provide an interesting relation between generalized eigenvectors and eigenvalues of \(X\) and \(Y\) whenever we have \(f(A; X, Y) = 0\).

**Proposition 2.4.** Let \(X\) and \(Y\) be two \(d\)-tuples of commuting operators. Suppose \(k\) is a positive integer, \(\lambda = (\lambda_1, \ldots, \lambda_d), \omega = (\omega_1, \ldots, \omega_d) \in \mathbb{C}^d\) and \(u, v \in \mathcal{H}\) such that
\[
(X_j - \lambda_j)^k u = (Y_j - \omega_j)^k v = 0
\]
for all \(1 \leq j \leq d\). Then for any polynomial \(f \in \text{Rad}(\mathcal{I}(A; X, Y))\), we have
\[
f(\bar{\lambda}, \omega) \langle Av, u \rangle = 0. \tag{2.3}
\]

**Proof.** We first assume that \(f \in \mathcal{I}(A; X, Y)\). Using Taylor’s expansion, we find polynomials \(g_1, \ldots, g_d\) and \(h_1, \ldots, h_d\) such that
\[
f(\bar{\lambda}, \omega) - f(x, y) = \sum_{j=1}^{d} (x_j - \bar{\lambda}_j) g_j(x, y) + \sum_{j=1}^{d} h_j(x, y)(y_j - \omega_j).
\]
Take any integer \(M \geq 1 + 2d(k - 1)\). By the multinomial expansion, there exist polynomials \(G_1, \ldots, G_d\) and \(H_1, \ldots, H_d\) such that
\[
\left(f(\bar{\lambda}, \omega) - f(x, y)\right)^M = \sum_{j=1}^{d} (x_j - \bar{\lambda}_j)^k G_j(x, y) + \sum_{j=1}^{d} H_j(x, y)(y_j - \omega_j)^k.
\]
The left-hand side, by the binomial expansion, can be written as
\[
(f(\bar{\lambda}, \omega))^M + f(x, y) H(x, y)
\]
for some polynomial \(H\). Since \(f(A; X, Y) = 0\), using Equation \(2.2\) and Proposition \(2.2\), we conclude that
\[
(f(\bar{\lambda}, \omega))^M \cdot A = \sum_{j=1}^{d} (X_j^* - \bar{\lambda}_j)^k G_j(A; X, Y) + \sum_{j=1}^{d} H_j(A; X, Y)(Y_j - \omega_j)^k.
\]
Consequently,

\[
(f(\dot{\lambda}, \omega))^N \langle Av, u \rangle = \sum_{j=1}^{d} \langle G_j(A; X, Y)v, (X_j - \lambda_j)^k u \rangle + \sum_{j=1}^{d} \langle H_j(A; X, Y)(Y_j - \omega_j)^k v, u \rangle = 0,
\]

which implies (2.3).

In the general case, there exists an integer \( N \geq 1 \) such that \( f^N \) belongs to \( \mathcal{J}(A; X, Y) \). By the case we have just proved, \( (f(\dot{\lambda}, \omega))^N \langle Av, u \rangle = 0 \), which again implies (2.3). This completes the proof of the proposition.

**Remark 2.5.** In the case of a single operator, Proposition 2.4 provides a generalization of [2, Lemmas 18 and 19]. Our proof here is even simpler and more transparent.

Question 1 and Question 2 in the introduction concern operators acting on a finite dimensional Hilbert space. It turns out that this condition can be replaced by a weaker one. Recall that a linear operator \( T \) is called algebraic if there exist complex constants \( c_0, c_1, \ldots, c_\ell \) such that

\[
c_0 I + c_1 T + \cdots + c_\ell T^\ell = 0.
\]

Algebraic operator roots of polynomials were investigated in [4]. We first discuss some preparatory results on algebraic operators acting on a general complex vector space \( V \). It is well known that if \( T \) is an algebraic linear operator on \( V \), then the spectrum \( \sigma(T) \) is finite and there exists a direct sum decomposition \( V = \bigoplus_{a \in \sigma(T)} V_a \), where each \( V_a \) is an invariant subspace for \( T \) (the subspace \( V_a \) is a closed subspace if \( V \) is a normed space and \( T \) is bounded) and \( T - aI \) is nilpotent on \( V_a \). Indeed, if the minimal polynomial of \( T \) is factored in the form

\[
p(z) = (z - a_1)^{m_1} \cdots (z - a_\ell)^{m_\ell},
\]

where \( a_1, \ldots, a_\ell \) are pairwise distinct and \( m_1, \ldots, m_\ell \geq 1 \), then \( \sigma(T) = \{a_1, \ldots, a_\ell\} \) and \( V_{a_j} = \ker(T - a_j)^{m_j} \) for \( 1 \leq j \leq \ell \). See, for example, [32, Section 6.3], which discusses operators acting on finite dimensional vector spaces. However, the arguments apply to algebraic operators on infinite dimensional vector spaces as well.
Suppose now \( T = [T_1, \ldots, T_d] \) is a tuple of commuting algebraic operators on \( \mathcal{V} \). We first decompose \( \mathcal{V} \) as above with respect to the spectrum \( \sigma(T_1) \). Since each subspace in the decomposition is invariant for all \( T_j \), we again decompose such subspace with respect to the spectrum \( \sigma(T_2) \). Continuing this process, we obtain a finite set \( \Lambda \subset \mathbb{C}^d \) and a direct sum decomposition \( \mathcal{V} = \bigoplus_{\lambda \in \Lambda} \mathcal{V}_\lambda \) such that for each \( \lambda = (\lambda_1, \ldots, \lambda_d) \in \Lambda \) and \( 1 \leq j \leq d \), the subspace \( \mathcal{V}_\lambda \) is invariant for \( T \) and \( T_j - \lambda_j I \) is nilpotent on \( \mathcal{V}_\lambda \). Let \( E_\lambda \) denote the canonical projection (possibly non-orthogonal) from \( \mathcal{V} \) onto \( \mathcal{V}_\lambda \). Then we have

\[
S = \sum_{\lambda \in \Lambda} \lambda \cdot E_\lambda = \left[ \sum_{\lambda \in \Lambda} \lambda_1 E_\lambda, \ldots, \sum_{\lambda \in \Lambda} \lambda_d E_\lambda \right]
\]

(2.4)

Then \( S \) is a tuple of commuting operators which commutes with \( T \), and \( T - S \) is nilpotent. For any multiindex \( \alpha \), we have

\[
S^\alpha = S_1^{\alpha_1} \cdots S_d^{\alpha_d} = \sum_{\lambda \in \Lambda} \lambda^\alpha E_\lambda.
\]

In the case \( \mathcal{V} \) is a normed space and \( T \) is bounded, each operator in the tuple \( S \) is bounded as well.

We now prove a very general result, which will provide affirmative answers to Questions 1 and 2 in the introduction.

**Theorem 2.6.** Let \( X \) and \( Y \) be two \( d \)-tuples of commuting algebraic operators on a Hilbert space \( \mathcal{H} \). Let \( U \) (respectively, \( V \)) be the commuting tuple associated with \( X \) (respectively, \( Y \)) as in (2.4). Then

\[
\text{Rad}(\mathcal{J}(A; X, Y)) \subseteq \mathcal{J}(A; U, V).
\]

(2.5)

**Proof.** Write \( X = [X_1, \ldots, X_d] \) and decompose \( \mathcal{H} = \bigoplus_{\lambda \in \Lambda} \mathcal{H}_\lambda \) such that for each \( \lambda = (\lambda_1, \ldots, \lambda_d) \in \Lambda \), the subspace \( \mathcal{H}_\lambda \) is invariant for \( X \) and \( X_j - \lambda_j I \) is nilpotent on \( \mathcal{H}_\lambda \). Let \( U_\lambda \) denote the canonical projection from \( \mathcal{H} \) onto \( \mathcal{H}_\lambda \). Then

\[
U^\alpha = \sum_{\lambda \in \Lambda} \lambda^\alpha \cdot U_\lambda.
\]

Similarly, write \( Y = [Y_1, \ldots, Y_d] \) and decompose \( \mathcal{H} = \bigoplus_{\omega \in \Omega} \mathcal{K}_\omega \). Let \( V_\omega \) be the canonical projection from \( \mathcal{H} \) onto \( \mathcal{K}_\omega \). Then

\[
V^\beta = \sum_{\omega \in \Omega} \omega^\beta \cdot V_\omega.
\]
Take any polynomial \( p \in \text{Rad}(J(A; X, Y)) \). For \( \lambda \in \Lambda, \omega \in \Omega \) and vectors \( u \in H_\lambda \) and \( v \in K_\omega \), there exists an integer \( k \geq 1 \) sufficiently large such that

\[
(X_j - \lambda_j I)^k u = (Y_j - \omega_j I)^k v = 0
\]

for all \( 1 \leq j \leq d \). Proposition 2.4 shows that

\[
p(\bar{\lambda}, \omega) \langle Av, u \rangle = 0,
\]

which implies

\[
p(\bar{\lambda}, \omega) U_\lambda^* AV_\omega = 0.
\]

Write \( p(x, y) = \sum_{\alpha, \beta} c_{\alpha, \beta} x^\alpha y^\beta \). We compute

\[
p(A; U, V) = \sum_{\alpha, \beta} c_{\alpha, \beta} U_\alpha^* A V^\beta
\]

\[
= \sum_{\alpha, \beta} c_{\alpha, \beta} \left( \sum_{\lambda \in \Lambda} \bar{\lambda}^\alpha U_\lambda^* \right) A \left( \sum_{\omega \in \Omega} \omega^\beta \cdot V_\omega \right)
\]

\[
= \sum_{\lambda \in \Lambda, \omega \in \Omega} \left( \sum_{\alpha, \beta} c_{\alpha, \beta} \bar{\lambda}^\alpha \omega^\beta \right) U_\lambda^* AV_\omega
\]

\[
= \sum_{\lambda \in \Lambda, \omega \in \Omega} p(\bar{\lambda}, \omega) U_\lambda^* AV_\omega = 0.
\]

We conclude that \( p \in J(A; U, V) \). Since \( p \in \text{Rad}(J(A; X, Y)) \) was arbitrary, the proof of the theorem is complete.

Theorem 2.6 enjoys numerous interesting applications that we now describe.

**Proof of Theorem 1.1.** We shall prove the theorem under a more general assumption that \( T \) is a tuple of commuting algebraic operators. Since \( T \) is \((A, m)-isometric\), the polynomial \( (\sum_{j=1}^d x_j y_j - 1)^m \) belongs to the ideal \( J(A; T, T) \). It follows that the polynomial \( p(x, y) = \sum_{j=1}^d x_j y_j - 1 \) belongs to the radical ideal of \( J(A; T, T) \). By Theorem 2.6 we may decompose \( T = S + N \), where \( N \) is a nilpotent tuple commuting with \( S \) and \( p(A; S, S) = 0 \), which means that \( S \) is a spherical \( A \)-isometry.

**Example 2.7.** Recall that an operator \( T \) is called \((m, n)-isosymmetric\) (see [33]) if \( T \) is a hereditary root of \( f(x, y) = (xy - 1)^m (x - y)^n \). Theorem 2.6 shows that any such algebraic \( T \) can be decomposed as \( T = S + N \), where \( N \) is nilpotent, \( S \) is isosymmetric (i.e. \((1, 1)-isosymmetric\)) and \( SN = NS \).
Example 2.8. Several researchers [27, 9] have investigated the so-called toral $m$-isometric tuples. It is straightforward to generalize this notion to toral $(A, m)$-isometric tuples, which are commuting $d$-tuples $T$ that satisfy

$$
\sum_{\substack{0 \leq \alpha_1 \leq m_1 \\
0 \leq \alpha_2 \leq m_2}} (-1)^{\|\alpha\|} \binom{m_1, \ldots, m_d}{\alpha} (T^\alpha)^* A T^\alpha = 0.
$$

for all $m_1 + \cdots + m_d = m$. Equivalently, $T$ is a common hereditary root of all polynomials of the form $(1 - x_1 y_1)^{m_1} \cdots (1 - x_d y_d)^{m_d}$ for $m_1 + \cdots + m_d = m$. This means that all these polynomials belong to the ideal $I(A; T, T)$. We see that toral $(A, 1)$-isometries are just commuting tuples $T$ such that each $T_j$ is an $A$-isometry, that is, $T_j^* A T_j = A$. Note that for any toral $(A, m)$-isometry $T$, the radical ideal $\text{Rad} (I(A; T, T))$ contains all the polynomials $\{1 - x_j y_j : j = 1, 2, \ldots, d\}$. Theorem 2.6 asserts that $T = S + N$, where $S$ is a toral $(A, 1)$-isometry and $N$ is a nilpotent tuple commuting with $S$.

3. On 2-isometric tuples

It is well known that any 2-isometry on a finite dimensional Hilbert space must actually be an isometry. On the other hand, there are many examples of finite dimensional 2-isometric tuples that are not spherical isometries. The following class of examples is given in Richter’s talk [31].

Example 3.1. If $\alpha = (\alpha_1, \ldots, \alpha_d) \in \partial B_d$ and $V_j : C^m \rightarrow C^n$ such that $\sum_{j=1}^d \alpha_j V_j = 0$, then $W = (W_1, \ldots, W_d)$ with

$$
W_j = \begin{pmatrix}
\alpha_j I_n & V_j \\
0 & \alpha_j I_m
\end{pmatrix}
$$

defines a 2-isometric $d$-tuple.

The following result was stated in [31] without a proof and as far as the author is aware of, it has not appeared in a published paper.

Theorem 3.2 (Richter-Sundberg). If $T$ is a 2-isometric tuple on a finite dimensional space, then

$$
T = U \oplus W,
$$

where $U$ is a spherical unitary and $W$ is a direct sum of operator tuples unitarily equivalent to those in Example 3.1.
In this section, we shall assume that \( A \) is self-adjoint and investigate \((A, 2)\)-isometric \( d \)-tuples. We obtain a characterization for such tuples that generalizes the above theorem. We first provide a generalization of Example 3.1. We call \( \mathbf{N} = (N_1, \ldots, N_d) \) an \((A, n)\)-nilpotent tuple if \( AN^\alpha = 0 \) for any indices \( \alpha \) with \(|\alpha| = n\).

**Proposition 3.3.** Assume that \( A \) is a self-adjoint operator. Let \( S \) be an \((A, 1)\)-isometry and \( \mathbf{N} \) an \((A, 2)\)-nilpotent tuple such that \( S \) commutes with \( \mathbf{N} \). Suppose \( S^*_j AN_j + \cdots + S^*_d AN_d = 0 \), then \( S + \mathbf{N} \) is an \((A, 2)\)-isometry.

**Proof.** By the assumption, we have \( AN_j N_k = N^*_j N^*_k A = 0 \) for \( 1 \leq j, k \leq d \),
\[
\sum_{j=1}^d S^*_j AS_j = A, \quad \text{and} \quad \sum_{j=1}^d S^*_j AN_j = \sum_{j=1}^d N^*_j AS_j = 0.
\]
It follows that
\[
\sum_{j=1}^d (S_j + N_j)^* A (S_j + N_j) = A + \sum_{j=1}^d N^*_j AN_j.
\]
We then compute
\[
\sum_{1 \leq k, j \leq d} (S_k + N_k)^*(S_j + N_j)^* A (S_j + N_j)(S_k + N_k)
\]
\[
= \sum_{k=1}^d (S^*_k + N^*_k)(A + \sum_{j=1}^d N^*_j AN_j)(S_k + N_k)
\]
\[
= \sum_{k=1}^d (S^*_k + N^*_k) A (S_k + N_k) + \sum_{1 \leq k, j \leq d} (S^*_k + N^*_k)N^*_j AN_j(S_k + N_k)
\]
\[
= A + \sum_{k=1}^d N^*_k AN_k + \sum_{1 \leq k, j \leq d} S^*_k N^*_j AN_j S_k
\]
\[
= A + \sum_{k=1}^d N^*_k AN_k + \sum_{j=1}^d N^*_j \left( \sum_{k=1}^d S^*_k AS_k \right) N_j
\]
\[
= A + \sum_{k=1}^d N^*_k AN_k + \sum_{j=1}^d N^*_j AN_j
\]
\[
= A + 2 \sum_{j=1}^d N^*_j AN_j
\]
\[ = 2 \sum_{j=1}^{d} (S_j + N_j)^* A (S_j + N_j) - A. \]

Consequently, the sum \( S + N \) is an \((A, 2)\)-isometric tuple.

**Remark 3.4.** We have provided a direct proof of Proposition 2.3. Using the hereditary functional calculus and the approach in [26], one may generalize the result to the case \( S \) being an \((A, m)\)-isometry and \( N \) an \((A, n)\)-nilpotent commuting with \( S \). Under such an assumption, if \( S_j^* A N_1 + \cdots + S_d^* A N_d = 0 \), then \( S + N \) is an \((A, m + 2n - 3)\)-isometry. We leave the details for the interested reader.

We now show that any algebraic \((A, 2)\)-isometric tuple has the form given in Proposition 3.3 and as a result, provide a proof of Richter-Sundberg’s theorem.

**Theorem 3.5.** Assume that \( A \) is a positive operator. Let \( T \) be an algebraic \((A, 2)\)-isometric tuple on \( \mathcal{H} \). Then there exists an \((A, 1)\)-isometric tuple \( S \) and a tuple \( N \) commuting with \( S \) such that \( T = S + N \), \( \sum_{j=1}^{d} S_j^* A N_j = 0 \), and \( AN_j N_j = 0 \) for all \( 1 \leq j \leq d \) (we call such \( N \) an \((A, 2)\)-nilpotent tuple).

In the case \( \mathcal{H} \) is finite dimensional and \( A = I \), the identity operator, we recover Theorem 3.2.

**Proof.** Recall that there exists a finite set \( \Lambda \subset C^d \) and a direct sum decomposition \( \mathcal{H} = \bigoplus_{\lambda \in \Lambda} \mathcal{H}_\lambda \) such that for each \( \lambda \in \Lambda \), the subspace \( \mathcal{H}_\lambda \) is invariant for \( T \) and \( T_j - \lambda_j I \) is nilpotent on \( \mathcal{H}_\lambda \). Let \( S \) be defined as in (2.4) and put \( N = T - S \). From the construction, \( N \) is nilpotent and Theorem 2.6 shows that \( S \) is \((A, 1)\)-isometric. We shall show that \( N \) satisfies the required properties.

Restricting on each invariant subspace \( \mathcal{H}_\lambda \), we only need to consider the case \( \mathcal{H} = \mathcal{H}_\lambda \) and so \( S = \lambda I \). Proposition 2.4 asserts that \(|\lambda|^2 - 1 \langle Av, u \rangle = 0 \) for all \( v, u \in \mathcal{H} \). If \(|\lambda| \neq 1 \), then \( A = 0 \) and the conclusion follows. Now we assume that \(|\lambda| = 1 \). Since \( N \) is nilpotent, there exists a positive integer \( r \) such that \( A N^\alpha = 0 \) whenever \(|\alpha| = r \). We claim that \( r \) may be taken to be 2. To prove the claim, we assume \( r \geq 3 \) and show that \( A N^\alpha = 0 \) for all \(|\alpha| = r - 1 \).
Since $T = \lambda I + N$ is $(A, 2)$-isometric, the tuple $N$ is an $A$-root of the polynomial

$$p(x, y) = \left( \sum_{j=1}^{d} (x_j + \bar{\lambda}_j)(y_j + \lambda_j) - 1 \right)^2 = \left( \sum_{j=1}^{d} x_jy_j + \lambda_jx_j + \bar{\lambda}_jy_j \right)^2.$$  

On the other hand, $N$ is an $A$-root of $x^\alpha$ and $y^\alpha$ for all $|\alpha| = r$. This shows that $p(x, y)$, $x^\alpha$ and $y^\alpha$ belong to $J(A; N, N)$ for all $|\alpha| = r$. To simplify the notation, we shall denote $J(A; N, N)$ by $J$ in the rest of the proof. Take any multiindex $\beta$ with $|\beta| = r - 2$. We write

$$x^\beta p(x, y)y^\beta = x^\beta \left( \sum_{j=1}^{d} \lambda_j x_j \right) \left( \sum_{\ell=1}^{d} \bar{\lambda}_\ell y_\ell \right)y^\beta + \sum_{|\gamma| \geq r} x^\gamma H_\gamma(x, y) + G_\gamma(x, y)y^\gamma$$

for some polynomials $H_\gamma$ and $G_\gamma$. Since the left-hand side and the second term on the right-hand side belong to $J$, which is an ideal, we conclude that

$$x^\beta \left( \sum_{j=1}^{d} \lambda_j x_j \right) \left( \sum_{\ell=1}^{d} \bar{\lambda}_\ell y_\ell \right)y^\beta \in J.$$  

Proposition 2.3 shows that both $\left( \sum_{\ell=1}^{d} \bar{\lambda}_\ell y_\ell \right)y^\beta$ and $x^\beta \left( \sum_{j=1}^{d} \lambda_j x_j \right)$ are in $J$. Now for any multiindex $\gamma$ with $|\gamma| = r - 3$, we compute

$$x^\gamma p(x, y)y^\gamma = x^\gamma \left( \sum_{j=1}^{d} x_jy_j \right)^2 y^\gamma + \sum_{|\beta| = r - 2} x^\beta \left( \sum_{j=1}^{d} \lambda_j x_j \right) P_\beta(x, y)$$

$$+ \sum_{|\beta| = r - 2} \left( \sum_{j=1}^{d} \lambda_j x_j \right) y^\beta Q_\beta(x, y).$$

Since the left-hand side and the last two sums on the right-hand side belong to $J$, it follows that $x^\gamma \left( \sum_{j=1}^{d} x_jy_j \right)^2 y^\gamma$ belongs to $J$. Another application of Proposition 2.3 then shows that $y_j y_\ell y^\gamma$ belongs to $J$ for all $1 \leq j, \ell \leq d$. That is, $y^\alpha$ belongs to $J$ whenever $|\alpha| = r - 1$ (as long as $r \geq 3$). As a consequence, we see that $y^\alpha$, and hence $x^\alpha$, belong to $J$ for all $|\alpha| = 2$. This together with the fact that $p(x, y) \in J$ forces $\sum_{j=1}^{d} \lambda_j x_j)(\sum_{\ell=1}^{d} \bar{\lambda}_\ell y_\ell)$ to belong to $J$, which implies that $\sum_{\ell=1}^{d} \bar{\lambda}_\ell y_\ell$ is in $J$. We have then shown $AN_jN_\ell = 0$ for all $1 \leq j, \ell \leq d$ and $\sum_{\ell=1}^{d} S^\ast_\ell AN_\ell = \sum_{\ell=1}^{d} \bar{\lambda}_\ell AN_\ell = 0$, as desired.
Now let us consider $T$ a 2-isometric tuple on a finite dimensional space $\mathcal{H}$. Recall that we have the decomposition $\mathcal{H} = \bigoplus_{\lambda \in \Lambda} \mathcal{H}_{\lambda}$ such that for each $\lambda \in \Lambda$, the subspace $\mathcal{H}_{\lambda}$ is invariant for $T$ and $T_{j} - \lambda_{j}I$ is nilpotent on $\mathcal{H}_{\lambda}$. By Proposition 2.4, we have $(\langle \omega, \lambda \rangle - 1)^{2}\langle v, u \rangle = 0$ for all $v \in \mathcal{H}_{\omega}$ and $u \in \mathcal{H}_{\lambda}$. It follows that $|\lambda| = 1$ for all $\lambda \in \Lambda$ and $\mathcal{H}_{\lambda} \perp \mathcal{H}_{\omega}$ whenever $\lambda \neq \omega$. As a result, each subspace $\mathcal{H}_{\lambda}$ is reducing for $T$. To complete the proof, it suffices to consider $\mathcal{H} = \mathcal{H}_{\lambda}$. We shall show that either $T$ is a spherical unitary or it is unitarily equivalent to a tuple given in Example 3.1. Indeed, we have $T = \lambda I + N$, where $\sum_{\ell=1}^{d} \lambda_{\ell} N_{\ell} = 0$ and $N_{j} N_{\ell} = 0$ for all $1 \leq j, \ell \leq d$. If $N = 0$, then $T$ is a spherical unitary. Otherwise, let $M = \ker(N_{1}) \cap \cdots \cap \ker(N_{d})$. Then $N_{\ell}(\mathcal{H}) \subseteq M$ for all $1 \leq \ell \leq d$. As a consequence, with respect to the orthogonal decomposition $\mathcal{H} = M \oplus M^{\perp}$, each $N_{\ell}$ has the form

$$N_{\ell} = \begin{pmatrix} 0 & V_{\ell} \\ 0 & 0 \end{pmatrix}$$

for some $V_{\ell} : M^{\perp} \to M$. Since $\sum_{\ell=1}^{d} \lambda_{\ell} N_{\ell} = 0$, we have $\sum_{\ell=1}^{d} \lambda_{\ell} V_{\ell} = 0$. It follows that $T$ is unitarily equivalent to an operator tuple in Example 3.1.

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References


