Decomposing algebraic m-isometric tuples

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Abstract

We show that any m-isometric tuple of commuting algebraic operators on a Hilbert space can be decomposed as a sum of a spherical isometry and a commuting nilpotent tuple. Our approach applies as well to tuples of algebraic operators that are hereditary roots of polynomials in several variables.

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1. Introduction

The notion of m-isometries was introduced and studied by Agler [3] back in the eighties. A bounded linear operator T on a complex Hilbert space $\mathcal{H}$ is called m-isometric if it satisfies the operator equation

$$\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}T^{*k}T^{k}=0,$$

where T^* is the adjoint operator of T. Equivalently, for all $v \in \mathcal{H}$,

$$\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}\|T^{k}v\|^{2}=0.$$

In a series of papers [5, 6, 7], Agler and Stankus gave an extensive study of m-isometric operators. It is clear that any 1-isometric operator is an isometry. Multiplication by z on the Dirichlet space over the unit disk is not an isometry but it is a 2-isometry. Richter [30] showed that any

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cyclic 2-isometry arises from multiplication by z on certain Dirichlet-type spaces. Very recently, researchers have been interested in algebraic properties, cyclicity and supercyclicity of m-isometries, among other things. See [28, 24, 14, 16, 15, 18, 13, 12, 26, 11, 22] and the references therein.

It was showed by Agler, Helton and Stankus [4, Section 1.4] that any m-isometry T on a finite dimensional Hilbert space admits a decomposition $T = S + N$, where S is a unitary and N is a nilpotent operator satisfying $SN = NS$. In [12], it was showed that if S is an isometry on any Hilbert space and N is a nilpotent operator of order n commuting with S then the sum $S + N$ is a strict $(2n-1)$-isometry. This result has been generalized to m-isometries by several authors [26, 11, 22].

Let A be a positive operator on $\mathcal{H}$. An operator T is called an (A, m)-isometry if it is a solution to the operator equation

$$\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}T^{*k}AT^{k} = 0.$$

Such operators were introduced and studied by Sid Ahmed and Saddi in [8], then by other authors [17, 25, 29, 23, 19, 10]. In the case $m = 1$, we call such operators A-isometries. Since A is positive, the map $v \mapsto \|v\|_A := \langle Av, v\rangle$ (where $\langle\cdot,\cdot\rangle$ denotes the inner product on $\mathcal{H}$) gives rise to a seminorm. In the case A is injective, $\|\cdot\|_A$ becomes a norm. It follows that an operator T is (A, m)-isometric if and only if T is m-isometric with respect to $\|\cdot\|_A$. As a result, several algebraic properties of (A, m)-isometries follow from the corresponding properties of m-isometries with more or less similar proofs (see [8, 10]). However, there are great differences between (A, m)-isometries and m-isometries, specially when A is not injective. For example, it is known [5] that the spectrum of an m-isometry must either be a subset of the unit circle or the entire closed unit disk. On the other hand, [10, Theorem 2.3] shows that for any compact set K on the plane that intersects the unit circle, there exist a non-zero positive operator A and an $(A, 1)$-isometry whose spectrum is exactly K. The following question was asked in [10].

Question 1. *Let T be an (A, m)-isometry on a finite dimensional Hilbert space. Is it possible to write T as a sum of an A-isometry and a commuting nilpotent operator?*

In this paper, we shall answer Question 1 in the affirmative. Indeed, we are able to prove a much more general result, in the setting of multivariable operator theory.

Gleason and Richter [20] considered the multivariable setting of m-isometries and studied their properties. A commuting d-tuple of operators $\mathbf{T} = [T_1, \ldots, T_d]$ is said to be an m-isometry if it satisfies the operator equation

$$\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}\sum_{|\alpha|=k}\frac{k!}{\alpha!}\,(T^{\alpha})^{*}T^{\alpha} = 0. \tag{1.1}$$

Here $\alpha = (\alpha_1, \ldots, \alpha_d)$ denotes a multiindex of non-negative integers. We have also used the standard multiindex notation: $|\alpha| = \alpha_1 + \cdots + \alpha_d$, $\alpha! = \alpha_1! \cdots \alpha_d!$ and $\mathbf{T}^{\alpha} = T_1^{\alpha_1} \cdots T_d^{\alpha_d}$. Note that 1-isometric tuples are called *spherical isometries*. It was shown in [20] that the d-shift on the Drury-Arveson space over the unit ball in $\mathbb{C}^d$ is d-isometric. This generalizes the single-variable fact that the unilateral shift on the Hardy space H^2 over the unit disk is an isometry. Gleason and Richter also studied spectral properties of m-isometric tuples and they constructed a list of examples of such operators, built from single-variable m-isometries. Many algebraic properties of m-isometric tuples have been discovered by the author in an unpublished work and independently by Gu [21]. As an application of our main result in this note, we shall answer the following question in the affirmative.

Question 2. *Let* $\mathbf{T}$ *be an* m*-isometric tuple acting on a finite dimensional Hilbert space. Is it possible to write* $\mathbf{T}$ *as a sum of a* 1*-isometric* $\mathbf{S}$ *(that is, a spherical isometry) and a nilpotent tuple* $\mathbf{N}$ *that commutes with* $\mathbf{S}$*?*

To state our main result, we first generalize the notion of (A, m)-isometric operators to tuples. Let A be any bounded operator on $\mathcal{H}$ (we do not need to assume that A is positive). A commuting tuple $\mathbf{T} = [T_1, \ldots, T_d]$ is said to be (A, m)-isometric if

$$\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}\sum_{|\alpha|=k}\frac{k!}{\alpha!}\,(T^{\alpha})^{*}AT^{\alpha} = 0. \tag{1.2}$$

It is clear that (I, m)-isometric tuples (here I stands for the identity operator) are the same as m-isometric tuples. We shall call $(A, 1)$-isometric tuples *spherical* A*-isometric*. They are tuples $\mathbf{T}$ that satisfies

$$T_1^{*}AT_1 + \cdots + T_d^{*}AT_d = A.$$

A main result in the paper is the following theorem.

Theorem 1.1. *Suppose* $\mathbf{T}$ *is an* (A,m)*-isometric tuple on a finite dimensional Hilbert space. Then there exist a spherical* A*-isometric tuple* $\mathbf{S}$ *and a nilpotent tuple* $\mathbf{N}$ *commuting with* $\mathbf{S}$ *such that* $\mathbf{T}=\mathbf{S}+\mathbf{N}$.

In the case of a single operator, Theorem 1.1 answers Question 1 in the affirmative. In the case $A=I$, we also obtain an affirmative answer to Question 2.

2. Hereditary calculus and applications

Our approach uses a generalization of the hereditary functional calculus developed by Agler [1, 2]. We begin with some definitions and notation. We use boldface lowercase letters, for example $\mathbf{x}$, $\mathbf{y}$, to denote d-tuples of complex variables. Let $\mathbb{C}[\mathbf{x},\mathbf{y}]$ denote the space of polynomials in commuting variables $\mathbf{x}$ and $\mathbf{y}$ with complex coefficients. Let A be a bounded linear operator on a Hilbert space $\mathcal{H}$ and $\mathbf{X}$, $\mathbf{Y}$ be two d-tuples of commuting bounded operators on $\mathcal{H}$. These two tuples may not commute with each other. We denote by $\mathbf{X}^*$ the tuple $[X_1^*,\ldots,X_d^*]$. Let $f\in\mathbb{C}[\mathbf{x},\mathbf{y}]$. If

$$f(\mathbf{x},\mathbf{y})=\sum_{\alpha,\beta}c_{\alpha,\beta}\mathbf{x}^{\alpha}\mathbf{y}^{\beta},$$

where the sum is finite, then we define

$$f(A;\mathbf{X},\mathbf{Y})=\sum_{\alpha,\beta}c_{\alpha,\beta}(\mathbf{X}^{\alpha})^*A\mathbf{Y}^{\beta}. \tag{2.1}$$

It is clear that the map $f\mapsto f(A;\mathbf{X},\mathbf{Y})$ is linear from $\mathbb{C}[\mathbf{x},\mathbf{y}]$ into $(\mathcal{B}(\mathcal{H}))^d$. If $g\in\mathbb{C}[\mathbf{x},\mathbf{y}]$ depending only on $\mathbf{x}$, then $g(A;\mathbf{X},\mathbf{Y})=g(\mathbf{X}^*)A$. On the other hand, if $h\in\mathbb{C}[\mathbf{x},\mathbf{y}]$ depending only on $\mathbf{y}$, then $h(A;\mathbf{X},\mathbf{Y})=A\,h(\mathbf{Y})$. Furthermore, if $F=g\,f\,h$, then

$$F(A;\mathbf{X},\mathbf{Y})=g(X^*)f(A;\mathbf{X},\mathbf{Y})h(\mathbf{Y}). \tag{2.2}$$

If $\mathbf{X}=\mathbf{Y}$, we shall write $f(A;\mathbf{X})$ instead of $f(A;\mathbf{X},\mathbf{X})$. In the case $A=I$, the identity operator, we shall use $f(\mathbf{X},\mathbf{Y})$ to denote $f(I;\mathbf{X},\mathbf{Y})$. Therefore, $f(\mathbf{X})$ denotes $f(I;\mathbf{X},\mathbf{X})$. We say that $\mathbf{X}$ is a *hereditary root* of f if $f(\mathbf{X})=0$.

Example 2.1. Define $p_m(\mathbf{x},\mathbf{y})=\big(\sum_{j=1}^d x_jy_j-1)^m\in\mathbb{C}[\mathbf{x},\mathbf{y}]$. It is then clear that $\mathbf{T}$ is m-isometric if and only if $\mathbf{T}$ is a hereditary root of p_m, that is, $p_m(\mathbf{T})=0$. Similarly, $\mathbf{T}$ is (A,m)-isometric if and only if $p_m(A;\mathbf{T})=0$.

Even though the map $f \mapsto f(A;\mathbf{X},\mathbf{Y})$ is not multiplicative in general, it turns out that its kernel is an ideal of $\mathbb{C}[\mathbf{x},\mathbf{y}]$. This observation will play an important role in our approach.

Proposition 2.2. *Let A be a bounded linear operator and let $\mathbf{X}$ and $\mathbf{Y}$ be two d-tuples of commuting operators. Define*

$$\mathcal{J}(A;\mathbf{X},\mathbf{Y}) = \big\{ f \in \mathbb{C}[\mathbf{x},\mathbf{y}] : f(A;\mathbf{X},\mathbf{Y}) = 0 \big\}.$$

Then $\mathcal{J}(A;\mathbf{X},\mathbf{Y})$ is an ideal of $\mathbb{C}[\mathbf{x},\mathbf{y}]$.

PROOF. For simplicity of the notation, throughout the proof, let us write $\mathcal{J}$ for $\mathcal{J}(A;\mathbf{X},\mathbf{Y})$. It is clear that $\mathcal{J}$ is a vector subspace of $\mathbb{C}[\mathbf{x},\mathbf{y}]$. Now let f be in $\mathcal{J}$ and g be in $\mathbb{C}[\mathbf{x},\mathbf{y}]$. We need to show that gf belongs to $\mathcal{J}$. By linearity, it suffices to consider the case g is a monomial $g(\mathbf{x},\mathbf{y}) = \mathbf{x}^{\alpha}\mathbf{y}^{\beta}$ for some multi-indices α and β. By 2.2,

$$(fg)(A;\mathbf{X},\mathbf{Y}) = (\mathbf{X}^{\alpha})^{*} \cdot f(A;\mathbf{X},\mathbf{Y}) \cdot \mathbf{Y}^{\beta} = 0,$$

since $f(A;\mathbf{X},\mathbf{Y}) = 0$. This shows that fg belongs to $\mathcal{J}$ as desired.

If f is a polynomial of $\mathbf{y}$ in the form $f(\mathbf{y}) = \sum_{\alpha} c_{\alpha}\mathbf{y}^{\alpha}$, we define $\bar{f}(\mathbf{x})$ as

$$\bar{f}(\mathbf{x}) = \sum_{\alpha} \bar{c}_{\alpha}\mathbf{x}^{\alpha}.$$

In the case A is positive and $\mathbf{X} = \mathbf{Y}$, we obtain an additional property of the ideal $\mathcal{J}(A;\mathbf{Y},\mathbf{Y})$ as follows.

Proposition 2.3. *Let A be a positive operator and $\mathbf{Y}$ be a d-tuple of commuting operators. Suppose $f_1,\ldots,f_m$ are polynomials of $\mathbf{y}$ such that the sum $\bar{f}_1(\mathbf{x})f_1(\mathbf{y}) + \cdots + \bar{f}_m(\mathbf{x})f_m(\mathbf{y})$ belongs to $\mathcal{J}(A;\mathbf{Y},\mathbf{Y})$. Then $f_1(\mathbf{y}),\ldots,f_m(\mathbf{y})$ also belong to $\mathcal{J}(A;\mathbf{Y},\mathbf{Y})$.*

PROOF. Note that $\bar{f}_j(\mathbf{Y}^{*}) = (f_j(\mathbf{Y}))^{*}$ for all j. By the hypotheses, we have

$$(f_1(\mathbf{Y}))^{*}Af_1(\mathbf{Y}) + \cdots + (f_m(\mathbf{Y}))^{*}Af_m(\mathbf{Y}) = 0,$$

which implies

$$[(A^{1/2}f_1(\mathbf{Y})]^{*}[A^{1/2}f_1(\mathbf{Y})] + \cdots + [A^{1/2}f_m(\mathbf{Y})]^{*}[A^{1/2}f_m(\mathbf{Y})] = 0.$$

It follows that for all j, we have $A^{1/2}f_j(\mathbf{Y}) = 0$, which implies $Af_j(\mathbf{Y}) = 0$. Therefore, $f_j(\mathbf{y}) \in \mathcal{J}(A;\mathbf{Y},\mathbf{Y})$ for all j.

Recall that the radical ideal of an ideal $\mathcal{J} \subset \mathbb{C}[\mathbf{x}, \mathbf{y}]$, denoted by $\mathrm{Rad}(\mathcal{J})$, is the set of all polynomials $p \in \mathbb{C}[\mathbf{x}, \mathbf{y}]$ such that $p^N \in \mathcal{J}$ for some positive integer N. In the following proposition, we provide an interesting relation between generalized eigenvectors and eigenvalues of $\mathbf{X}$ and $\mathbf{Y}$ whenever we have $f(A; \mathbf{X}, \mathbf{Y}) = 0$.

Proposition 2.4. *Let $\mathbf{X}$ and $\mathbf{Y}$ be two d-tuples of commuting operators. Suppose k is a positive integer, $\lambda = (\lambda_1, \dots, \lambda_d), \omega = (\omega_1, \dots, \omega_d) \in \mathbb{C}^d$ and $u, v \in \mathcal{H}$ such that*

$$(X_j - \lambda_j)^k u = (Y_j - \omega_j)^k v = 0$$

for all $1 \le j \le d$. Then for any polynomial $f \in \mathrm{Rad}(\mathcal{J}(A; \mathbf{X}, \mathbf{Y}))$, we have

$$f(\bar{\lambda}, \omega)\langle Av, u\rangle = 0. \tag{2.3}$$

Proof. We first assume that $f \in \mathcal{J}(A; \mathbf{X}, \mathbf{Y})$. Using Taylor's expansion, we find polynomials $g_1, \dots, g_d$ and $h_1, \dots, h_d$ such that

$$f(\bar{\lambda}, \omega) - f(\mathbf{x}, \mathbf{y}) = \sum_{j=1}^{d} (x_j - \bar{\lambda}_j) g_j(\mathbf{x}, \mathbf{y}) + \sum_{j=1}^{d} h_j(\mathbf{x}, \mathbf{y})(y_j - \omega_j).$$

Take any integer $M \ge 1 + 2d(k-1)$. By the multinomial expansion, there exist polynomials $G_1, \dots, G_d$ and $H_1, \dots, H_d$ such that

$$\Big(f(\bar{\lambda}, \omega) - f(\mathbf{x}, \mathbf{y})\Big)^M = \sum_{j=1}^{d} (x_j - \bar{\lambda}_j)^k G_j(\mathbf{x}, \mathbf{y}) + \sum_{j=1}^{d} H_j(\mathbf{x}, \mathbf{y})(y_j - \omega_j)^k.$$

The left-hand side, by the binomial expansion, can be written as

$$(f(\bar{\lambda}, \omega))^M + f(\mathbf{x}, \mathbf{y}) H(\mathbf{x}, \mathbf{y})$$

for some polynomial H. Since $f(A; \mathbf{X}, \mathbf{Y}) = 0$, using Equation (2.2) and Proposition 2.2, we conclude that

$$(f(\bar{\lambda}, \omega))^M \cdot A = \sum_{j=1}^{d} (X_j^* - \bar{\lambda}_j)^k G_j(A; \mathbf{X}, \mathbf{Y}) + \sum_{j=1}^{d} H_j(A; \mathbf{X}, \mathbf{Y})(Y_j - \omega_j)^k.$$

Consequently,

$$\begin{aligned}
&(f(\bar{\lambda},\omega))^M\langle Av,u\rangle\\
&=\sum_{j=1}^{d}\Big\langle G_j(A;\mathbf{X},\mathbf{Y})v,(X_j-\lambda_j)^k u\Big\rangle+\sum_{j=1}^{d}\Big\langle H_j(A;\mathbf{X},\mathbf{Y})(Y_j-\omega_j)^k v,u\Big\rangle\\
&=0,
\end{aligned}$$

which implies (2.3).

In the general case, there exists an integer $N\geq 1$ such that f^N belongs to $\mathcal{J}(A;\mathbf{X},\mathbf{Y})$. By the case we have just proved, $(f(\bar{\lambda},\omega))^N\langle Av,u\rangle=0$, which again implies (2.3). This completes the proof of the proposition.

Remark 2.5. In the case of a single operator, Proposition 2.4 provides a generalization of [4, Lemmas 18 and 19]. Our proof here is even simpler and more transparent.

Question 1 and Question 2 in the introduction concern operators acting on a finite dimensional Hilbert space. It turns out that this condition can be replaced by a weaker one. Recall that a linear operator T is called *algebraic* if there exist complex constants $c_0,c_1,\ldots,c_\ell$ such that

$$c_0I+c_1T+\cdots+c_\ell T^\ell=0.$$

Algebraic operator roots of polynomials were investigated in [4].

We first discuss some preparatory results on algebraic operators acting on a general complex vector space $\mathcal{V}$. It is well known that if T is an algebraic linear operator on $\mathcal{V}$, then the spectrum $\sigma(T)$ is finite and there exists a direct sum decomposition $\mathcal{V}=\oplus_{a\in\sigma(T)}\mathcal{V}_a$, where each $\mathcal{V}_a$ is an invariant subspace for T (the subspace $\mathcal{V}_a$ is a closed subspace if $\mathcal{V}$ is a normed space and T is bounded) and $T-aI$ is nilpotent on $\mathcal{V}_a$. Indeed, if the minimal polynomial of T is factored in the form

$$p(z)=(z-a_1)^{m_1}\cdots(z-a_\ell)^{m_\ell},$$

where $a_1,\ldots,a_\ell$ are pairwise distinct and $m_1,\ldots,m_\ell\geq 1$, then $\sigma(T)=\{a_1,\ldots,a_\ell\}$ and $\mathcal{V}_{a_j}=\ker(T-a_j)^{m_j}$ for $1\leq j\leq\ell$. See, for example, [32, Section 6.3], which discusses operators acting on finite dimensional vector spaces. However, the arguments apply to algebraic operators on infinite dimensional vector spaces as well.

Suppose now $\mathbf{T} = [T_1, \dots, T_d]$ is a tuple of commuting algebraic operators on $\mathcal{V}$. We first decompose $\mathcal{V}$ as above with respect to the spectrum $\sigma(T_1)$. Since each subspace in the decomposition is invariant for all T_j, we again decompose such subspace with respect to the spectrum $\sigma(T_2)$. Continuing this process, we obtain a finite set $\Lambda \subset \mathbb{C}^d$ and a direct sum decomposition $\mathcal{V} = \oplus_{\lambda\in\Lambda}\mathcal{V}_\lambda$ such that for each $\lambda = (\lambda_1, \dots, \lambda_d) \in \Lambda$ and $1 \le j \le d$, the subspace $\mathcal{V}_\lambda$ is invariant for $\mathbf{T}$ and $T_j - \lambda_j I$ is nilpotent on $\mathcal{V}_\lambda$. Let E_λ denote the canonical projection (possibly non-orthogonal) from $\mathcal{V}$ onto $\mathcal{V}_\lambda$. Then we have $\sum_{\lambda\in\Lambda} E_\lambda = I$, $E_\lambda^2 = E_\lambda$, and $E_\lambda E_\gamma = 0$ if $\lambda \neq \gamma$. Define

$$\mathbf{S} = \sum_{\lambda\in\Lambda} \lambda \cdot E_\lambda = \Big[\sum_{\lambda\in\Lambda} \lambda_1 E_\lambda, \dots, \sum_{\lambda\in\Lambda} \lambda_d E_\lambda\Big] \tag{2.4}$$

Then $\mathbf{S}$ is a tuple of commuting operators which commutes with $\mathbf{T}$, and $\mathbf{T}-\mathbf{S}$ is nilpotent. For any multiindex α, we have

$$\mathbf{S}^\alpha = S_1^{\alpha_1} \cdots S_d^{\alpha_d} = \sum_{\lambda\in\Lambda} \lambda^\alpha E_\lambda.$$

In the case $\mathcal{V}$ is a normed space and $\mathbf{T}$ is bounded, each operator in the tuple $\mathbf{S}$ is bounded as well.

We now prove a very general result, which will provide affirmative answers to Questions 1 and 2 in the introduction.

Theorem 2.6. *Let $\mathbf{X}$ and $\mathbf{Y}$ be two d-tuples of commuting algebraic operators on a Hilbert space $\mathcal{H}$. Let $\mathbf{U}$ (respectively, $\mathbf{V}$) be the commuting tuple associated with $\mathbf{X}$ (respectively, $\mathbf{Y}$) as in* (2.4). *Then*

$$\mathrm{Rad}(\mathcal{J}(A; \mathbf{X}, \mathbf{Y})) \subseteq \mathcal{J}(A; \mathbf{U}, \mathbf{V}). \tag{2.5}$$

Proof. Write $\mathbf{X} = [X_1, \dots, X_d]$ and decompose $\mathcal{H} = \oplus_{\lambda\in\Lambda}\mathcal{H}_\lambda$ such that for each $\lambda = (\lambda_1, \dots, \lambda_d) \in \Lambda$, the subspace $\mathcal{H}_\lambda$ is invariant for $\mathbf{X}$ and $X_j - \lambda_j I$ is nilpotent on $\mathcal{H}_\lambda$. Let U_λ denote the canonical projection from $\mathcal{H}$ onto $\mathcal{H}_\lambda$. Then $\mathbf{U} = \sum_{\lambda\in\Lambda} \lambda \cdot U_\lambda$ and for any multiindex α, we have

$$\mathbf{U}^\alpha = \sum_{\lambda\in\Lambda} \lambda^\alpha \cdot U_\lambda.$$

Similarly, write $\mathbf{Y} = [Y_1, \dots, Y_d]$ and decompose $\mathcal{H} = \oplus_{\omega\in\Omega}\mathcal{K}_\omega$. Let V_ω be the canonical projection from $\mathcal{H}$ onto $\mathcal{K}_\omega$. Then $\mathbf{V} = \sum_{\omega\in\Omega} \omega \cdot V_\omega$ and for any multiindex β,

$$\mathbf{V}^\beta = \sum_{\omega\in\Omega} \omega^\beta \cdot V_\omega.$$

Take any polynomial $p \in \mathrm{Rad}(\mathcal{J}(A; \mathbf{X}, \mathbf{Y}))$. For $\lambda \in \Lambda$, $\omega \in \Omega$ and vectors $u \in \mathcal{H}_\lambda$ and $v \in \mathcal{K}_\omega$, there exists an integer $k \geq 1$ sufficiently large such that

$$(X_j - \lambda_j I)^k u = (Y_j - \omega_j I)^k v = 0$$

for all $1 \leq j \leq d$. Proposition 2.4 shows that $p(\bar{\lambda}, \omega)\langle Av, u\rangle = 0$, which implies

$$p(\bar{\lambda}, \omega) U_\lambda^* A V_\omega = 0.$$

Write $p(\mathbf{x}, \mathbf{y}) = \sum_{\alpha,\beta} c_{\alpha,\beta} \mathbf{x}^\alpha \mathbf{y}^\beta$. We compute

$$\begin{aligned} p(A; \mathbf{U}, \mathbf{V}) &= \sum_{\alpha,\beta} c_{\alpha,\beta} \mathbf{U}^{*\alpha} A \mathbf{V}^\beta \\ &= \sum_{\alpha,\beta} c_{\alpha,\beta} \Big(\sum_{\lambda \in \Lambda} \bar{\lambda}^\alpha U_\lambda^* \Big) A \Big(\sum_{\omega \in \Omega} \omega^\beta \cdot V_\omega \Big) \\ &= \sum_{\lambda \in \Lambda, \omega \in \Omega} \Big(\sum_{\alpha,\beta} c_{\alpha,\beta} \bar{\lambda}^\alpha \omega^\beta \Big) U_\lambda^* A V_\omega \\ &= \sum_{\lambda \in \Lambda, \omega \in \Omega} p(\bar{\lambda}, \omega) U_\lambda^* A V_\omega = 0. \end{aligned}$$

We conclude that $p \in \mathcal{J}(A; \mathbf{U}, \mathbf{V})$. Since $p \in \mathrm{Rad}(\mathcal{J}(A; \mathbf{X}, \mathbf{Y}))$ was arbitrary, the proof of the theorem is complete.

Theorem 2.6 enjoys numerous interesting applications that we now describe.

Proof of Theorem 1.1. We shall prove the theorem under a more general assumption that $\mathbf{T}$ is a tuple of commuting algebraic operators. Since $\mathbf{T}$ is (A, m)-isometric, the polynomial $(\sum_{j=1}^d x_j y_j - 1)^m$ belongs to the ideal $\mathcal{J}(A; \mathbf{T}, \mathbf{T})$. It follows that the polynomial $p(x, y) = \sum_{j=1}^d x_j y_j - 1$ belongs to the radical ideal of $\mathcal{J}(A; \mathbf{T}, \mathbf{T})$. By Theorem 2.6, we may decompose $\mathbf{T} = \mathbf{S} + \mathbf{N}$, where $\mathbf{N}$ is a nilpotent tuple commuting with $\mathbf{S}$ and $p(A; \mathbf{S}, \mathbf{S}) = 0$, which means that $\mathbf{S}$ is a spherical A-isometry.

Example 2.7. Recall that an operator T is called (m, n)-isosymmetric (see [33]) if T is a hereditary root of $f(x, y) = (xy - 1)^m (x - y)^n$. Theorem 2.6 shows that any such algebraic T can be decomposed as $T = S + N$, where N is nilpotent, S is isosymmetric (i.e. $(1, 1)$-isosymmetric) and $SN = NS$.

Example 2.8. Several researchers [27, 9] have investigated the so-called toral m-isometric tuples. It is straightforward to generalize this notion to toral (A, m)-isometric tuples, which are commuting d-tuples $\mathbf{T}$ that satisfy

$$\sum_{\substack{0\leq\alpha_1\leq m_1\\ \cdots\\ 0\leq\alpha_d\leq m_d}} (-1)^{|\alpha|}\binom{m_1,\ldots,m_d}{\alpha}(\mathbf{T}^\alpha)^*\,A\,\mathbf{T}^\alpha = 0.$$

for all $m_1+\cdots+m_d = m$. Equivalently, $\mathbf{T}$ is a common hereditary root of all polynomials of the form $(1-x_1y_1)^{m_1}\cdots(1-x_dy_d)^{m_d}$ for $m_1+\cdots+m_d = m$. This means that all these polynomials belong to the ideal $\mathfrak{J}(A;\mathbf{T},\mathbf{T})$. We see that toral $(A,1)$-isometries are just commuting tuples $\mathbf{T}$ such that each T_j is an A-isometry, that is, $T_j^*AT_j = A$. Note that for any toral (A,m)-isometry $\mathbf{T}$, the radical ideal $\mathrm{Rad}(\mathfrak{J}(A;\mathbf{T},\mathbf{T}))$ contains all the polynomials $\big\{1-x_jy_j : j = 1,2,\ldots,d\big\}$. Theorem 2.6 asserts that $\mathbf{T} = \mathbf{S}+\mathbf{N}$, where $\mathbf{S}$ is a toral $(A,1)$-isometry and $\mathbf{N}$ is a nilpotent tuple commuting with $\mathbf{S}$.

3. On 2-isometric tuples

It is well known that any 2-isometry on a finite dimensional Hilbert space must actually be an isometry. On the other hand, there are many examples of finite dimensional 2-isometric tuples that are not spherical isometries. The following class of examples is given in Richter's talk [31].

Example 3.1. If $\alpha = (\alpha_1,\ldots,\alpha_d) \in \partial\mathbb{B}_d$ and $V_j : \mathbb{C}^m \to \mathbb{C}^n$ such that $\sum_{j=1}^d \overline{\alpha}_jV_j = 0$, then $\mathbf{W} = (W_1,\ldots,W_d)$ with

$$W_j = \begin{pmatrix}\alpha_jI_n & V_j\\ 0 & \alpha_jI_m\end{pmatrix}$$

defines a 2-isometric d-tuple.

The following result was stated in [31] without a proof and as far as the author is aware of, it has not appeared in a published paper.

Theorem 3.2 (Richter-Sundberg). *If $\mathbf{T}$ is a 2-isometric tuple on a finite dimensional space, then*

$$\mathbf{T} = \mathbf{U}\oplus\mathbf{W},$$

where $\mathbf{U}$ is a spherical unitary and $\mathbf{W}$ is a direct sum of operator tuples unitarily equivalent to those in Example 3.1.

In this section, we shall assume that A is self-adjoint and investigate $(A,2)$-isometric d-tuples. We obtain a characterization for such tuples that generalizes the above theorem. We first provide a generalization of Example 3.1. We call $\mathbf{N} = (N_1, \ldots, N_d)$ an (A,n)-nilpotent tuple if $A\mathbf{N}^\alpha = 0$ for any indices α with $|\alpha| = n$.

Proposition 3.3. *Assume that A is a self-adjoint operator. Let $\mathbf{S}$ be an $(A,1)$-isometry and $\mathbf{N}$ an $(A,2)$-nilpotent tuple such that $\mathbf{S}$ commutes with $\mathbf{N}$. Suppose $S_1^*AN_1 + \cdots + S_d^*AN_d = 0$, then $\mathbf{S}+\mathbf{N}$ is an $(A,2)$-isometry.*

Proof. By the assumption, we have $AN_jN_k = N_j^*N_k^*A = 0$ for $1 \le j,k \le d$, $\sum_{j=1}^d S_j^*AS_j = A$, and $\sum_{j=1}^d S_j^*AN_j = \sum_{j=1}^d N_j^*AS_j = 0$. It follows that

$$\sum_{j=1}^d (S_j+N_j)^*A\,(S_j+N_j) = A + \sum_{j=1}^d N_j^*AN_j.$$

We then compute

$$\begin{aligned}
&\sum_{1\le k,j\le d} (S_k+N_k)^*(S_j+N_j)^*A\,(S_j+N_j)(S_k+N_k)\\
&\qquad = \sum_{k=1}^d (S_k^*+N_k^*)(A+\sum_{j=1}^d N_j^*AN_j)(S_k+N_k)\\
&\qquad = \sum_{k=1}^d (S_k^*+N_k^*)A\,(S_k+N_k) + \sum_{1\le k,j\le d}(S_k^*+N_k^*)N_j^*AN_j(S_k+N_k)\\
&\qquad = A + \sum_{k=1}^d N_k^*AN_k + \sum_{1\le k,j\le d} S_k^*N_j^*AN_jS_k\\
&\qquad = A + \sum_{k=1}^d N_k^*AN_k + \sum_{j=1}^d N_j^*\Big(\sum_{k=1}^d S_k^*AS_k\Big)N_j\\
&\qquad = A + \sum_{k=1}^d N_k^*AN_k + \sum_{j=1}^d N_j^*AN_j\\
&\qquad = A + 2\sum_{j=1}^d N_j^*AN_j
\end{aligned}$$

$$= 2\sum_{j=1}^{d}(S_j + N_j)^* A\,(S_j + N_j) - A.$$

Consequently, the sum $\mathbf{S} + \mathbf{N}$ is an $(A, 2)$-isometric tuple.

Remark 3.4. We have provided a direct proof of Proposition 2.3. Using the hereditary functional calculus and the approach in [26], one may generalize the result to the case $\mathbf{S}$ being an (A, m)-isometry and $\mathbf{N}$ an (A, n)-nilpotent commuting with $\mathbf{S}$. Under such an assumption, if $S_1^* A N_1 + \cdots + S_d^* A N_d = 0$, then $\mathbf{S} + \mathbf{N}$ is an $(A, m + 2n - 3)$-isometry. We leave the details for the interested reader.

We now show that any algebraic $(A, 2)$-isometric tuple has the form given in Proposition 3.3 and as a result, provide a proof of Richter-Sundberg's theorem.

Theorem 3.5. *Assume that A is a positive operator. Let $\mathbf{T}$ be an algebraic $(A, 2)$-isometric tuple on $\mathcal{H}$. Then there exists an $(A, 1)$-isometric tuple $\mathbf{S}$ and a tuple $\mathbf{N}$ commuting with $\mathbf{S}$ such that $\mathbf{T} = \mathbf{S}+\mathbf{N}$, $\sum_{\ell=1}^{d} S_\ell^* A N_\ell = 0$, and $A N_j N_\ell = 0$ for all $1 \leq j, \ell \leq d$ (we call such N an $(A, 2)$-nilpotent tuple).*

In the case $\mathcal{H}$ is finite dimensional and $A = I$, the identity operator, we recover Theorem 3.2.

Proof. Recall that there exists a finite set $\Lambda \subset \mathbb{C}^d$ and a direct sum decomposition $\mathcal{H} = \oplus_{\lambda\in\Lambda}\mathcal{H}_\lambda$ such that for each $\lambda \in \Lambda$, the subspace $\mathcal{H}_\lambda$ is invariant for $\mathbf{T}$ and $T_j - \lambda_j I$ is nilpotent on $\mathcal{H}_\lambda$. Let $\mathbf{S}$ be defined as in (2.4) and put $\mathbf{N} = \mathbf{T} - \mathbf{S}$. From the construction, $\mathbf{N}$ is nilpotent and Theorem 2.6 shows that $\mathbf{S}$ is $(A, 1)$-isometric. We shall show that $\mathbf{N}$ satisfies the required properties.

Restricting on each invariant subspace $\mathcal{H}_\lambda$, we only need to consider the case $\mathcal{H} = \mathcal{H}_\lambda$ and so $\mathbf{S} = \lambda\mathbf{I}$. Proposition 2.4 asserts that $(|\lambda|^2-1)\langle Av, u\rangle = 0$ for all $v, u \in \mathcal{H}$. If $|\lambda| \neq 1$, then $A = 0$ and the conclusion follows. Now we assume that $|\lambda| = 1$. Since $\mathbf{N}$ is nilpotent, there exists a positive integer r such that $A\,\mathbf{N}^\alpha = 0$ whenever $|\alpha| = r$. We claim that r may be taken to be 2. To prove the claim, we assume $r \geq 3$ and show that $A\,\mathbf{N}^\alpha = 0$ for all $|\alpha| = r - 1$.

Since $\mathbf{T} = \lambda\mathbf{I} + \mathbf{N}$ is $(A, 2)$-isometric, the tuple $\mathbf{N}$ is an A-root of the polynomial

$$p(\mathbf{x}, \mathbf{y}) = \Big(\sum_{j=1}^{d}(x_j + \bar{\lambda}_j)(y_j + \lambda_j) - 1\Big)^2 = \Big(\sum_{j=1}^{d} x_j y_j + \lambda_j x_j + \bar{\lambda}_j y_j\Big)^2.$$

On the other hand, $\mathbf{N}$ is an A-root of $\mathbf{x}^\alpha$ and $\mathbf{y}^\alpha$ for all $|\alpha| = r$. This shows that $p(\mathbf{x}, \mathbf{y})$, $\mathbf{x}^\alpha$ and $\mathbf{y}^\alpha$ belong to $\mathcal{J}(A; \mathbf{N}, \mathbf{N})$ for all $|\alpha| = r$. To simplify the notation, we shall denote $\mathcal{J}(A; \mathbf{N}, \mathbf{N})$ by $\mathcal{J}$ in the rest of the proof. Take any multiindex β with $|\beta| = r - 2$. We write

$$\mathbf{x}^\beta p(\mathbf{x}, \mathbf{y})\mathbf{y}^\beta = \mathbf{x}^\beta\Big(\sum_{j=1}^{d}\lambda_j x_j\Big)\Big(\sum_{\ell=1}^{d}\bar{\lambda}_\ell y_\ell\Big)\mathbf{y}^\beta + \sum_{|\gamma|\geq r} x^\gamma\, H_\gamma(\mathbf{x}, \mathbf{y}) + G_\gamma(\mathbf{x}, \mathbf{y})\,\mathbf{y}^\gamma$$

for some polynomials H_γ and G_γ. Since the left-hand side and the second term on the right-hand side belong to $\mathcal{J}$, which is an ideal, we conclude that

$$\mathbf{x}^\beta\Big(\sum_{j=1}^{d}\lambda_j x_j\Big)\Big(\sum_{\ell=1}^{d}\bar{\lambda}_\ell y_\ell\Big)\mathbf{y}^\beta \in \mathcal{J}.$$

Proposition 2.3 shows that both $\Big(\sum_{\ell=1}^{d}\bar{\lambda}_\ell y_\ell\Big)\mathbf{y}^\beta$ and $\mathbf{x}^\beta\Big(\sum_{j=1}^{d}\lambda_j x_j\Big)$ are in $\mathcal{J}$. Now for any multiindex γ with $|\gamma| = r - 3$, we compute

$$\begin{aligned}\mathbf{x}^\gamma p(\mathbf{x}, \mathbf{y})\mathbf{y}^\gamma = \mathbf{x}^\gamma\big(\sum_{j=1}^{d} x_j y_j\big)^2\mathbf{y}^\gamma &+ \sum_{|\beta|=r-2}\mathbf{x}^\beta\Big(\sum_{j=1}^{d}\lambda_j x_j\Big)P_\beta(\mathbf{x}, \mathbf{y})\\ &+ \sum_{|\beta|=r-2}\Big(\sum_{j=1}^{d}\lambda_j x_j\Big)\mathbf{y}^\beta Q_\beta(\mathbf{x}, \mathbf{y}).\end{aligned}$$

Since the left-hand side and the last two sums on the right-hand side belong to $\mathcal{J}$, it follows that $\mathbf{x}^\gamma(\sum_{j=1}^{d} x_j y_j)^2\mathbf{y}^\gamma$ belongs to $\mathcal{J}$. Another application of Proposition 2.3 then shows that $y_j y_\ell \mathbf{y}^\gamma$ belongs to $\mathcal{J}$ for all $1 \leq j, \ell \leq d$. That is, $\mathbf{y}^\alpha$ belongs to $\mathcal{J}$ whenever $|\alpha| = r - 1$ (as long as $r \geq 3$). As a consequence, we see that $\mathbf{y}^\alpha$, and hence $\mathbf{x}^\alpha$, belong to $\mathcal{J}$ for all $|\alpha| = 2$. This together with the fact that $p(\mathbf{x}, \mathbf{y}) \in \mathcal{J}$ forces $(\sum_{j=1}^{d}\lambda_j x_j)(\sum_{\ell=1}^{d}\bar{\lambda}_\ell y_\ell)$ to belong to $\mathcal{J}$, which implies that $\sum_{\ell=1}^{d}\bar{\lambda}_\ell y_\ell$ is in $\mathcal{J}$. We have then shown $AN_j N_\ell = 0$ for all $1 \leq j, \ell \leq d$ and $\sum_{\ell=1}^{d} S_\ell^* A N_\ell = \sum_{\ell=1}^{d}\bar{\lambda}_\ell A N_\ell = 0$, as desired.

Now let us consider $\mathbf{T}$ a 2-isometric tuple on a finite dimensional space $\mathcal{H}$. Recall that we have the decomposition $\mathcal{H} = \oplus_{\lambda\in\Lambda}\mathcal{H}_\lambda$ such that for each $\lambda \in \Lambda$, the subspace $\mathcal{H}_\lambda$ is invariant for $\mathbf{T}$ and $T_j - \lambda_j I$ is nilpotent on $\mathcal{H}_\lambda$. By Proposition 2.4, we have $(\langle\omega,\lambda\rangle - 1)^2\langle v,u\rangle = 0$ for all $v \in \mathcal{H}_\omega$ and $u \in \mathcal{H}_\lambda$. It follows that $|\lambda| = 1$ for all $\lambda \in \Lambda$ and $\mathcal{H}_\lambda \perp \mathcal{H}_\omega$ whenever $\lambda \neq \omega$. As a result, each subspace $\mathcal{H}_\lambda$ is reducing for $\mathbf{T}$. To complete the proof, it suffices to consider $\mathcal{H} = \mathcal{H}_\lambda$. We shall show that either $\mathbf{T}$ is a spherical unitary or it is unitarily equivalent to a tuple given in Example 3.1. Indeed, we have $\mathbf{T} = \lambda\mathbf{I} + \mathbf{N}$, where $\sum_{\ell=1}^d \bar{\lambda}_\ell N_\ell = 0$ and $N_j N_\ell = 0$ for all $1 \leq j,\ell \leq d$. If $\mathbf{N} = 0$, then $\mathbf{T}$ is a spherical unitary. Otherwise, let $\mathcal{M} = \ker(N_1) \cap \cdots \cap \ker(N_d)$. Then $N_\ell(\mathcal{H}) \subseteq \mathcal{M}$ for all $1 \leq \ell \leq d$. As a consequence, with respect to the orthogonal decomposition $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp$, each N_ℓ has the form

$$N_\ell = \begin{pmatrix} 0 & V_\ell \\ 0 & 0 \end{pmatrix}$$

for some $V_\ell : \mathcal{M}^\perp \to \mathcal{M}$. Since $\sum_{\ell=1}^d \bar{\lambda}_\ell N_\ell = 0$, we have $\sum_{\ell=1}^d \bar{\lambda}_\ell V_\ell = 0$. It follows that $\mathbf{T}$ is unitarily equivalent to an operator tuple in Example 3.1.

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