DIAGONAL TOEPLITZ OPERATORS ON WEIGHTED BERGMAN SPACES

TRIEU LE

Abstract. In this paper we discuss some algebraic properties of diagonal Toeplitz operators on weighted Bergman spaces of the unit ball in $\mathbb{C}^n$. We give the affirmative answer to the zero-product problem when all but possibly one of the involving Toeplitz operators are diagonal with respect to the standard orthonormal basis. We also study Toeplitz operators which commute with diagonal Toeplitz operators.

1. INTRODUCTION

For any integer $n \geq 1$, let $\mathbb{C}^n$ denote the Cartesian product of n copies of $\mathbb{C}$. For any $z = (z_1, \ldots, z_n)$ and $w = (w_1, \ldots, w_n)$ in $\mathbb{C}^n$, we write $\langle z, w\rangle = z_1\overline{w}_1 + \cdots + z_n\overline{w}_n$ and $|z| = \sqrt{|z_1|^2 + \cdots + |z_n|^2}$ for the inner product and the associated Euclidean norm. Let $\mathbb{B}_n$ denote the open unit ball consisting of all $z \in \mathbb{C}^n$ with $|z| < 1$. Let $\mathbb{S}_n$ denote the unit sphere consisting of all $z \in \mathbb{C}^n$ with $|z| = 1$.

Let ν denote the Lebesgue measure on $\mathbb{B}_n$ normalized so that $\nu(\mathbb{B}_n) = 1$. Fix a real number $\alpha > -1$. The weighted Lebesgue measure ν_α on $\mathbb{B}_n$ is defined by $\mathrm{d}\nu_\alpha(z) = c_\alpha(1-|z|^2)^\alpha \mathrm{d}\nu(z)$, where c_α is a normalizing constant so that $\nu_\alpha(\mathbb{B}_n) = 1$. A direct computation shows that $c_\alpha = \dfrac{\Gamma(n+\alpha+1)}{\Gamma(n+1)\Gamma(\alpha+1)}$. For $1 \leq p \leq \infty$, let L^p_α denote the space $L^p(\mathbb{B}_n, \mathrm{d}\nu_\alpha)$. Note that L^∞_α is the same as $L^\infty = L^\infty(\mathbb{B}_n, \mathrm{d}\nu)$.

The weighted Bergman space A^p_α consists of all holomorphic functions in $\mathbb{B}_n$ which are also in L^p_α. It is well-known that A^p_α is a closed subspace of L^p_α. In this paper we are only interested in the case $p = 2$. We denote the inner product in L^2_α by $\langle , \rangle_\alpha$.

For any multi-index $m = (m_1, \ldots, m_n)$ of non-negative integers, we write $|m| = m_1 + \cdots + m_n$ and $m! = m_1! \cdots m_n!$. For any $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$, we write $z^m = z_1^{m_1} \cdots z_n^{m_n}$. The standard orthonormal basis for A^2_α is $\{e_m : m \in \mathbb{N}^n\}$, where

$$e_m(z) = \Big[\frac{\Gamma(n+|m|+\alpha+1)}{m!\,\Gamma(n+\alpha+1)}\Big]^{1/2} z^m, \ m \in \mathbb{N}^n, z \in \mathbb{B}_n.$$

2000 *Mathematics Subject Classification.* Primary 47B35.

Key words and phrases. Toeplitz operators, Zero-product problem, Weighted Bergman spaces.

For a more detailed discussion of A^2_α, see Chapter 2 in [12].

Since A^2_α is a closed subspace of the Hilbert space L^2_α, there is an orthogonal projection P_α from L^2_α onto A^2_α. For any function $f \in L^2_\alpha$ the Toeplitz operator with symbol f is denoted by T_f, which is densely defined on A^2_α by $T_f\varphi = P_\alpha(f\varphi)$ for bounded holomorphic functions φ on $\mathbb{B}_n$. If f is a bounded function then T_f is a bounded operator on A^2_α with $\|T_f\| \leq \|f\|_\infty$ and $(T_f)^* = T_{\bar{f}}$. If f and g are bounded functions such that either f or $\bar{g}$ is holomorphic in the open unit ball then $T_gT_f = T_{gf}$. These properties can be verified directly from the definition of Toeplitz operators.

There is an extensive literature on Toeplitz operators on the Hardy space H^2 of the unit circle. We refer the reader to [9] for definitions of H^2 and their Toeplitz operators. In the context of Toeplitz operators on H^2, it was showed by A. Brown and P. Halmos [4] back in the sixties that if f and g are bounded functions on the unit circle then T_gT_f is another Toeplitz operator if and only if either f or $\bar{g}$ is holomorphic. From this it is readily deduced that if $f, g \in L^\infty$ such that $T_gT_f = 0$ then one of the symbols must be the zero function. A more general question concerning products of several Toeplitz operators is the so-called "zero-product problem".

Problem 1.1. *Suppose $f_1, \ldots, f_N$ are functions in L^∞ such that $T_{f_1} \cdots T_{f_N}$ is the zero operator. Does it follow that one of the functions f_j's must be the zero function?*

For Toeplitz operators on H^2, the affirmative answer was proved by K.Y. Guo [8] for $N = 5$ and by C. Gu [7] for $N = 6$. Problem 1.1 for general N remains open. For Toeplitz operators on the Bergman space of the unit disk, P. Ahern and Ž. Čučković [2] answered Problem 1.1 affirmatively for the case $N = 2$ with an additional assumption that both symbols are bounded harmonic functions. In fact they studied a type of Brown-Halmos Theorem for Toeplitz operators on the Bergman space. They showed that if f and g are bounded harmonic functions and h is a bounded C^2 function whose invariant Laplacian is also bounded (later Ahern [1] removed this condition on h) then the equality $T_gT_f = T_h$ holds only in the trivial case, that is, when f or $\bar{g}$ is holomorphic. This result was generalized to Toeplitz operators on the Bergman space of the unit ball in $\mathbb{C}^n$ by B. Choe and K. Koo in [5] with an assumption about the continuity of the symbols on an open subset of the boundary. They were also able to show that if $f_1, \ldots, f_{n+3}$ (here $N = n+3$) are bounded harmonic functions that have Lipschitz continuous extensions to the whole boundary of the unit ball then $T_{f_1} \cdots T_{f_{n+3}} = 0$ implies that one of the symbols must be zero. The answer in the general case remains unknown, even for two Toeplitz operators.

In this paper we provide the affirmative answer to Problem 1.1 when all but possibly one of the Toeplitz operators are diagonal with respect to the standard orthonormal basis. The only result in this direction which we are aware of is by Ahern and Čučković in their 2004 paper [3]. There, among other things, they showed that in the one dimensional case, if T_f or T_g is diagonal and $T_gT_f = 0$ then either f or g is the zero function. It is not clear how their method will work for products of more than two Toeplitz operators or in the setting of weighted Bergman spaces in higher dimensions. Our result is the following theorem.

Theorem 1.2. *Suppose $f_1, \ldots, f_N$ and $g_1, \ldots, g_M$ are bounded measurable functions on $\mathbb{B}_n$ so that none of them is the zero function and that the corresponding Toeplitz operators on A^2_α are diagonal with respect to the standard orthonormal basis. Suppose $f \in L^2_\alpha$ such that the operator $T_{f_1} \cdots T_{f_N} T_f T_{g_1} \cdots T_{g_M}$ (which is densely defined on A^2_α) is the zero operator. Then f must be the zero function.*

A function f on $\mathbb{B}_n$ is called a radial function if there is a function $\tilde{f}$ on $[0,1)$ such that $f(z) = \tilde{f}(|z|)$ for all $z \in \mathbb{B}_n$. It is well-known that if $f \in L^2_\alpha$ is a radial function then T_f is a diagonal operator (in the rest of the paper, a densely defined operator on A^2_α is called diagonal if it is diagonal with respect to the standard orthonormal basis). In [6, Theorem 6], Čučković and Rao showed that if f and g are $L^\infty(\mathbb{B}_1, \mathrm{d}\nu)$ functions and g is radial and non-constant then $T_fT_g = T_gT_f$ implies that f is a radial function. The situation in higher dimensions turns out to be more complicated. Since non-radial functions may give rise to diagonal Toeplitz operators (see Theorem 3.1), a version of Čučković and Rao's result that one may hope for is that if T_f and T_g commute and g is a non-constant radial function then T_f is also diagonal. However this is not true when $n \geq 2$. As we will see in the next theorem, there is a function $f \in L^\infty$ so that T_f is not diagonal and $T_fT_g = T_gT_f$ for all radial functions $g \in L^\infty$. We will also show that there is a function $g \in L^\infty$ such that T_g is diagonal and for any $f \in L^\infty$, $T_fT_g = T_gT_f$ implies T_f is diagonal. So the set

$$\begin{aligned} G = \{g \in L^\infty :\ & T_g \text{ is diagonal} \\ & \text{and for } f \in L^\infty,\ T_fT_g = T_gT_f \text{ implies } T_f \text{ is diagonal}\} \end{aligned}$$

is non-empty and does not contain any radial functions when $n \geq 2$ (in the one dimensional case, this set is exactly the set of all non-constant radial functions in L^∞). It would be interesting if one can give a complete description of G as in the one dimensional case. For now we are not aware of such a description. In Theorem 4.1 we will give a sufficient condition for a function g to belong to G.

Theorem 1.3. *Suppose $n \geq 2$. The there exist*

(1) *a function $f \in L^\infty$ such that $T_fT_h = T_hT_f$ for all radial functions h in L^∞, and*

(2) *a function $g \in L^\infty$ such that T_g is diagonal and for any $h \in L^\infty$, $T_gT_h = T_hT_g$ implies T_h is diagonal.*

2. SOME FUNCTION-THEORETIC RESULTS

In this section we will prove some function-theoretic results which are useful for our analysis of Toeplitz operators in the next two sections. For any $1 \leq j \leq n$, let $\sigma_j : \mathbb{N} \times \mathbb{N}^{n-1} \to \mathbb{N}^n$ be the map defined by the formula $\sigma_j(s, (m_1, \ldots, m_{n-1})) = (m_1, \ldots, m_{j-1}, s, m_j, \ldots, m_{n-1})$ for all $s \in \mathbb{N}$ and $(m_1, \ldots, m_{n-1}) \in \mathbb{N}^{n-1}$. If M is a subset of $\mathbb{N}^n$ and $1 \leq j \leq n$, we define

$$\widetilde{M}_j = \Big\{\tilde{m} = (m_1, \ldots, m_{n-1}) \in \mathbb{N}^{n-1} : \sum_{\substack{s \in \mathbb{N} \\ \sigma_j(s,\tilde{m}) \in M}} \frac{1}{s+1} = \infty\Big\}.$$

We say that M has property (P) if one of the following statements holds.

(1) $M = \emptyset$, or
(2) $M \neq \emptyset$, $n = 1$ and $\sum\limits_{s \in M} \dfrac{1}{s+1} < \infty$, or
(3) $M \neq \emptyset$, $n \geq 2$ and for any $1 \leq j \leq n$, the set $\widetilde{M}_j$ has property (P) as a subset of $\mathbb{N}^{n-1}$.

The following observations are then immediate.

(1) If $M \subset \mathbb{N}$ and M does not have property (P) then $\sum_{s \in M} \frac{1}{s+1} = \infty$. If $M \subset \mathbb{N}^n$ with $n \geq 2$ and M does not have property (P) then $\widetilde{M}_j$ does not have property (P) as a subset of $\mathbb{N}^{n-1}$ for some $1 \leq j \leq n$.
(2) If M_1 and M_2 are subsets of $\mathbb{N}^n$ that both have property (P) then $M_1 \cup M_2$ also has property (P).
(3) If $M \subset \mathbb{N}^n$ has property (P) and $l \in \mathbb{Z}^n$ then $(M + l) \cap \mathbb{N}^n$ also has property (P). Here $M + l = \{m + l : m \in M\}$.
(4) If $M \subset \mathbb{N}^n$ has property (P) then $\mathbb{N} \times M$ also has property (P) as a subset of $\mathbb{N}^{n+1}$.
(5) The set $\mathbb{N}^n$ does not have property (P) for all $n \geq 1$. This together with (2) shows that if $M \subset \mathbb{N}^n$ has property (P) then $\mathbb{N}^n \backslash M$ does not have property (P).

For any function $f \in L^1(\mathbb{B}_n, \mathrm{d}\nu)$, we define

$$S(f)(z_1, \ldots z_n) = \frac{1}{(2\pi)^n} \int_0^{2\pi} \!\!\cdots\! \int_0^{2\pi} f(e^{i\theta_1} z_1, \ldots, e^{i\theta_n} z_n) \mathrm{d}\theta_1 \cdots \mathrm{d}\theta_n,$$

for $z = (z_1, \ldots, z_n) \in \mathbb{B}_n$. Then $S(f) \in L^1(\mathbb{B}_n, \mathrm{d}\nu)$, $S(f)(z_1, \ldots, z_n) = S(f)(|z_1|, \ldots, |z_n|)$ for $z \in \mathbb{B}_n$ and $f = S(f)$ as $L^1(\mathbb{B}_n, \mathrm{d}\nu)$–functions if and only if $f(z_1, \ldots, z_n) = f(|z_1|, \ldots, |z_n|)$ for almost all $z \in \mathbb{B}_n$. Also, for any

$m \in \mathbb{N}^n$, we have

$$\begin{aligned}
&\int_{\mathbb{B}_n} S(f)(z) z^m \bar{z}^m \mathrm{d}\nu(z) \\
&= \int_{\mathbb{B}_n} \Big\{ \frac{1}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} f(e^{i\theta_1} z_1, \ldots, e^{i\theta_n} z_n) \mathrm{d}\theta_1 \cdots \mathrm{d}\theta_n \Big\} z^m \bar{z}^m \mathrm{d}\nu(z) \\
&= \frac{1}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} \Big\{ \int_{\mathbb{B}_n} f(e^{i\theta_1} z_1, \ldots, e^{i\theta_n} z_n) z^m \bar{z}^m \mathrm{d}\nu(z) \Big\} \mathrm{d}\theta_1 \cdots \mathrm{d}\theta_n \qquad (2.1) \\
&= \frac{1}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} \Big\{ \int_{\mathbb{B}_n} f(z) z^m \bar{z}^m \mathrm{d}\nu(z) \Big\} \mathrm{d}\theta_1 \cdots \mathrm{d}\theta_n \\
&= \int_{\mathbb{B}_n} f(z) z^m \bar{z}^m \mathrm{d}\nu(z).
\end{aligned}$$

The next proposition shows that if f is a function in $L^1(\mathbb{B}_n, \mathrm{d}\nu)$ then the set of all multi-indexes m such that $\int_{\mathbb{B}_n} f(z) z^m \bar{z}^m \mathrm{d}\nu(z) = 0$ either has property (P) or is all of $\mathbb{N}^n$.

Proposition 2.1. *Suppose $f \in L^1(\mathbb{B}_n, \mathrm{d}\nu)$ so that the set*

$$Z(f) = \big\{ m \in \mathbb{N}^n : \int_{\mathbb{B}_n} f(z) z^m \bar{z}^m \mathrm{d}\nu(z) = 0 \big\}$$

does not have property (P). Then $S(f)(z) = 0$ for almost all $z \in \mathbb{B}_n$ and as a consequence, $Z(f) = \mathbb{N}^n$.

Proof. Since $Z(f) = Z(S(f))$ (which follows from the computation preceding the proposition), we may assume without loss of generality, that $f = S(f)$. So $f(z_1, \ldots, z_n) = f(|z_1|, \ldots, |z_n|)$ for $z = (z_1, \ldots, z_n) \in \mathbb{B}_n$. We will show that $f(z) = 0$ for almost all $z \in \mathbb{B}_n$ by induction on the dimension n.

Consider first $n = 1$. For any $m \in Z(f)$, we have

$$0 = \int_{\mathbb{B}_1} f(z) z^m \bar{z}^m \mathrm{d}\nu(z) = \int_{\mathbb{B}_1} f(|z|) |z|^{2m} \mathrm{d}\nu(z) = \int_0^1 f(t^{1/2}) t^m \mathrm{d}t.$$

Since $Z(f)$ does not have property (P), we have $\sum_{m \in Z(f)} \frac{1}{m+1} = \infty$. It then follows from the Muntz-Szasz Theorem (see Theorem 15.26 in [11]) that $f(t^{1/2}) = 0$ for almost all $t \in [0, 1)$. Hence $f(z) = 0$ for almost all $z \in \mathbb{B}_1$.

Now suppose the conclusion of the proposition holds for all $n \le N$, where N is an integer greater than or equal to 1. Suppose that $f \in L^1(\mathbb{B}_{N+1}, \mathrm{d}\nu)$ satisfies the hypothesis of the proposition and that $f = S(f)$. Since $Z(f)$ does not have property (P), $\widetilde{Z(f)}_j$ does not have property (P) for some $1 \le j \le n$. Without loss of generality we may assume that $j = 1$. For any $z \in \mathbb{B}_{N+1}$ we write $z = (z_1, \sqrt{1-|z_1|^2}\ w)$ where $z_1 \in \mathbb{B}_1$ and $w \in \mathbb{B}_N$. We then have $\mathrm{d}\nu(z) = (N+1)(1-|z_1|^2)^N \mathrm{d}\nu(z_1)\mathrm{d}\nu(w)$ (the factor $N+1$ comes from the normalization of the Lebesgue measure so that unit balls have total mass 1). For each $\tilde{m} \in \widetilde{Z(f)}_1$ let $K_{\tilde{m}} = \big\{m_1 \in \mathbb{N} : (m_1, \tilde{m}) \in Z(f)\big\}$. Then $\sum\limits_{m_1 \in K_{\tilde{m}}} \frac{1}{m_1+1} = \infty$. For $\tilde{m} \in \widetilde{Z(f)}_1$ and $m_1 \in K_{\tilde{m}}$, we have $m = (m_1, \tilde{m}) \in Z(f)$. Hence

$$
\begin{aligned}
0 &= \int\limits_{\mathbb{B}_{N+1}} f(z) z^m \bar{z}^m \mathrm{d}\nu(z) \\
&= (N+1) \int\limits_{\mathbb{B}_1} \Big\{ \int\limits_{\mathbb{B}_N} f(z_1, \sqrt{1-|z_1|^2}\ w) w^{\tilde{m}} \bar{w}^{\tilde{m}} \mathrm{d}\nu(w) \Big\} \\
&\qquad\qquad \times |z_1|^{2m_1} (1-|z_1|^2)^{N+|\tilde{m}|} \mathrm{d}\nu(z_1).
\end{aligned}
$$

By the case $n = 1$ which was showed above, we conclude that for each $\tilde{m} \in \widetilde{Z(f)}_1$, for almost all $z_1 \in \mathbb{B}_1$,

$$
\int\limits_{\mathbb{B}_N} f(z_1, \sqrt{1-|z_1|^2}\ w) w^{\tilde{m}} \bar{w}^{\tilde{m}} \mathrm{d}\nu(w) = 0. \tag{2.2}
$$

So there is a null set $E \subset \mathbb{B}_1$ such that (2.2) holds for all $\tilde{m} \in \widetilde{Z(f)}_1$ and all $z_1 \in \mathbb{B}_1 \backslash E$. Now applying the induction hypothesis, we see that for each $z_1 \in \mathbb{B}_1 \backslash E$, $f(z_1, \sqrt{1-|z_1|^2}\ w) = 0$ for almost all $w \in \mathbb{B}_N$. This implies $f(z) = 0$ for almost all $z \in \mathbb{B}_{N+1}$. □

Proposition 2.2. *Let $M \subset \mathbb{N}^n$ be a subset that has property (P). Suppose $f \in L^1(\mathbb{B}_n, \mathrm{d}\nu)$ so that $\int\limits_{\mathbb{B}_n} f(z) z^m \bar{z}^k \mathrm{d}\nu(z) = 0$ whenever $m, k \in \mathbb{N}^n \backslash M$. Then $f(z) = 0$ for almost all $z \in \mathbb{B}_n$.*

Proof. Let $l \in \mathbb{Z}^n$ be an arbitrary $n-$tuple of integers. Then $H_l = M \cup ((M-l) \cap \mathbb{N}^n)$ has property (P). Thus $K_l = \mathbb{N}^n \backslash H_l$ does not have property (P). Because of the identities

$$
\begin{aligned}
K_l &= \{m \in \mathbb{N}^n : m \notin M \text{ and } m \notin M - l\} \\
&= \{m \in \mathbb{N}^n : m \notin M \text{ and } m + l \notin M\}
\end{aligned}
$$

and the assumption about f, we have $\int\limits_{\mathbb{B}^n} f(z) z^m \bar{z}^{m+l} \mathrm{d}\nu(z) = 0$ for all $m \in K_l$ with $m + l \succeq 0$ (here, for any $j = (j_1, \ldots, j_n) \in \mathbb{Z}^n$, by $j \succeq 0$ we mean

$j_1 \geq 0, \ldots, j_n \geq 0)$. Since K_l does not have property (P), Proposition 2.1 then implies that the above identity holds true for all multi-indexes $m \in \mathbb{N}^n$ with $m+l \succeq 0$. Since l was arbitrary, we conclude that $\int\limits_{\mathbb{B}^n} f(z) z^m \bar{z}^k \mathrm{d}\nu(z) = 0$ for all multi-indexes m and k in $\mathbb{N}^n$. Since the span of $\{z^m \bar{z}^k : m, k \in \mathbb{N}^n\}$ is dense in $C(\bar{\mathbb{B}}_n)$, it follows that $f(z) = 0$ for almost all $z \in \mathbb{B}_n$. $\square$

Corollary 2.3. *Let f be a function in $L^1(\mathbb{B}_n, \mathrm{d}\nu)$. Suppose there exists a subset $\widetilde{M}$ of $\mathbb{N}^{n-1}$ which has property (P) so that for any $\tilde{m}, \tilde{k} \in \mathbb{N}^n \backslash \widetilde{M}$ and any integer $l \neq 0$, there is a subset $N(\tilde{m}, \tilde{k}, l) \subset \mathbb{N}$ which does not have property (P) such that $\int\limits_{\mathbb{B}_n} f(z) z^{(s,\tilde{m})} \bar{z}^{(s+l,\tilde{k})} \mathrm{d}\nu(z) = 0$ for all $s \in N(\tilde{m}, \tilde{k}, l)$ with $s \geq -l$. Then $f(z_1, z_2, \ldots, z_n) = f(|z_1|, z_2, \ldots, z_n)$ for almost all $z \in \mathbb{B}_n$.*

Proof. For any $z = (z_1, \ldots, z_n) \in \mathbb{B}_n$, define

$$g(z_1, z_2, \ldots, z_n) = \frac{1}{2\pi} \int\limits_0^{2\pi} f(e^{i\theta} z_1, z_2, \ldots, z_n) \mathrm{d}\theta.$$

Then we have $g(z_1, z_2, \ldots, z_n) = g(|z_1|, z_2, \ldots, z_n)$ for all $z \in \mathbb{B}_n$. Furthermore, for $m = (m_1, \ldots, m_n), k = (k_1, \ldots, k_n) \in \mathbb{N}^n$ with $m_1 \neq k_1$,

$$\begin{aligned}
&\int\limits_{\mathbb{B}_n} g(z) z^m \bar{z}^k \mathrm{d}\nu(z) \\
&= \int\limits_{\mathbb{B}_n} g(|z_1|, \ldots, z_n) z^m \bar{z}^k \mathrm{d}\nu(z) \qquad (2.3)\\
&= \int\limits_{\mathbb{B}_n} \Big\{ \frac{1}{2\pi} \int\limits_0^{2\pi} (e^{i\theta} z_1)^{m_1} (e^{-i\theta} \bar{z}_1)^{k_1} \mathrm{d}\theta \Big\} g(|z_1|, \ldots, z_n) z_2^{m_2} \cdots z_n^{m_n} \bar{z}_2^{k_2} \cdots \bar{z}_n^{k_n} \mathrm{d}\nu(z) \\
&\quad \text{(by the invariance of } \nu \text{ under rotations, see also [10, Formula 1.4.2 (2)])} \\
&= 0.
\end{aligned}$$

For $m, k \in \mathbb{N}^n$ with $m_1 = k_1$, an argument similar to (2.1) shows that $\int\limits_{\mathbb{B}_n} g(z) z^m \bar{z}^k \mathrm{d}\nu(z) = \int\limits_{\mathbb{B}_n} f(z) z^m \bar{z}^k \mathrm{d}\nu(z)$.

By our assumption about f and the above identities, for any $\tilde{m}, \tilde{k}$ in $\mathbb{N}^{n-1} \backslash \widetilde{M}$ and any integer l, we have

$$\int\limits_{\mathbb{B}_n} (f(z) - g(z)) z^{(s,\tilde{m})} \bar{z}^{(s+l,\tilde{k})} \mathrm{d}\nu(z) = 0 \qquad (2.4)$$

for all $s \in N(\tilde{m}, \tilde{k}, l)$ with $s \geq -l$ (for $l = 0$ we put $N(\tilde{m}, \tilde{k}, 0) = \mathbb{N}$). Since the set $N(\tilde{m}, \tilde{k}, l)$ does not have property (P), Proposition 2.1 shows that (2.4) holds for all $s \in \mathbb{N}$ and $l \in \mathbb{Z}$ with $s + l \geq 0$. Hence we have

$$\int_{\mathbb{B}_n} (f(z) - g(z)) z^{(m_1, \tilde{m})} \bar{z}^{(k_1, \tilde{k})} \mathrm{d}\nu(z) = 0$$

for all $m_1, k_1 \in \mathbb{N}$ and all $\tilde{m}, \tilde{k} \in \mathbb{N}^{n-1} \backslash \widetilde{M}$. This means the identity $\int_{\mathbb{B}_n} (f(z) - g(z)) z^m \bar{z}^k \mathrm{d}\nu(z) = 0$ holds for all $m, k \in \mathbb{N}^n \backslash (\mathbb{N} \times \widetilde{M})$. Since $\mathbb{N} \times \widetilde{M}$ has property (P), Proposition 2.2 shows that $f(z) = g(z)$ for almost all $z \in \mathbb{B}_n$. Hence $f(z_1, z_2, \ldots, z_n) = f(|z_1|, z_2, \ldots, z_n)$ for almost all $z \in \mathbb{B}_n$.

The above proof also works for the case $n = 1$ in which there is no $\widetilde{M}$. $\square$

Corollary 2.4. *Let f be a function in $L^1(\mathbb{B}_n, \mathrm{d}\nu)$. Suppose for each $1 \leq j \leq n$ there exists a subset set $\widetilde{M_j}$ of $\mathbb{N}^{n-1}$ which has property (P) so that for any $\tilde{m}, \tilde{k}$ in $\mathbb{N}^{n-1} \backslash \widetilde{M_j}$ and any integer $l \neq 0$ there is a subset $N_j(\tilde{m}, \tilde{k}, l) \subset \mathbb{N}$ which does not have property (P) such that $\int_{\mathbb{B}_n} f(z) z^m \bar{z}^k \mathrm{d}\nu(z) = 0$ for any $m = \sigma_j(s, \tilde{m})$ and $k = \sigma_j(s + l, \tilde{k})$, where $s \in N_j(\tilde{m}, \tilde{k}, l)$ with $s \geq -l$. Then $f(z_1, \ldots, z_n) = f(|z_1|, \ldots, |z_n|)$ for almost all $z \in \mathbb{B}_n$.*

Proof. Apply Corollary 2.3 n times. $\square$

3. DIAGONAL TOEPLITZ OPERATORS

The following criterion for diagonal Toeplitz operators is probably well-known but since we are not aware of an appropriate reference, we give here a proof which is based on Corollary 2.4.

Theorem 3.1. *Suppose $f \in L^\infty$. Then the Toeplitz operator T_f is diagonal if and only if $f(z_1, \ldots, z_n) = f(|z_1|, \ldots, |z_n|)$ for almost all $z = (z_1, \ldots, z_n)$ in $\mathbb{B}_n$.*

Proof. If T_f is a diagonal operator then $\langle T_f e_m, e_k \rangle_\alpha = 0$ for all $m, k \in \mathbb{N}^n$ with $m \neq k$. Thus $\int_{\mathbb{B}_n} f(z) z^m \bar{z}^k (1 - |z|^2)^\alpha \mathrm{d}\nu(z) = 0$ for all $m, k \in \mathbb{N}^n$ with $m \neq k$. Corollary 2.4 (with $\widetilde{M_j} = \emptyset$ and $N_j(\tilde{m}, \tilde{k}, l) = \mathbb{N}$ for each $1 \leq j \leq n$, $l \in \mathbb{Z} \backslash \{0\}$, and $\tilde{m}, \tilde{k} \in \mathbb{N}^{n-1}$) implies that $f(z_1, \ldots, z_n) = f(|z_1|, \ldots, |z_n|)$ for almost all $z = (z_1, \ldots, z_n) \in \mathbb{B}_n$.

Now suppose $f(z_1, \ldots, z_n) = f(|z_1|, \ldots, |z_n|)$ for almost all $z \in \mathbb{B}_n$. Then as in (2.3) we see that $\langle T_f e_m, e_k \rangle_\alpha = 0$ for all $m, k \in \mathbb{N}^n$ with $m \neq k$. So T_f is diagonal with respect to the standard orthonormal basis $\{e_m : m \in \mathbb{N}^n\}$.

In fact, $T_f = \sum\limits_{m\in\mathbb{N}^n} \omega_\alpha(f,m) e_m \otimes e_m$, where the eigenvalues are given by

$$\begin{aligned}\omega_\alpha(f,m) &= \langle T_f e_m, e_m\rangle_\alpha \\ &= \frac{\Gamma(n+|m|+\alpha+1)}{m!\,\Gamma(n+\alpha+1)} \int\limits_{\mathbb{B}_n} f(z) z^m \bar{z}^m \mathrm{d}\nu_\alpha(z) \qquad (3.1)\\ &= c_\alpha \frac{\Gamma(n+|m|+\alpha+1)}{m!\,\Gamma(n+\alpha+1)} \int\limits_{\mathbb{B}_n} f(|z_1|,\ldots,|z_n|) z^m \bar{z}^m (1-|z|^2)^\alpha \mathrm{d}\nu(z)\end{aligned}$$

for $m \in \mathbb{N}^n$. □

We are now ready for our proof of Theorem 1.2.

Proof of Theorem 1.2. For any $h \in \{\bar{f}_1,\ldots,\bar{f}_N, g_1,\ldots,g_M\}$, the Toeplitz operator T_h is diagonal by assumption. Theorem 3.1 then shows that for almost all z in $\mathbb{B}_n$, $h(z_1,\ldots,z_n) = h(|z_1|,\ldots,|z_n|)$. Let $Z(h) = \{m \in \mathbb{N}^n : \omega_\alpha(h,m) = 0\}$. Then since h is not the zero function, Proposition 2.1 shows that $Z(h)$ must have property (P). Put $Z = Z(\bar{f}_1)\cup\cdots\cup Z(\bar{f}_N)\cup Z(g_1)\cup\cdots\cup Z(g_M)$. Then Z has property (P). For any m and k in $\mathbb{N}^n\backslash Z$, there are nonzero numbers α_m and β_k such that $T_{g_1}\cdots T_{g_M} e_m = \alpha_m e_m$ and $T_{\bar{f}_N}\cdots T_{\bar{f}_1} e_k = \beta_k e_k$. Therefore,

$$\begin{aligned}\langle f e_m, e_k\rangle_\alpha &= \langle T_f e_m, e_k\rangle_\alpha \\ &= \frac{1}{\alpha_m \bar{\beta}_k} \langle T_f T_{g_1}\cdots T_{g_M} e_m, T_{\bar{f}_N}\cdots T_{\bar{f}_1} e_k\rangle_\alpha \\ &= \frac{1}{\alpha_m \bar{\beta}_k} \langle T_{\bar{f}_1}\cdots T_{\bar{f}_N} T_f T_{g_1}\cdots T_{g_M} e_m, e_k\rangle_\alpha = 0.\end{aligned}$$

This implies that $\int\limits_{\mathbb{B}_n} f(z) z^m \bar{z}^k (1-|z|^2)^\alpha \mathrm{d}\nu(z) = 0$ for all $m,k \in \mathbb{N}^n\backslash Z$. Since Z has property (P), Proposition 2.2 then shows that $f(z)(1-|z|^2)^\alpha = 0$ for almost all $z \in \mathbb{B}_n$. Hence f is the zero function. □

4. COMMUTING WITH DIAGONAL TOEPLITZ OPERATORS

Suppose $g \in L^\infty$ so that $g(z_1,\ldots,z_n) = g(|z_1|,\ldots,|z_n|)$ for almost all $z \in \mathbb{B}_n$ then from Theorem 3.1, T_g is diagonal and $T_g = \sum_{m\in\mathbb{N}^n} \omega_\alpha(g,m) e_m \otimes e_m$. Note that $\omega_\alpha(\bar{g},m) = \bar{\omega}_\alpha(g,m)$ for any $m \in \mathbb{N}^n$. Suppose f is a function in L^∞. Then $T_f T_g = T_g T_f$ if and only if $\langle T_f T_g e_m, e_k\rangle_\alpha = \langle T_g T_f e_m, e_k\rangle_\alpha$ for all $m,k \in \mathbb{N}^n$. This is equivalent to

$$\omega_\alpha(g,m)\langle T_f e_m, e_k\rangle_\alpha = \langle T_f e_m, \omega_\alpha(\bar{g},k) e_k\rangle_\alpha$$

for all $m,k \in \mathbb{N}^n$, which is in turn equivalent to

$$(\omega_\alpha(g,m) - \omega_\alpha(g,k))\langle T_f e_m, e_k\rangle_\alpha = 0 \qquad (4.1)$$

for all $m,k \in \mathbb{N}^n$.

The following theorem gives a sufficient condition for $g \in L^\infty$ to belong to G. Recall that the set G is given by

$$G = \{g \in L^\infty : \ T_g \text{ is diagonal} \\ \text{and for } f \in L^\infty,\ T_f T_g = T_g T_f \text{ implies } T_f \text{ is diagonal}\}$$

The hypothesis of Theorem 4.1 is easier to check when the function g is invariant under permutations of the variables.

Theorem 4.1. *Suppose $g \in L^\infty$ so that $g(z_1, \ldots, z_n) = g(|z_1|, \ldots, |z_n|)$ for almost all $z \in \mathbb{B}_n$. Suppose that for any $1 \le j \le n$, any $\tilde{m}, \tilde{k}$ in $\mathbb{N}^{n-1}$ and any integer $l > 0$, the set*

$$Z_j(\tilde{m}, \tilde{k}, l) = \big\{s \in \mathbb{N} : \omega_\alpha(g, \sigma_j(s, \tilde{m})) = \omega_\alpha(g, \sigma_j(s + l, \tilde{k}))\big\}$$

has property (P). Let $f \in L^\infty$ so that $T_f T_g = T_g T_f$. Then for almost all $z \in \mathbb{B}_n$ we have $f(z_1, \ldots, z_n) = f(|z_1|, \ldots, |z_n|)$. As a consequence, the operator T_f is diagonal.

Proof. For any $1 \le j \le n$, any $\tilde{m}, \tilde{k} \in \mathbb{N}^{n-1}$ and any integer $l > 0$, let $N_j(\tilde{m}, \tilde{k}, l) = \mathbb{N} \backslash Z_j(\tilde{m}, \tilde{k}, l)$. Then $N_j(\tilde{m}, \tilde{k}, l)$ does not have property (P). For any s in $N_j(\tilde{m}, \tilde{k}, l)$, let $m = \sigma_j(s, \tilde{m})$ and $k = \sigma_j(s + l, \tilde{k})$. Then $\omega_\alpha(g, m) \ne \omega_\alpha(g, k)$, so equation (4.1) implies that $\langle T_f e_m, e_k\rangle_\alpha = 0$ and $\langle T_f e_k, e_m\rangle_\alpha = 0$. Thus,

$$\int\limits_{\mathbb{B}_n} f(z) z^m \bar{z}^k (1 - |z|^2)^\alpha \mathrm{d}\nu(z) = 0 \text{ and } \int\limits_{\mathbb{B}_n} f(z) z^k \bar{z}^m (1 - |z|^2)^\alpha \mathrm{d}\nu(z) = 0.$$

Applying Corollary 2.4, we get $f(z_1, \ldots, z_n) = f(|z_1|, \ldots, |z_n|)$ for almost all $z \in \mathbb{B}_n$. $\square$

We next give examples of functions that satisfy the requirements of Theorem 1.3.

Proof of Theorem 1.3. To show (1), we consider the function $f(z) = z_1 \bar{z}_2$. For $m, k \in \mathbb{N}^n$,

$$\int\limits_{\mathbb{B}_n} f(z) z^m \bar{z}^k (1 - |z|^2)^\alpha \mathrm{d}\nu(z) = \int\limits_{\mathbb{B}_n} z^{m+\delta_1} \bar{z}^{k+\delta_2} (1 - |z|^2)^\alpha \mathrm{d}\nu(z) = 0$$

unless $m + \delta_1 = k + \delta_2$ (here $\delta_j = (0, \ldots, 0, 1, 0, \ldots, 0)$ where the number 1 is in the jth slot, for $1 \le j \le n$). So $\langle T_f e_m, e_k\rangle_\alpha = 0$ whenever $m + \delta_1 \ne k + \delta_2$.

Now suppose that $h \in L^\infty$ is a radial function, say $h(z) = \tilde{h}(|z|)$ for some bounded measurable function $\tilde{h}$ on $[0, 1)$. Then by (3.1),

$$\omega_\alpha(h, m) = c_\alpha \frac{\Gamma(n + |m| + \alpha + 1)}{m!\ \Gamma(n + \alpha + 1)} \int\limits_{\mathbb{B}_n} \tilde{h}(|z|) z^m \bar{z}^m (1 - |z|^2)^\alpha \mathrm{d}\nu(z),$$

for any $m \in \mathbb{N}^n$. Using polar coordinates and the explicit formula of c_α, we see that

$$\omega_\alpha(h,m) = \frac{\Gamma(n+|m|+\alpha+1)}{\Gamma(\alpha+1)\Gamma(n+|m|)} \int_0^1 r^{n+|m|-1}(1-r)^\alpha \tilde{h}(r^{1/2})\mathrm{d}r.$$

So $\omega_\alpha(h,m) = \omega_\alpha(h,k)$ whenever $|m| = |k|$. This shows that when $m+\delta_1 = k+\delta_2$ we have $\omega_\alpha(h,m) = \omega_\alpha(h,k)$. Therefore equation (4.1) (with h in place of g) holds for all $m,k \in \mathbb{N}^n$, which implies $T_f T_h = T_h T_f$.

To show (2), we consider the function $g(z) = |z_1|^2 \cdots |z_n|^2$. Then for any $m \in \mathbb{N}^n$, we have

$$\begin{aligned}
\omega_\alpha(g,m) &= c_\alpha \frac{\Gamma(n+|m|+\alpha+1)}{m!\,\Gamma(n+\alpha+1)} \int_{\mathbb{B}_n} g(z) z^m \bar{z}^m (1-|z|^2)^\alpha \mathrm{d}\nu(z)\\
&= c_\alpha \frac{\Gamma(n+|m|+\alpha+1)}{m!\,\Gamma(n+\alpha+1)} \int_{\mathbb{B}_n} z^{m+(1,\ldots,1)} \bar{z}^{m+(1,\ldots,1)} (1-|z|^2)^\alpha \mathrm{d}\nu(z)\\
&= \frac{\Gamma(n+|m|+\alpha+1)}{m!\,\Gamma(n+\alpha+1)} \frac{(m+(1,\ldots,1))!\,\Gamma(n+\alpha+1)}{\Gamma(n+|m+(1,\ldots,1)|+\alpha+1)}\\
&= \frac{(m_1+1)\cdots(m_n+1)}{(m_1+\cdots+m_n+\alpha+2n)\cdots(m_1+\cdots+m_n+\alpha+n+1)}.
\end{aligned}$$

We will show that g satisfies the hypothesis of Theorem 4.1. Since the function g is independent of the order of the variables, we only need to show that for any $\tilde{m} = (m_1,\ldots,m_{n-1}), \tilde{k} = (k_1,\ldots,k_{n-1}) \in \mathbb{N}^{n-1}$ and any integer $l > 0$ the set $N_1(\tilde{m},\tilde{k},l) = \{s \in \mathbb{N} : \omega_\alpha(g,(s,\tilde{m})) = \omega_\alpha(g,(s+l,\tilde{k}))\}$ has property (P). In fact, we will show that $N_1(\tilde{m},\tilde{k},l)$ has at most $n+1$ elements. Consider the polynomial

$$\begin{aligned}
p(w) &= (w+1)(w+|\tilde{k}|+\alpha+2n)\prod_{j=1}^{n-1}(m_j+1)(w+|\tilde{k}|+\alpha+2n-j)\\
&- (w+l+1)(w+l+|\tilde{m}|+\alpha+2n)\prod_{j=1}^{n-1}(k_j+1)(w+l+|\tilde{m}|+\alpha+2n-j).
\end{aligned}$$

Then $N_1(\tilde{m},\tilde{k},l)$ is exactly the set of all non-negative integer roots of p. Since $p(-1) \neq 0$, $p(w)$ is not identically zero. This shows that p has at most $n+1$ distinct roots. $\square$

Acknowledgments This work was completed when the author was attending the Fall 2007 Thematic Program on Operator Algebras at the Fields Institute for Research in Mathematical Sciences.

References

[1] Patrick Ahern, *On the range of the Berezin transform*, J. Funct. Anal. **215** (2004), no. 1, 206–216. MR 2085115 (2005e:47067)

[2] Patrick Ahern and Željko Čučković, *A theorem of Brown-Halmos type for Bergman space Toeplitz operators*, J. Funct. Anal. **187** (2001), no. 1, 200–210. MR 1867348 (2002h:47040)

[3] ______, *Some examples related to the Brown-Halmos theorem for the Bergman space*, Acta Sci. Math. (Szeged) **70** (2004), no. 1-2, 373–378. MR 2072710 (2005d:47046)

[4] Arlen Brown and Paul R. Halmos, *Algebraic properties of Toeplitz operators*, J. Reine Angew. Math. **213** (1963/1964), 89–102. MR 0160136 (28 #3350)

[5] Boorim Choe and Hyungwoon Koo, *Zero products of Toeplitz operators with harmonic symbols*, J. Funct. Anal. **233** (2006), no. 2, 307–334. MR 2214579 (2006m:47049)

[6] Željko Čučković and N. V. Rao, *Mellin transform, monomial symbols, and commuting Toeplitz operators*, J. Funct. Anal. **154** (1998), no. 1, 195–214. MR 1616532 (99f:47033)

[7] Caixing Gu, *Products of several Toeplitz operators*, J. Funct. Anal. **171** (2000), no. 2, 483–527. MR 1745626 (2001d:47039)

[8] Kun Yu Guo, *A problem on products of Toeplitz operators*, Proc. Amer. Math. Soc. **124** (1996), no. 3, 869–871. MR 1307521 (96f:47050)

[9] Rubén A. Martínez-Avendaño and Peter Rosenthal, *An introduction to operators on the Hardy-Hilbert space*, Graduate Texts in Mathematics, vol. 237, Springer, New York, 2007. MR 2270722 (2007k:47051)

[10] Walter Rudin, *Function theory in the unit ball of* $\mathbf{C}^n$, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Science], vol. 241, Springer-Verlag, New York, 1980. MR 601594 (82i:32002)

[11] ______, *Real and complex analysis*, third ed., McGraw-Hill Book Co., New York, 1987. MR 924157 (88k:00002)

[12] Kehe Zhu, *Spaces of holomorphic functions in the unit ball*, Graduate Texts in Mathematics, vol. 226, Springer-Verlag, New York, 2005. MR 2115155 (2006d:46035)

Trieu Le, Department of Pure Mathematics, University of Waterloo, 200 University Avenue West, Waterloo, Ontario, Canada N2L 3G1

E-mail address: `t29le@math.uwaterloo.ca`