ON THE ESSENTIAL COMMUTANT OF TOEPLITZ OPERATORS IN SEVERAL COMPLEX VARIABLES

Abstract. Using the joint local mean oscillation, Xia [11] showed that the essential commutant of $\mathfrak{T}(\mathcal{L})$ - the algebra generated by all Toeplitz operators T_g where g is bounded and has at most one discontinuity, is $\mathfrak{T}(\mathrm{QC})$. Even though Xia's method cannot be used, we are able to generalize his result to Toeplitz operators in higher dimensions with a different approach. This result is stronger than the well-known fact stating that the essential commutant of the full Toeplitz algebra $\mathfrak{T}$ is $\mathfrak{T}(\mathrm{QC})$.

1. Introduction

For an integer $n \geq 1$, let $\mathbb{C}^n$ denote the Cartesian product of n copies of $\mathbb{C}$. For $z = (z_1, \dots, z_n)$ and $w = (w_1, \dots, w_n)$ in $\mathbb{C}^n$, we use $\langle z, w\rangle = z_1\overline{w}_1 + \cdots + z_n\overline{w}_n$ and $|z| = \sqrt{|z_1|^2 + \cdots + |z_n|^2}$ for the inner product and the associated Euclidean norm. Let B^n denote the open unit ball which consists of points $z \in \mathbb{C}^n$ with $|z| < 1$. Let S^n denote the unit sphere which is the boundary of B^n. We denote by σ the unitary-invariant measure on S^n so normalized that $\sigma(S^n) = 1$.

Let $\mathcal{A}(B^n)$ be the algebra of all functions that are analytic in the open unit ball B^n and continuous on the closed unit ball $\bar{B^n}$. We write L^p for $L^p(S^n, d\sigma)$ and H^p for the Hardy subspace of L^p for $1 \leq p \leq \infty$. For each $\zeta \in S^n$, $k_\zeta(w) = (1 - |\zeta|^2)^{n/2}(1 - \langle w, \zeta\rangle)^{-n}$ is a normalized reproducing kernel for H^2. We have $\|k_\zeta\| = 1$ and $\langle \varphi, k_\zeta\rangle = (1 - |\zeta|^2)^{n/2}\varphi(z)$ for all $\varphi \in H^2$. Let $P : L^2 \longrightarrow H^2$ denote the orthogonal projection. For any $f \in L^\infty$, the Toeplitz operator T_f and the Hankel operator H_f are defined by $T_f\varphi = P(f\varphi)$ and $H_f\varphi = f\varphi - P(f\varphi)$, respectively, for all $\varphi \in H^2$. We have $T_{gf} - T_gT_f = H^*_{\bar{g}}H_f$ for all $f, g \in L^\infty$. Let $\mathfrak{B}(H^2)$ denote the C^*-algebra of all bounded linear operators on H^2. For any subset $\mathcal{G}$ of L^∞, we denote by $\mathfrak{T}(\mathcal{G})$ the C^*-subalgebra of $\mathfrak{B}(H^2)$ generated by the set $\{T_f : f \in \mathcal{G}\}$. Then $\mathfrak{T} = \mathfrak{T}(L^\infty)$ is the full Toeplitz algebra. Let $\mathcal{K}$ denote the ideal of all compact operators on H^2. It is well-known that $\mathcal{K} \subset \mathfrak{T}$. We us π to denote the canonical map from $\mathfrak{T}$ onto the Calkin algebra $\mathfrak{T}/\mathcal{K}$.

For any $\zeta \in S^n$, let K_ζ be the closed two-sided ideal of $\mathfrak{T}$ generated by the set $\{T_f : f \text{ is continuous on } S^n \text{ and } f(\zeta) = 0\}$. Then each K_ζ contains the compact operators $\mathcal{K}$ and $\bigcap\limits_{\zeta \in S^n} K_\zeta = \mathcal{K}$. This follows from the localization in $\mathfrak{T}$, see [5, p. 176-177].

For any subset $\mathcal{S}$ of $\mathfrak{B}(H^2)$, we denote by $\mathrm{EssCom}(\mathcal{S})$ the essential commutant of $\mathcal{S}$, that is,

$$\mathrm{EssCom}(\mathcal{S}) = \{A \in \mathfrak{B}(H^2) : AT - TA \text{ is compact for all } T \in \mathcal{S}\}.$$

For any $z, w \in \bar{B^n}$, let $d(z,w) = |1 - \langle z, w\rangle|^{1/2}$. Then d is a metric on S^n which is called the nonisotropic metric, see [9, 5.1]. Because of the denominators of the reproducing kernels, it is this metric, not the usual Euclidean metric, which is closely associated to the theory of Toeplitz and Hankel operators on S^n. Note that for $|\zeta| = 1$ and $|z| \le 1$, $(d(\zeta, z))^2 \le |\zeta - z| \le \sqrt{2}\, d(\zeta, z)$.

For $\zeta \in S^n$ and $r > 0$, let $Q(\zeta, r) = \{z \in S^n : d(z,\zeta) < r\}$ be the open ball of radius r centered at ζ in the metric d. There is a constant A_0 depending on n so that

$$2^{-n} r^{2n} \le \sigma(Q(\zeta, r)) \le A_0 r^{2n}. \tag{1.1}$$

For any function $f \in L^1$ and any $\zeta \in S^n$, the mean oscillation of f at ζ is

$$\mathrm{LMO}(f)(\zeta) = \lim_{\delta \downarrow 0} \sup \left\{ \frac{1}{\sigma(Q_r)} \int\limits_{Q_r} |f - f_{Q_r}|\, d\sigma : Q_r \subset Q(\zeta, \delta), r < \delta \right\},$$

where Q_r denotes a ball of radius r centered at a point on S^n in the $d-$metric and $f_{Q_r} = \dfrac{1}{\sigma(Q_r)} \displaystyle\int\limits_{Q_r} f\, d\sigma$.

A function $f \in L^1$ is said to have bounded mean oscillation if

$$\|f\|_{\mathrm{BMO}} = \sup\{\mathrm{LMO}(f)(\zeta) : \zeta \in S^n\} < \infty.$$

A function $f \in L^1$ is said to have vanishing mean oscillation if $\mathrm{LMO}(f)(\zeta) = 0$ for all $\zeta \in S^n$.

Define

$$\mathrm{BMO} = \{f \in L^1 : f \text{ has bounded mean oscillation}\},$$
$$\mathrm{VMO} = \{f \in L^1 : f \text{ has vanishing mean oscillation}\}.$$

Let QC $=$ VMO$\cap L^\infty$. Davidson proved in [2] that for $n = 1$, $\mathrm{EssCom}(L^\infty) = \mathfrak{T}(\mathrm{QC})$. The key result from which Davidson derived the above identity is the following Theorem.

Theorem 1.1. *Consider $n = 1$. If S is a bounded operator on H^2 which is not the sum of a Toeplitz operator and a compact operator then there is a function $h \in H^\infty$ such that $T_hS - ST_h$ is not compact. The function h can be taken to have at most one discontinuity.*

For any $\zeta \in S^n$, let $\mathcal{L}(\zeta)$ be the collections of bounded measurable functions on S^n which are continuous on $S^n\backslash\{\zeta\}$ and let $\mathcal{H}(\zeta) = H^\infty \cap \mathcal{L}(\zeta)$.

Xia proved in [11] the following Theorem which is a local version of a result of Sarason and used this together with Theorem 1.1 to deduce that $\mathrm{EssCom}(\mathcal{L}) = \mathfrak{T}(QC)$ when $n = 1$. Note that $\mathcal{L}$ is indeed smaller than L^∞, which was also showed by Xia in the same paper.

Theorem 1.2. *Let $f \in L^\infty$ and let $\zeta \in S^1$.*

(*a*) *If $[T_f, T_h] \in K_\zeta$ for all $h \in \mathcal{H}(\zeta)$ then* $\mathrm{LMO}(Qf)(\zeta) = 0$.

(*b*) *If* $\mathrm{LMO}(Qf)(\zeta) = 0$, *then $[T_f, T_g] \in K_\zeta$ for all $g \in H^\infty$.*

Here $Q = 1 - P$ is the orthogonal projection of L^2 onto $L^2 \ominus H^2$.

We would like to generalize Xia's results to Toeplitz operators on the Hardy spaces of the unit sphere in higher dimensions. To do that, we need the high dimensional versions of Theorem 1.1 and Theorem 1.2.

A result quite similar to Theorem 1.1 was proved by Ding and Sun in [4] but there they did not have the continuity of h. In the following Theorem we do have the continuity of h which is necessary in Xia's proof.

Theorem 1.3. *Let $n \geq 1$. Suppose S is a bounded operator on $\mathfrak{B}(H^2)$ which is not the sum of a Toeplitz operator and a compact operator. Then there is a function $h \in H^\infty$ so that $T_hS - ST_h$ is not compact. The function h can be taken to have at most one discontinuity.*

One of the important ingredients needed to prove Theorem 1.2 is the identity $1 - T_{\xi_z}T^*_{\xi_z} = k_z \otimes k_z$, where $z \in B^1$ and $\xi_z(\tau) = (z-\tau)(1-\overline{z}\tau)^{-1}$, where $\tau \in S^1$. We do not know if such a similar identity exists in higher dimensions where there is no continuous inner function (note that each ξ_z is a continuous inner function on S^1.) So we cannot use Xia's method for $n \geq 2$. Nevertheless, with a different approach, we can generalize Theorem 1.2. Note that when $n = 1$, by checking directly, we can show that $H_f k_z = (Qf - (Qf)(z))k_z$ and hence $\lim\limits_{\substack{|z|<1\\ z\to\zeta}} \|H_f k_z\| = 0$ if and only if $\mathrm{LMO}(Qf)(\zeta) = 0$ for any $\zeta \in S^1$, see [12, Theorem 6] for more details. Therefore, the following Theorem is a version of Theorem 1.2 for $n \geq 2$.

Theorem 1.4. *Let $f, g \in L^\infty$ and $\zeta \in S^n$.*

(a) *If $[T_f, T_h] \in \mathcal{K}_\zeta$ for all $h \in \mathcal{H}(\zeta)$, then $\lim\limits_{\substack{|z|<1\\ z\to\zeta}} \|H_f k_z\| = 0$.*

(b) *If $\lim\limits_{\substack{|z|<1\\ z\to\zeta}} \|H_f k_z\|\|H_{\bar{g}} k_z\| = 0$, then $H_{\bar{g}}^* H_f = T_{gf} - T_g T_f \in \mathcal{K}_\zeta$.*

An open question, which appears to be highly non-trivial, is whether or not the converse of Theorem 1.4 (b) holds true in the case $n \geq 2$. We know that this converse holds true in the case $n = 1$, see [12], and the proof uses the identity $1 - T_{\xi_z} T_{\xi_z}^* = k_z \otimes k_z$ in an essential way. A related question is that, in the case $n \geq 2$, if $H_{\bar{g}}^* H_f$ is compact, where $f, g \in L^\infty$, does it follow that

$$\lim_{|z|\uparrow 1} \|H_f k_z\|\|H_{\bar{g}}^* k_z\| = 0 \ ? \tag{1.2}$$

From Theorem 1.4 (b) and the fact that $\mathcal{K} = \bigcap\limits_{\zeta\in S^n} K_\zeta$, it follows that (1.2) is sufficient for the compactness of $H_{\bar{g}}^* H_f$ (see also [13, Theorem 3].) In the case $n \geq 2$, the question whether or not (1.2) is necessary for the compactness of $H_{\bar{g}}^* H_f$ seems to be highly non-trivial. In this context, Theorem 1.4(a), which is the main result of the paper, can be viewed as partial progress toward answering these open questions.

Using Theorem 1.3 and Theorem 1.4, we can prove Xia's result on essential commutants of Toeplitz operators on the Hardy space of the unit sphere in high dimensions.

Corollary 1.1. *Let $\mathcal{H}$ denote the subalgebra of H^∞ generated by $\bigcup\limits_{\zeta\in S^n} \mathcal{H}(\zeta)$. If $f \in L^\infty$ so that $[T_f, T_h]$ is compact for all $h \in \mathcal{H}$, then $\lim\limits_{|z|\uparrow 1} \|H_f k_z\| = 0$. As a consequence, the essential commutant of $\mathfrak{T}(\mathcal{H})$ is $\mathfrak{T}(\mathcal{C})$, where $\mathcal{C} = \{f \in L^\infty : H_f$ is compact$\}$.*

Corollary 1.2. *Let $\mathcal{L}$ denote the subalgebra of L^∞ generated by $\bigcup\limits_{\zeta\in S^n} \mathcal{L}(\zeta)$. Then the essential commutant of $\mathfrak{T}(\mathcal{L})$ equals $\mathfrak{T}(\mathrm{QC})$.*

In [11], Xia also proved the one dimensional version of the following Theorem. We are presenting here the proof for $n \geq 2$.

Theorem 1.5. *Suppose $\mathcal{S}$ is a subset of $\mathfrak{T}$ and suppose that $\mathcal{S}$ is separable in the operator-norm topology. Then there is a real valued function $f \in L^\infty$ so that f is not in* VMO *and $[T_f, S]$ is compact for all $S \in \mathcal{S}$. Furthermore, given such an $\mathcal{S}$, there is a $\zeta = \zeta(\mathcal{S})$ such that there is an $f \in \mathcal{L}(\zeta)$ which satisfies the above requirements.*

In the rest of the paper, we will prove a couple of Lemmas before giving the proof of Theorem 1.3 in Section 2 and proofs of Theorem 1.4, Corollary 1.1 and Corollary 1.2 in Section 3. In section 4, we construct the function f that satisfies the requirement of Theorem 1.5.

2. Operators Essentially Commuting with Analytic Toeplitz Operators

We begin this section by a Lemma about pointwise approximation of a positive lower semi-continuous function on S^n by a sequence of functions in the ball algebra $\mathcal{A}(B^n)$. For a proof, see [7, Theorem 3.5 and Remark 3.6].

Lemma 2.1. *Suppose φ is a lower semi-continuous function on S^n and $\varphi > 0$. Then there is a sequence $\{\varphi_m\}_{m=1}^\infty$ of functions in $\mathcal{A}(B^n)$ with the following properties:*

$$\begin{aligned} &1) \quad |\varphi_m| \leq \varphi \text{ on } S^n, \text{ and} \\ &2) \quad \{|\varphi_m|\}_{m=1}^\infty \text{ converges to } \varphi \text{ a.e. on } S^n. \end{aligned}$$

The following Lemma is a construction of the holomorphic function which appears in Theorem 1.3.

Lemma 2.2. *Suppose $\{E_m\}_{m=1}^\infty$ is a sequence of mutually disjoint measurable subsets of S^n which cluster at only one point $\zeta \in S^n$. Suppose $\{\varphi_m\}_{m=1}^\infty$ is a sequence of functions in $\mathcal{A}(B^n)$ so that $\|\varphi_m(1 - \chi_{E_m})\|_\infty \leq \alpha^m$ and $\|\varphi_m\|_\infty \leq \beta$ for all m, where $\beta > 0$ and $0 < \alpha < 1$. Suppose S is a bounded operator on $\mathfrak{B}(H^2)$ so that $[T_{\varphi_m}, S]$ is compact and $\|[T_{\varphi_m}, S]\| > c > 0$ for all m, where c is a fixed constant. Then there is a function h in H^∞ which is continuous on $S^n \backslash \{\zeta\}$ so that $\|\pi([T_h, S])\| \geq c$.*

Proof. For each m, put $A_m = [T_{\varphi_m}, S]$. Then A_m is a compact operator for each m.

Let $E = \bigcap\limits_{k=1}^{\infty} \bigcup\limits_{m=k}^{\infty} E_m$. Then $\sigma(E) = 0$ since $\sum\limits_{m=1}^{\infty} \sigma(E_m) \leq \sigma(S^n) = 1$.

For $w \in S^n \backslash E$, there is a k so that $w \in S^n \backslash (\bigcup_{m=k}^{\infty} E_m) = \bigcap\limits_{m=k}^{\infty} (S^n \backslash E_m)$. Hence $|\varphi_m(w)| \leq \alpha^m$ for all $m \geq k$. So the sequence $\{\varphi_m\}_{m=1}^{\infty}$ converges to 0 almost everywhere. By the Lebesgue Dominated Convergence Theorem, it follows that the sequences $\{T_{\varphi_m}\}_{m=1}^{\infty}$ and $\{T_{\overline{\varphi}_m}\}_{m=1}^{\infty}$ converge to 0 in the strong operator topology. Therefore $\{A_m\}_{m=1}^{\infty}$ and $\{A_m^*\}_{m=1}^{\infty}$ converge to 0 in the strong operator topology. Now, since $\|A_m\| \leq 2\beta\|S\|$ for all m, passing to a subsequence of $\{A_m\}_{m=1}^{\infty}$ if necessary, we may assume that the numerical sequence $\{\|A_m\|\}_{m=1}^{\infty}$ converges. So $\lim\limits_{m\to\infty} \|A_m\|$ exists and is not less than c.

By [6, Lemma 2.1], there is an increasing sequence of positive integers $\{m(k)\}_{k=1}^{\infty}$ so that the sum

$$A = \sum_{k=1}^{\infty} A_{m(k)} = \lim_{N\to\infty} \sum_{k=1}^{N} A_{m(k)}$$

exists in the strong operator topology and

$$\|\pi(A)\| = \|\pi(\sum_{k=1}^{\infty} A_{m(k)})\| = \lim_{m\to\infty} \|A_m\| \geq c.$$

Now put $h = \sum\limits_{k=1}^{\infty} \varphi_{m(k)}$.

For any neighborhood U of ζ in S^n, there is a $k(U)$ so that $E_{m(k)} \subset U$ for all $k \geq k(U)$. Thus

$$\begin{aligned} \sum_{k=1}^{\infty} \|\varphi_{m(k)}\chi_{S^n\backslash\overline{U}}\|_\infty &\leq \sum_{k=1}^{k(U)} \|\varphi_{m(k)}\chi_{S^n\backslash\overline{U}}\|_\infty + \sum_{k=k(U)+1}^{\infty} \|\varphi_{m(k)}\chi_{S^n\backslash\overline{U}}\|_\infty \\ &\leq \sum_{k=1}^{k(U)} \|\varphi_{m(k)}\chi_{S^n\backslash\overline{U}}\|_\infty + \sum_{k=k(U)+1}^{\infty} \alpha^{m(k)} \\ &\leq \beta k(U) + 1/(1-\alpha). \end{aligned}$$

So $\{\sum\limits_{k=1}^{N} \varphi_{m(k)}\}_{N=1}^{\infty}$ converges uniformly to h on $S^n\backslash\overline{U}$. Because this holds true for any neighborhood U of ζ, it follows that h is in H^∞ and continuous on $S^n\backslash\{\zeta\}$.

Now in the strong operator topology,

$$\begin{aligned} A &= \lim_{N\to\infty} \sum_{k=1}^{N} A_{m(k)} \\ &= \lim_{N\to\infty} \sum_{k=1}^{N} [T_{\varphi_{m(k)}}, S] \\ &= [T_{\lim\limits_{N\to\infty} \sum_{k=1}^{N} \varphi_{m(k)}}, S] \\ &= [T_h, S]. \end{aligned}$$

So $\|\pi([T_h, S])\| = \|\pi(A)\| \geq c > 0$. □

In what follows, by a characteristic function we mean the characteristic function of a Lebesgue measurable set of S^n. If g is a characteristic function, we also use g to denote its support. We use w–lim to denote limits in the weak operator topology of $\mathfrak{B}(H^2)$. The first two conclusions in the following Lemma were proved in [2] for functions on the unit circle. The high dimensional case is similar and was proved in [4]. For our purpose of proving Theorem 1.3, we have added the third conclusion.

Lemma 2.3. *Suppose $F : L^\infty \to \mathfrak{B}(H^2)$ is a linear map which has the following properties:*

- (P1) *If g_m's are in L^∞ with $\|g_m\|_\infty \leq M$ and $g_m \to g_0$ almost everywhere, then w–$\lim F(g_m) = F(g_0)$.*
- (P2) *If f_1, f_2 are characteristic functions and have disjoint closed supports, then $F(f_1)F(f_2)$ and $F(f_1)F(f_2)^*$ are compact.*
- (P3) *There is a characteristic function g so that $\|\pi(F(g))\| > \alpha > 0$.*

Then there exists a sequence of characteristic functions of disjoint closed supports $\{\chi_m\}$ so that $\|F(\chi_m g)\| > C(\alpha)/2 > 0$ for all m, where $C(\alpha)$ depends on α and the dimension n. These sets can be chosen to cluster at only one point.

It then follows that there is a sequence of continuous functions $\{h_m\}$ so that $h_m \geq 0$, $\|h_m\|_\infty \leq 2$, $\|h_m(1-\chi_m g)\|_\infty \leq 4^{-m-1}$ and $\|F(h_m)\| > C(\alpha)/2$ for all m.

Consequently, if F is of the form $F(g) = T_g S - T_{gf}$, where S is a bounded operator on H^2 and f is a bounded measurable function, then there exists a sequence $\{\varphi_m\}$ of functions in $\mathcal{A}(B^n)$ so that $\|\varphi_m\|_\infty \leq 2$, $\|\varphi_m(1-\chi_m g)\|_\infty \leq 2^{-m}$ and $\|F(\varphi_m)\| > C(\alpha)/8$.

Proof. The existence of the sequences $\{\chi_m\}_{m=1}^{\infty}$, $\{h_m\}_{m=1}^{\infty}$ and the constant $C(\alpha)$ was established in [4, Lemma 4].

Now, fix an integer m. Choose a positive number β so that $\beta < 4^{-m-1}$, and $\beta\|F(1)\| < C(\alpha)/4$.

Let $g_m = h_m + \beta$. Then g_m is continuous, positive on S^n, $\|g_m\|_\infty \leq 4$ and $|g_m(1-\chi_m g)| \leq 4^{-m}$.

Also,

$$\begin{aligned}\|F(g_m)\| &\geq \|F(h_m)\| - \|F(\beta)\| \\ &= \|F(h_m)\| - \beta\|F(1)\| \\ &> C(\alpha)/4.\end{aligned}$$

From Lemma 2.1, there is a sequence $\{\vartheta_j\}$ of functions in $\mathcal{A}(B^n)$ so that $|\vartheta_j|^2 \leq g_m$ on S^n, and $|\vartheta_j|^2 \to g_m$ a.e. on S^n.

Since $F(|\vartheta_j|^2) \to F(g_m)$ in the weak operator topology by $(P1)$ and $\|F(g_m)\| > C(\alpha)/4$, there is a $j(m)$ so that $\|F(|\vartheta_{j(m)}|^2)\| > C(\alpha)/4$.

Let $\varphi_m = \vartheta_{j(m)}$. Then $\varphi_m \in \mathcal{A}(B^n)$, $\|\varphi_m\|_\infty \leq 2$, $|\varphi_m(1-\chi_m g)| \leq 2^{-m}$ and $\|F(|\varphi_m|^2)\| > C(\alpha)/4$.

Now if $F(|\varphi_m|^2) = T_{|\varphi_m|^2}S - T_{|\varphi_m|^2 f}$ for all m, then

$$\begin{aligned}C(\alpha)/4 &< \|F(|\varphi_m|^2)\| \\ &= \|T_{\varphi_m\overline{\varphi}_m}S - T_{\varphi_m\overline{\varphi}_m f}\| \\ &= \|T_{\overline{\varphi}_m}T_{\varphi_m}S - T_{\overline{\varphi}_m}T_{\varphi_m f}\| \text{ (since } \varphi_m \text{ is analytic)} \\ &= \|T_{\overline{\varphi}_m}(T_{\varphi_m}S - T_{\varphi_m f})\| \\ &\leq \|T_{\overline{\varphi}_m}\|\|T_{\varphi_m}S - T_{\varphi_m f}\| \\ &\leq 2\|F(\varphi_m)\|.\end{aligned}$$

Hence, $\|F(\varphi_m)\| > C(\alpha)/8$. □

Now suppose S is a bounded operator so that $[T_\varphi, S]$ is compact for all $\varphi \in \mathcal{A}(B^n)$. So $\pi(S)$ and $\pi(T_\varphi)$ commute in the Calkin algebra. Since $\pi(T_\varphi)$ is normal, Fuglede Theorem implies that $\pi(S)$ commutes with $\pi(T_\varphi)^* = \pi(T_\varphi^*) = \pi(T_{\overline{\varphi}})$ for all $\varphi \in \mathcal{A}(B^n)$. Thus S essentially commutes with the C^*-algebra generated by $\{T_\varphi : \varphi \in \mathcal{A}(B^n)\}$. This implies that $[T_\varphi, S]$ is compact for any continuous function φ.

Now fix a sequence $\{\xi_m\}_{m=1}^{\infty}$ of functions in $\mathcal{A}(B^n)$ so that

(C1) $\|\xi_m\|_\infty \leq 1$ for all $m \in \mathbb{N}$,
(C2) $|\xi_m| \to 1$, a.e. on S^n as $m \to \infty$,

(C3) $T_{\bar{\xi}_m} \to 0$ in the weak operator topology as $m \to \infty$.

On the unit circle, we can choose $\xi_m(w) = w^m$ for $w \in S^1$ and (C2) can be replaced by $|\xi_m| \equiv 1$. In higher dimensions, there is no function in $\mathcal{A}(B^n)$ whose absolute values on the unit sphere are identically 1. This is the reason why we require condition (C2) which is much weaker. The existence of such a sequence can be proved as follows.

The existence of a sequence satisfying (C1)-(C3). Suppose ϑ is an inner function in the ball B^n. We then also use ϑ to denote the radial limit of ϑ which is defined almost everywhere on S^n. For each $0 < r < 1$, put $\vartheta_r(z) = \vartheta(rz),\ z \in B^n$. Then $\vartheta_r \in \mathcal{A}(B^n),\ \|\vartheta_r\|_\infty \leq 1$ and $\vartheta_r \to \vartheta$ a.e. on S^n when $r \to 1$. Let $\delta > 0$ be given. By Egoroff's Theorem, there is a measurable set $E_\delta \subset S^n$ with $\sigma(E_\delta) < \delta$ such that $\vartheta_r \to \vartheta$ uniformly on $S^n \backslash E_\delta$. Since $|\vartheta| = 1$ a.e. on S^n, we can require that $|\vartheta(w)| = 1$ for all $w \in S^n \backslash E_\delta$. Also for any $\epsilon > 0$, there is an r so that $|\vartheta(w) - \vartheta_r(w)| \leq \epsilon$ for all $w \in S^n \backslash E_\delta$.

Now take η to be any inner function with $\eta(0) = 0$. Then for any $u \in L^1(S^n)$,

$$\lim_{m\to\infty} \int_{S^n} \eta^m u \, d\sigma = 0, \tag{2.1}$$

see [8, Lemma 5.1].

Apply the above remark to each η^m, we get a measurable set E_m with $\sigma(E_m) < 2^{-m}$ and $|\eta| = 1$ on E_m and a function $\xi_m \in \mathcal{A}(B^n)$ so that $\|\xi_m\|_\infty \leq 1$ and $|\eta^m(w) - \xi_m(w)| \leq m^{-1}$ for all $w \in S^n \backslash E_m$. Then clearly the sequence $\{\xi_m\}$ satisfies (C1). We claim that it also satisfies (C2) and (C3).

Let

$$E = \bigcap_{k=1}^{\infty} \bigcup_{m=k}^{\infty} E_m.$$

Then $\sigma(E) = 0$ because $\sum\limits_{m=1}^{\infty} \sigma(E_m) < \infty$.

Now for any $w \in S^n \backslash E$, there is a k so that $w \in S^n \backslash E_m$ for all $m \geq k$. Hence

$$\begin{aligned} \big|1 - |\xi_m(w)|\big| &= \big||\eta^m(w)| - |\xi_m(w)|\big| \\ &\leq \big|\eta^m(w) - \xi_m(w)\big| \\ &\leq m^{-1}, \end{aligned}$$

for all $m \geq k$.

Thus $\lim_{m\to\infty} |\xi_m(w)| = 1$. So $|\xi_m| \to 1$ a.e. on S^n as $m \to \infty$.

In addition, for any $f, g \in H^2$,

$$\begin{aligned}|\langle T_{\bar{\xi}_m} f, g\rangle| &= |\langle P(\overline{\xi}_m f), g\rangle| \\ &= |\int_{S^n} \xi_m \overline{f} g \, d\sigma| \\ &\leq |\int_{S^n} (\xi_m - \eta^m)\overline{f} g \, d\sigma| + |\int_{S^n} \eta^m \overline{f} g \, d\sigma| \\ &\leq m^{-1}\|f\|\|g\| + 2\int_{E_m} |\overline{f} g| \, d\sigma + |\int_{S^n} \eta^m \overline{f} g \, d\sigma|.\end{aligned}$$

Letting $m \to \infty$ and using (2.1) and the fact that $\sigma(E_m) < 2^{-m}$, we get $\lim_{m\to\infty} |\langle T_{\bar{\xi}_m} f, g\rangle| = 0$. Hence $\{T_{\bar{\xi}_m}\}$ converges to 0 in the weak operator topology as $m \to \infty$. □

Put $\sigma_m = T_{\bar{\xi}_m} S T_{\xi_m}$, for $m = 1, 2, \ldots$. Then $\{\sigma_m\}_{m=1}^{\infty}$ is a bounded sequence of operators in $\mathfrak{B}(H^2)$. By passing to a subsequence if necessary, we can assume that the sequence $\{\sigma_m\}_{m=1}^{\infty}$ converges in the weak operator topology. Let $T = w\text{–}\lim_{m\to\infty} \sigma_m$. The following Lemma is about properties of the operator T.

Lemma 2.4. *Let T be as in the above remark. Then the followings hold true.*

(a) *There is an $f \in L^\infty$ so that $T = T_f$.*
(b) *For any continuous function h on S^n,*

$$w\text{–}\lim_{m\to\infty} T_{\bar{\xi}_m} T_h S T_{\xi_m} = T_{hf}.$$

(c) *For any continuous function h on S^n,*

$$w\text{–}\lim_{m\to\infty} T_{\bar{\xi}_m}(T_{h\xi_m} S - S T_{h\xi_m}) = T_h S - T_{hf}.$$

Consequently, for any $\alpha < \|T_h S - T_{hf}\|$, there is an m_α so that $\|T_{h\xi_{m_\alpha}} S - S T_{h\xi_{m_\alpha}}\| > \alpha$.

Proof. (a) For any function $b \in \mathcal{A}(B^n)$, we have

$$\begin{aligned} T - T_{\overline{b}}TT_b &= w\text{–}\lim_{m\to\infty} (T_{\overline{\xi}_m} ST_{\xi_m} - T_{\overline{b}}T_{\overline{\xi}_m} ST_{\xi_m} T_b) \\ &= w\text{–}\lim_{m\to\infty} T_{\overline{\xi}_m}(S - T_{\overline{b}}ST_b)T_{\xi_m} \\ &= w\text{–}\lim_{m\to\infty} \left\{T_{\overline{\xi}_m} T_{\overline{b}}(T_bS - ST_b)T_{\xi_m} + T_{\overline{\xi}_m} T_{1-|b|^2} ST_{\xi_m}\right\} \\ &= w\text{–}\lim_{m\to\infty} T_{\overline{\xi}_m} T_{1-|b|^2} ST_{\xi_m}, \end{aligned}$$

because $T_{\overline{b}}(T_bS - ST_b)$ is compact and $T_{\overline{\xi}_m}$ converges to 0 in the weak operator topology so $w\text{–}\lim_{m\to\infty} T_{\overline{\xi}_m} T_{\overline{b}}(T_bS - ST_b)T_{\xi_m} = 0$.

From this, we have

$$T - \sum_{j=1}^{n} T_{\overline{b}_j} TT_{b_j} = w\text{–}\lim_{n\to\infty} T_{\overline{\xi}_m} T_{(1-\sum_{j=1}^{n}|b_j|^2)} ST_{\xi_m},$$

for any $b_1, \dots, b_n \in \mathcal{A}(B^n)$.

In particular, for $b_j(z) = z_j,\ j = 1, \dots, n$, we get

$$T - \sum_{j=1}^{n} T_{\overline{z}_j} TT_{z_j} = 0.$$

By [3, Theorem 2.6], there is an $f \in L^\infty$ so that $T = T_f$.

Thus,

$$w\text{–}\lim_{m\to\infty} \sigma_m = w\text{–}\lim_{m\to\infty} T_{\overline{\xi}_m} ST_{\xi_m} = T_f.$$

(b) Since the span of $\{\overline{h}_1 h_2 : h_1, h_2 \in \mathcal{A}(B^n)\}$ is dense in $C(S^n)$, it suffices to prove the identity for $h = \overline{h}_1 h_2$ with $h_1, h_2 \in \mathcal{A}(B^n)$.

For each positive integer m,

$$\begin{aligned} T_{\overline{\xi}_m} T_{\overline{h}_1 h_2} ST_{\xi_m} &= T_{\overline{\xi}_m} T_{\overline{h}_1} T_{h_2} ST_{\xi_m} \\ &= T_{\overline{\xi}_m} T_{\overline{h}_1}[T_{h_2}, S]T_{\xi_m} + T_{\overline{\xi}_m} T_{\overline{h}_1} ST_{h_2} T_{\xi_m} \\ &= T_{\overline{\xi}_m} T_{\overline{h}_1}[T_{h_2}, S]T_{\xi_m} + T_{\overline{h}_1} T_{\overline{\xi}_m} ST_{\xi_m} T_{h_2}. \end{aligned}$$

Since $[T_{h_2}, S]$ is compact, the first term in the sum goes to 0 in the weak operator topology when $m \to \infty$. So

$$w\text{–}\lim_{m\to\infty} T_{\overline{\xi}_m} T_{\overline{h}_1 h_2} ST_{\xi_m} = T_{\overline{h}_1} T_f T_{h_2} = T_{\overline{h}_1 h_2 f}.$$

(c) Now for any continuous function h on S^n,

$$\begin{aligned} T_{\bar{\xi}_m}(T_{h\xi_m}S - ST_{h\xi_m}) &= T_{h|\xi_m|^2}S - T_{\bar{\xi}_m}ST_hT_{\xi_m} \\ &= (T_hS - T_{h(1-|\xi_m|^2)}S) \\ &\qquad - (T_{\bar{\xi}_m}T_hST_{\xi_m} + T_{\bar{\xi}_m}[S,T_h]T_{\xi_m}) \\ &= (T_hS - T_{\bar{\xi}_m}T_hST_{\xi_m}) \\ &\qquad - (T_{h(1-|\xi_m|^2)}S + T_{\bar{\xi}_m}[S,T_h]T_{\xi_m}). \end{aligned}$$

Since $\{|\xi_m|\}$ converges to 1 a.e.,

$$w\text{–}\lim_{m\to\infty} T_{h(1-|\xi_m|^2)}S = 0.$$

Since $[S,T_h]$ is compact and $T_{\bar{\xi}_m} \to 0$ weakly,

$$w\text{–}\lim_{m\to\infty} T_{\bar{\xi}_m}[S,T_h]T_{\xi_m} = 0.$$

Also from (b),

$$w\text{–}\lim_{m\to\infty} T_{\bar{\xi}_m}T_hST_{\xi_m} = T_{hf}.$$

So

$$w\text{–}\lim_{m\to\infty} T_{\bar{\xi}_m}(T_{h\xi_m}S - ST_{h\xi_m}) = T_hS - T_{hf}.$$

Since the norm is lower semi-continuous with respect to the weak operator topology, for any $\alpha < \|T_hS - T_{hf}\|$, there is an m_α so that

$$\|T_{\bar{\xi}_{m_\alpha}}(T_{h\xi_{m_\alpha}}S - ST_{h\xi_{m_\alpha}})\| > \alpha.$$

But $\|T_{\bar{\xi}_{m_\alpha}}(T_{h\xi_{m_\alpha}}S - ST_{h\xi_{m_\alpha}})\| \le \|T_{h\xi_{m_\alpha}}S - ST_{h\xi_{m_\alpha}}\|$, hence

$$\|T_{h\xi_{m_\alpha}}S - ST_{h\xi_{m_\alpha}}\| > \alpha. \qquad \square$$

Now we are in the position of proving Theorem 1.3.

Proof of theorem 1.3. Suppose S is a bounded operator on H^2 which is not the sum of a Toeplitz operator and a compact operator. We need to find a holomorphic function h so that $[T_h, S] = T_hS - ST_h$ is not compact. If there is a function $\varphi \in \mathcal{A}(B^n)$ so that $[T_\varphi, S]$ is not compact, take $h = \varphi$. Now suppose $[T_\varphi, S]$ is compact for all $\varphi \in \mathcal{A}(B^n)$. Let f be the function in Lemma 2.4. Let $F : L^\infty \to \mathfrak{B}(H^2)$ be defined by

$$F(g) = T_gS - T_{gf},\ g \in L^\infty.$$

Then F satisfies the properties $(P1)$ and $(P2)$ in Lemma 2.3, see [2]. Since S is not the sum of a Toeplitz operator and a compact operator, $F(1) = S - T_f$ is not compact. The last conclusion of Lemma 2.3 gives a sequence $\{\varphi_m\}_{m=1}^{\infty}$ of functions in $\mathcal{A}(B^n)$ and a sequence $\{\chi_m\}_{m=1}^{\infty}$ of mutually disjoint closed sets so that $\|\varphi_m\|_\infty < 2,\ |\varphi_m(1-\chi_m)| < 2^{-m}$, and

$$\|T_{\varphi_m}S - T_{\varphi_m f}\| = \|F(\varphi_m)\| > C(\alpha)/8 \text{ for all } m \in \mathbb{N},$$

where $\alpha = \|\pi(F(1))\|/2$. Furthermore, the sets $\{\chi_m\}$ cluster at only one point of S^n, say ζ.

For each k, apply part (c) of Lemma 2.4 to φ_k, we get a function $\vartheta_k = \varphi_k \xi_{m_k}$ in $\mathcal{A}(B^n)$ so that $\|\vartheta_k\|_\infty \leq 2,\ \|\vartheta_k(1-\chi_k)\|_\infty \leq 2^{-m}$, and $\|[T_{\vartheta_k}, S]\| > C(\alpha)/8$.

By Lemma 2.2, there is a function h in H^∞ such that h is continuous on $S^n \backslash \{\zeta\}$ and $[T_h, S]$ is not compact. The proof of the Theorem is thus completed. □

3. Local commutativity and essential commutant

In order to prove Theorem 1.4, we need a couple of results. The first Lemma gives a necessary condition for an operator to be in $\mathcal{K}_\zeta$.

Lemma 3.1. *Let ζ be in S^n and A be an element in $\mathcal{K}_\zeta$. Then for any $\epsilon > 0$, there is an open neighborhood V_ζ of ζ depending on A and ϵ so that for any continuous function η with* $\mathrm{supp}(\eta) \subset V_\zeta$ *and $\|\eta\|_\infty \leq 1$, we have $\|\pi(T_\eta A)\| < \epsilon$.*

Proof. Let $\epsilon > 0$ be given. There is an $\tilde{A}$ in the ideal of $\mathfrak{T}$ generated by $\{T_\varphi : \varphi \in C(S^n), \varphi(\zeta) = 0\}$ so that $\|A - \tilde{A}\| < \epsilon/2$. Now there are continuous functions $\varphi_1, \ldots, \varphi_m$ with $\varphi_1(\zeta) = \cdots = \varphi_m(\zeta) = 0$ and operators $B_1, \ldots, B_m$ and $C_1, \ldots, C_m$ in $\mathfrak{T}$ so that $\tilde{A} = \sum_{j=1}^m B_j T_{\varphi_j} C_j$.

Since each φ_j is continuous, the commutator $[B_j, T_{\varphi_j}]$ is compact, so $\tilde{A} - \sum_{j=1}^m T_{\varphi_j} B_j C_j \in \mathcal{K}$.

For any $\eta \in L^\infty(S^n)$ and $\|\eta\|_\infty \leq 1$, the semi-commutator $T_{\eta\varphi_j} - T_\eta T_{\varphi_j}$ is compact for each j, so we have

$$\begin{aligned}\|\pi(T_\eta A)\| &\leq \epsilon/2 + \|\pi(T_\eta \tilde{A})\| \\ &= \epsilon/2 + \|\pi(\sum_{j=1}^m T_\eta T_{\varphi_j} B_j C_j)\| \\ &= \epsilon/2 + \|\pi(\sum_{j=1}^m T_{\eta\varphi_j} B_j C_j)\| \\ &\leq \epsilon/2 + M\|\eta(\sum_{j=1}^m |\varphi_j|)\|_\infty,\end{aligned}$$

where M is a positive constant, depending on B_j's, C_j's and m. Since $\varphi = \sum_{j=1}^m |\varphi_j|$ is a continuous function on S^n with $\varphi(\zeta) = 0$, there is an open neighborhood V_ζ of ζ so that for all $\omega \in V_\zeta$, $0 \leq \varphi(\omega) \leq \epsilon(2M)^{-1}$.

So if η is any continuous function on S^n with $\operatorname{supp}(\eta) \subset V_\zeta$ and $\|\eta\|_\infty \leq 1$ then $\|\eta\varphi\|_\infty \leq \epsilon(2M)^{-1}$. Hence, $\|\pi(T_\eta A)\| \leq \epsilon$. □

The next Lemma asserts that some commutators of a certain kind cannot belong to $\mathcal{K}_\zeta$ unless they are compact.

Lemma 3.2. *Let $\zeta \in S^n$ be given. Suppose $f \in L^\infty$ and $g \in \mathcal{L}(\zeta)$. Then $[T_f, T_g]$ is in $\mathcal{K}_\zeta$ if and only if it is in $\mathcal{K}$.*

Proof. Because each $g \in \mathcal{L}(\zeta)$ is continuous on $S^n \backslash \{\zeta\}$, for any continuous function η on S^n with $\eta(\zeta) = 1$, we have $\pi(T_\eta[T_f, T_g]) = \pi([T_f, T_g])$.

Now if $[T_f, T_g]$ is in $\mathcal{K}_\zeta$, Lemma 3.1 together with the above identity show that $\|\pi([T_f, T_g])\| < \epsilon$ for all $\epsilon > 0$. So $[T_f, T_g] \in \mathcal{K}$.

The converse is obvious since $\mathcal{K}$ is contained in $\mathcal{K}_\zeta$. □

Lemma 3.3. *Suppose g is a function in L^∞ with $\|g\|_\infty \leq 1$. Suppose $\zeta \in S^n$ and $\{z_m\}$ in B^n so that $\lim_{m\to\infty} |z_m - \zeta| = 0$, hence $\lim_{m\to\infty} d(z_m, \zeta) = 0$ as well. Also suppose that $\|H_g k_{z_m}\| \geq a > 0$ for all $m \in \mathbb{N}$, where $0 < a < 1$ is a constant. Then there exists a function $h \in H^\infty$ which is continuous on $S^n \backslash \{\zeta\}$ so that $[T_h, T_g]$ is not compact.*

Proof. For each m, there is a $\delta_m > 0$ so that

$$\int_{\overline{Q}(\zeta,\delta_m)} |k_{z_m}|^2 d\sigma < a^4/9,$$

where $\overline{Q}(\zeta,\delta_m)=\{\omega\in S^n : d(\omega,\zeta)\le\delta_m\}$.

We may choose δ_m such that $\delta_m\to 0$ as $m\to\infty$.

For each m, let

$$\epsilon_m = d(z_m,\zeta)+3a^{-2}(1-|z_m|^2)^{1/4}.$$

Then $\epsilon_m\to 0$ as $m\to\infty$. So we can choose a subsequence $\{\epsilon_{m_l}\}_{l=1}^{\infty}$ so that $\epsilon_{m_{l+1}}<\epsilon_{m_l}$, and $\epsilon_{m_{l+1}}<\delta_{m_l}$ for all $l\in\mathbb{N}$.

Let

$$B_l=\{\omega\in S^n :\ \epsilon_{m_{l+1}}<d(\omega,\zeta)<\epsilon_{m_l}\}.$$

Then these are mutually disjoint open sets of S^n. For each l,

$$\begin{aligned}\int\limits_{S^n\setminus B_l}|gk_{z_{m_l}}|^2d\sigma &\le \int\limits_{\overline{Q}(\zeta,\epsilon_{m_{l+1}})}|k_{z_{m_l}}|^2d\sigma+\int\limits_{S^n\setminus Q(\zeta,\epsilon_{m_l})}|k_{z_{m_l}}|^2d\sigma\\ &\le a^4/9+\int\limits_{S^n\setminus Q(\zeta,\epsilon_{m_l})}|k_{z_{m_l}}|^2d\sigma.\end{aligned}\tag{3.1}$$

Now for $\omega\in S^n\setminus Q(\zeta,\epsilon_{m_l})$, we have

$$\begin{aligned}d(\omega,z_{m_l})&\ge d(\omega,\zeta)-d(z_{m_l},\zeta)\\ &\ge \epsilon_{m_l}-d(z_{m_l},\zeta)\\ &=3a^{-2}(1-|z_{m_l}|^2)^{1/4}.\end{aligned}$$

So

$$3a^{-2}(1-|z_{m_l}|^2)^{1/4}\le d(\omega,z_{m_l})=|1-\langle\omega,z_{m_l}\rangle|^{1/2}.$$

Hence,

$$|k_{z_{m_l}}(\omega)|^2=\frac{(1-|z_{m_l}|^2)^n}{|1-\langle\omega,z_{m_l}\rangle|^{2n}}\le(a^2/3)^{4n}\le a^4/9.$$

Thus,

$$\int\limits_{S^n\setminus Q(\zeta,\epsilon_{m_l})}|k_{z_{m_l}}|^2d\sigma\le a^4/9.\tag{3.2}$$

Inequalities (3.1) and (3.2) give

$$\int\limits_{S^n\setminus B_l}|gk_{z_{m_l}}|^2d\sigma\le a^4/9+a^4/9<4a^4/9.$$

So

$$\|\chi_{S^n\backslash B_l} g k_{z_{m_l}}\| < 2a^2/3. \tag{3.3}$$

Now, since $\|H_g k_{z_{m_l}}\| \geq a$, we have

$$\left\langle H_g k_{z_{m_l}}, H_g k_{z_{m_l}} \right\rangle \geq a^2.$$

On the other hand,

$$\begin{aligned} |\langle H_g k_{z_{m_l}}, H_{g\chi_{S^n\backslash B_l}} k_{z_{m_l}}\rangle| &\leq \|H_{g\chi_{S^n\backslash B_l}} k_{z_{m_l}}\| \\ &\leq \|g\chi_{S^n\backslash B_l} k_{zm_l}\| \\ &< 2a^2/3 \quad \text{ by (3.3).} \end{aligned}$$

So

$$|\langle H_g k_{z_{m_l}}, H_{g\chi_{B_l}} k_{z_{m_l}}\rangle| > a^2 - 2a^2/3 = a^2/3.$$

Now write $g = (f^{(1)} - f^{(2)}) + i(f^{(3)} - f^{(4)})$, where $0 \leq f^{(j)} \leq 1$ for $1 \leq j \leq 4$. For each l, the above inequality shows that there is a $j = j(l) \in \{1, 2, 3, 4\}$ so that $|\langle H_g k_{z_{m_l}}, H_{f^{(j)}\chi_{B_l}} k_{z_{m_l}}\rangle| > a^2/12$. By approximating $f^{(j)}\chi_{B_l}$ almost everywhere on S^n by continuous functions with compact supports in B_l, we can find a continuous function ϕ_l so that $0 \leq \phi_l \leq 1$, $\text{supp}(\phi_l) \subset B_l$ and

$$|\langle H_g k_{z_{m_l}}, H_{\phi_l} k_{z_{m_l}}\rangle| > a^2/12.$$

Let α be any positive number less than $a^2/24$. For each $l \in \mathbb{N}$, put $\eta_l = \max\{\phi_l, \alpha^l\}$.

Then η_l is continuous, $\alpha^l \leq \eta_l \leq 1 + \alpha^l$, $\eta_l(w) = \alpha^l$ for $w \in S^n\backslash B_l$ and $\|\eta_l - \phi_l\|_\infty = \alpha^l$.

So

$$\begin{aligned} |\langle H_g k_{z_{m_l}}, H_{\eta_l - \phi_l} k_{z_{m_l}}\rangle| &\leq \|H_{\eta_l - \phi_l} k_{z_{m_l}}\| \\ &\leq \|\eta_l - \phi_l\|_\infty \\ &= \alpha^l. \end{aligned}$$

Thus, for $l \in \mathbb{N}$,

$$|\langle H_g k_{z_{m_l}}, H_{\eta_l} k_{z_{m_l}}\rangle| > a^2/12 - \alpha^l > a^2/12 - a^2/24 = a^2/24.$$

From Lemma 2.1, for each l there is a function $\vartheta_l \in \mathcal{A}(B^n)$ so that

(a) $|\vartheta_l(\omega)|^2 \leq \eta_l(\omega)$ for $\omega \in S^n$, and
(b) $|\langle H_g k_{z_{m_l}}, H_{|\vartheta_l|^2} k_{z_{m_l}}\rangle| > a^2/24.$

Property (a) implies that $\|\vartheta_l\|_\infty \le 2$, and $|\vartheta_l(1-\chi_{B_l})| \le (\sqrt{\alpha})^l$. Property (b) implies that

$$\|H^*_{|\vartheta_l|^2}H_g\| > a^2/24.$$

Now

$$\begin{aligned} H^*_{|\vartheta_l|^2}H_g &= T_{|\vartheta_l|^2 g} - T_{|\vartheta_l|^2}T_g \\ &= T_{\overline{\vartheta}_l}T_gT_{\vartheta_l} - T_{\overline{\vartheta}_l}T_{\vartheta_l}T_g \\ &= -T_{\overline{\vartheta}_l}[T_{\vartheta_l}, T_g]. \end{aligned}$$

Since $\|T_{\overline{\vartheta}_l}\| \le 2$, we get

$$\|[T_{\vartheta_l}, T_g]\| > a^2/48.$$

If there is an l so that $[T_{\vartheta_l}, T_g]$ is not compact, let $h = \vartheta_l$. Otherwise, apply Lemma 2.2, we get a function $h \in H^\infty$ so that h is continuous on $S^n\backslash\{\zeta\}$ and

$$\|\pi([T_h, T_g])\| \ge a^2/48. \qquad \square$$

Before going on to the proof of Theorem 1.4, we need one more Lemma about functions in $\mathcal{A}(B^n)$. This is a kind of Urysohn's Lemma for $\mathcal{A}(B^n)$.

Lemma 3.4. *Suppose $\zeta \in S^n$ and $0 < \delta,\ \epsilon < 1$ are given. Then there exists a function $\eta \in \mathcal{A}(B^n)$ so that $\eta(\zeta) = \|\eta\|_\infty = 1$ and $|\eta(\omega)| \le \epsilon$ for $\omega \in S^n$ with $d(\omega, \zeta) \ge \delta$.*

Proof. Let φ be any continuous function on S^n so that $0 < \varphi \le 2$ and for any $\omega \in S^n$, $\varphi(\omega) = 2$ if $d(\omega,\zeta) \le \delta/4$ and $\varphi(\omega) \le \epsilon$ if $d(\omega,\zeta) \ge \delta/2$.

By Lemma 2.1, there are functions $\{\varphi_m\}_{m=1}^\infty \subset \mathcal{A}(B^n)$ such that

$$|\varphi_m| < \varphi \text{ on } S^n,$$
$$\text{and } |\varphi_m| \to \varphi \text{ a.e. } \sigma.$$

So there exist $m_0 \in \mathbb{N}$ so that $\|\varphi_{m_0}\|_\infty \ge 1$. Let $\tilde{\zeta} \in S^n$ such that $|\varphi_{m_0}(\tilde{\zeta})| = \|\varphi_{m_0}\|_\infty$. Then $d(\tilde{\zeta}, \zeta) \le \delta/2$.

Put $\vartheta = \dfrac{\varphi_{m_0}}{\varphi_{m_0}(\tilde{\zeta})}$. Then $\vartheta(\tilde{\zeta}) = \|\vartheta\|_\infty = 1$ and $|\vartheta(\omega)| \le \dfrac{\varphi(\omega)}{\|\varphi_{m_0}\|_\infty} \le \epsilon$ if $d(\omega,\zeta) \ge \delta/2$.

Now take U to be any rotation on S^n so that $U\zeta = \tilde{\zeta}$. Put $\eta = \vartheta \circ U$.

Then $\eta \in \mathcal{A}(B^n)$, $\|\eta\|_\infty = 1$, $\eta(\zeta) = \vartheta(U\zeta) = \vartheta(\tilde{\zeta}) = 1$, and for any $\omega \in S^n$ with $d(\omega, \zeta) \geq \delta$,

$$\begin{aligned} d(U\omega, \zeta) &= d(\omega, U^{-1}\zeta) \\ &\geq d(\omega, \zeta) - d(\zeta, U^{-1}\zeta) \\ &= d(\omega, \zeta) - d(\zeta, \tilde{\zeta}) \\ &\geq \delta - \delta/2 \\ &= \delta/2, \end{aligned}$$

so $|\eta(\omega)| = |\vartheta(U\omega)| < \epsilon$. □

Proof of Theorem 1.4. (a) If it were not true that $\lim\limits_{\substack{|z|<1 \\ z\to\zeta}} \|H_f k_z\| = 0$ then there would be a sequence $\{z_m\}$ of points in B^n and a constant $a > 0$ such that $\lim\limits_{m\to\infty} |z_m - \zeta| = 0$ and $\|H_f k_{z_m}\| \geq a$ for all m. By Lemma 3.3, there is a function $h \in \mathcal{H}(\zeta)$ so that $[T_f, T_h]$ is not compact. Lemma 3.2 then implies that $[T_f, T_h]$ is not in $\mathcal{K}_\zeta$ either, which is a contradiction.

(b) Now suppose $\lim\limits_{\substack{|z|<1 \\ z\to\zeta}} \|H_f k_z\| \|H_{\overline{g}} k_z\| = 0$. Let $\epsilon > 0$ be given. There is a $\delta > 0$ so that for all $z \in B^n$ with $d(z, \zeta) < 3\delta$,

$$\|H_f k_z\| \|H_{\overline{g}} k_z\| < \epsilon. \tag{3.4}$$

By Lemma 3.4, there is a function $\eta \in \mathcal{A}(B^n)$ with $\|\eta\|_\infty = \eta(\zeta) = 1$ and $|\eta(w)| < \epsilon' = \dfrac{\epsilon}{\|\overline{g}\|_\infty \|f\|_\infty + 1}$ if $w \in S^n$ with $d(w, \zeta) \geq \delta$.

We claim that

$$\limsup_{|z|\uparrow 1} \|H_{f\eta} k_z\| \|H_{\overline{g}} k_z\| \leq \epsilon. \tag{3.5}$$

To prove this, it suffices to show that for any $\tilde{\omega} \in S^n$,

$$\limsup_{z\to\tilde{\omega}} \|H_{f\eta} k_z\| \|H_{\overline{g}} k_z\| \leq \epsilon.$$

We have

$$\begin{aligned} H_{f\eta} k_z - \eta H_f k_z &= (f\eta k_z - T_{f\eta} k_z) - \eta(f k_z - T_f k_z) \\ &= \eta T_f k_z - T_{f\eta} k_z \\ &= (T_\eta T_f - T_{f\eta}) k_z \text{ (since } \eta \text{ is holomorphic)} \\ &= -H_{\overline{\eta}}^* H_f k_z. \end{aligned}$$

So

$$\|H_{f\eta}k_z - \eta H_f k_z\| = \|H_{\overline{\eta}}^* H_f k_z\|.$$

Since $H_{\overline{\eta}}^*$ is compact because η is continuous and $k_z \to 0$ weakly as $|z| \to 1$, we have $\lim_{|z|\to 1} \|H_{\overline{\eta}}^* H_f k_z\| = 0$.

So for any $\tilde{\omega} \in S^n$ with $d(\tilde{\omega}, \zeta) < 3\delta$,

$$\limsup_{\substack{|z|<1 \\ z\to\tilde{\omega}}} \|H_{f\eta}k_z\|\|H_g k_z\| = \limsup_{\substack{|z|<1 \\ z\to\tilde{\omega}}} \|\eta H_f k_z\|\|H_{\overline{g}} k_z\| \le \epsilon,$$

by $\|\eta\|_\infty = 1$ and (3.4).

Now fix an $\tilde{\omega} \in S^n$ with $d(\tilde{\omega}, \zeta) \ge 2\delta$. For any $z \in B^n$,

$$\begin{aligned} \|f\eta k_z\|_2 &\le \big\{\epsilon'\|\chi_{S^n\setminus Q(\zeta,\delta)}k_z\|_2 + \|\chi_{Q(\zeta,\delta)}k_z\|_2\big\}\|f\|_\infty \\ &\le \big\{\epsilon' + \|\chi_{Q(\zeta,\delta)}k_z\|_2\big\}\|f\|_\infty. \end{aligned}$$

For $w \in S^n$ with $d(w, \zeta) < \delta$ and for $z \in B^n$ with $d(z, \tilde{\omega}) < \delta/2$, we have

$$\begin{aligned} d(w, z) &\ge d(\tilde{\omega}, \zeta) - d(w, \zeta) - d(z, \tilde{\omega}) \\ &\ge 2\delta - \delta - \delta/2 = \delta/2. \end{aligned}$$

Thus the function $\chi_{Q(\zeta,\delta)}k_z$ is bounded by $(2/\delta)^{2n}$, which is independent of z when $z \to \tilde{w}$. On the other hand, $k_z \to 0$ a.e. as $z \to \tilde{\omega}$. So by the Lebesgue Dominated Convergence Theorem, $\lim_{z\to\tilde{\omega}} \|\chi_{Q(\zeta,\delta)}k_z\|_2 = 0$.

It then follows that

$$\limsup_{\substack{|z|<1 \\ z\to\tilde{\omega}}} \|f\eta k_z\|_2 \le \|f\|_\infty \epsilon'.$$

Hence

$$\limsup_{\substack{|z|<1 \\ z\to\tilde{\omega}}} \|H_{f\eta}k_z\| \le \|f\|_\infty \epsilon',$$

and so

$$\limsup_{\substack{|z|<1 \\ z\to\tilde{\omega}}} \|H_{f\eta}k_z\|\|H_{\overline{g}}k_z\| \le \|\overline{g}\|_\infty \|f\|_\infty \epsilon' \le \epsilon.$$

Therefore, (3.5) has been proved. From the proof of [13, Theorem 3], for any $\gamma > 0$ there is a constant $C_\gamma > 0$ so that

$$\|\pi(H_{\overline{g}}^* H_{f\eta})\| \le C(C_\gamma \epsilon + \gamma), \tag{3.6}$$

where C depends only on the dimension.

Now

$$\begin{aligned} H_{\bar g}^* H_{f\eta} &= T_{gf\eta} - T_g T_{f\eta} \\ &= T_{gf} T_\eta - T_g T_f T_\eta \quad \text{(since } \eta \text{ is analytic)} \\ &= (T_{gf} - T_g T_f) T_\eta \\ &= (T_{gf} - T_g T_f) + (T_{gf} - T_g T_f) T_{\eta - 1}. \end{aligned}$$

Since $\eta - 1$ is continuous on S^n and $\eta(\zeta) - 1 = 0$, the second term is in $\mathcal{K}_\zeta$. Let π_ζ denote the canonical map from $\mathfrak{T}$ onto $\mathfrak{T}/\mathcal{K}_\zeta$. Inequality (3.6) then gives

$$\begin{aligned} \|\pi_\zeta(T_{fg} - T_g T_f)\| &= \|\pi_\zeta(H_{\bar g}^* H_{f\eta})\| \\ &\leq \|\pi(H_{\bar g}^* H_{f\eta})\| \\ &\leq C(C_\gamma \epsilon + \gamma). \end{aligned}$$

Since ϵ and γ were arbitrary and C depends only on the dimension, we conclude that $\pi_\zeta(T_{gf} - T_g T_f) = 0$. □

Proof of Corollary 1.1. The first conclusion follows directly from Theorem 1.4 (a). Now suppose S is a bounded operator that essentially commutes with all Toeplitz operators $T_h,\ h \in \mathcal{H}$. Theorem 1.3 implies that $S = T_f + K$ where $f \in L^\infty$ and K is a compact operator. Now T_f essentially commutes with all $T_h,\ h \in \mathcal{H}$. Hence $\lim_{|z|\uparrow 1} \|H_f k_z\| = 0$. By [13, Theorem 5], H_f is compact. So $\mathrm{EssCom}(\mathfrak{T}(\mathcal{H})) = \mathfrak{T}(\mathcal{C})$ where $\mathcal{C} = \{f \in L^\infty : H_f \text{ is compact}\}$.

□

Proof of Corollary 1.2. Suppose S is a bounded operator that essentially commutes with all Toeplitz operators T_g, where $g \in \mathcal{L}$. Then in particular, S essentially commutes with all T_h, where $h \in \mathcal{H}$. Theorem 1.3 implies that $S = T_f + K$, where K is a compact operator. Now T_f essentially commutes with all T_g, where $g \in \mathcal{L}$. Since $\mathcal{L}$ is $*-$symmetric, $T_{\bar f} = T_f^*$ also essentially commutes with all $T_g,\ g \in \mathcal{L}$. As in the proof of Corollary 1.1, both H_f and $H_{\bar f}$ are compact. Hence $f \in \mathrm{VMO}$, see [3, p. 365] or [10, p. 465]. □

4. Essential Commutant of a Separable Subset of $\mathfrak{T}$

In this section, we will prove Theorem 1.5 by constructing a real valued function f that satisfies all the requirements. We begin with the following Lemma.

Lemma 4.1. *Let $0 < a < 1$ and let η be a continuous function on $[a, 1]$ with $\eta(a) = \eta(1)$. Define $\varphi : [0, 1] \longrightarrow \mathbb{R}$ by $\varphi(0) = 0$ and $\varphi(t) = \eta(a^{-m}t)$ if $a^{m+1} < t \leq a^m$. Then φ is continuous on $(0, 1]$ and it satisfies the following properties:*

(a) *For any $m \geq 0$ and $0 < t \leq a^m$, we have $\varphi(t) = \varphi(a^{-m}t)$.*
(b) *For any $\epsilon > 0$, there is a $\delta > 0$ so that for any s, t in $(0, 1]$ with $|1 - s^{-1}t| \leq \delta$, we have $|\varphi(s) - \varphi(t)| \leq \epsilon$.*
(c) *Suppose $n \geq 2$. For any complex number α there is a constant c depending on η and α so that for any $0 < r < 1$,*

$$\int\limits_{\substack{|1-z|\leq r^2 \\ |z|<1}} (1 - |z|^2)^{n-2} \big|\varphi(|1 - z|^{1/2}) - \alpha\big| \; dA(z) \geq cr^{2n}, \tag{4.1}$$

where $dA(z) = \pi^{-1}dxdy$.

Proof. (a) Suppose $a^{s+1} < a^{-m}t \leq a^s$ for some $s \geq 0$. Then $a^{s+m+1} < t \leq a^{s+m}$. So by the definition of φ, we have

$$\varphi(t) = \eta(a^{-s-m}t) = \eta(a^{-s}(a^{-m}t)) = \varphi(a^{-m}t).$$

(b) Let $\epsilon > 0$ be given. Since φ is uniformly continuous on $[a^2, 1]$, there is a $\delta_0 > 0$ so that $|\varphi(u) - \varphi(v)| < \epsilon$ for all $u, v \in [a^2, 1]$ with $|u - v| < \delta_0$. This δ_0 depends only on η, a and ϵ. Let $\delta = \min\{\delta_0, 1 - a\}$.

For $s, t \in (0, 1]$ with $|1 - s^{-1}t| < \delta$, we have $1 - \delta < s^{-1}t < 1 + \delta$. But $a < 1 - \delta$ and $1 + \delta < 2 - a \leq a^{-1}$, so $a < s^{-1}t < a^{-1}$. This implies that there is an $m \geq 0$ so that $s, t \in (a^{m+2}, a^m]$.

Now $a^{-m}s, a^{-m}t \in (a^2, 1]$ and

$$\begin{aligned} |a^{-m}s - a^{-m}t| &= a^{-m}s|1 - s^{-1}t| \\ &|1 - s^{-1}t| \quad (\text{since } s \leq a^m) \\ &< \delta_0. \end{aligned}$$

Hence $|\varphi(a^{-m}s) - \varphi(a^{-m}t)| < \epsilon$. By (a), we have

$$|\varphi(s) - \varphi(t)| = |\varphi(a^{-m}s) - \varphi(a^{-m}t)| < \epsilon.$$

(c) By change of variable $u = 1 - z$, the integral on the left hand side of (4.1) becomes

$$\begin{aligned}
&\int\limits_{\substack{|u|\le r^2\\ |1-u|\le 1}} \big(1-|1-u|^2\big)^{n-2}\big|\varphi(|u|^{1/2})-\alpha\big|\; dA(u)\\
&\qquad = \pi^{-1}\int\limits_{\substack{0\le t\le r^2\\ |1-te^{i\theta}|\le 1}} \big(1-|1-te^{i\theta}|^2\big)^{n-2}\big|\varphi(|t|^{1/2})-\alpha\big| t\; dtd\theta\\
&\qquad = \pi^{-1}\int\limits_{\substack{0\le t\le r^2\\ 0\le 2\cos\theta - t}} \big(2t\cos\theta - t^2\big)^{n-2}\big|\varphi(|t|^{1/2})-\alpha\big| t\; dtd\theta.
\end{aligned}$$

For $|\theta| \le \dfrac{\pi}{4}$ and $0 \le t \le 1$ we have

$$\begin{aligned}
2t\cos\theta - t^2 &\ge \sqrt{2}t - t^2\\
&= (t-t^2) + (\sqrt{2}-1)t\\
&\ge (\sqrt{2}-1)t.
\end{aligned}$$

Hence the above integral is not less than

$$\begin{aligned}
&\pi^{-1}\int\limits_0^{r^2}\int\limits_{-\pi/4}^{\pi/4} (\sqrt{2}-1)^{n-2}t^{n-2}\big|\varphi(t^{1/2})-\alpha\big| t\; d\theta dt\\
&\qquad = 2^{-1}(\sqrt{2}-1)^{n-2}\int\limits_0^{r^2} t^{n-1}\big|\varphi(t^{1/2})-\alpha\big|\; dt\\
&\qquad = (\sqrt{2}-1)^{n-2}\int\limits_0^{r} s^{2n-1}|\varphi(s)-\alpha|\; ds
\end{aligned}$$

Now choose an $m \ge 0$ so that $a^{m+1} < r \le a^m$. Then

$$\begin{aligned}
\int\limits_0^r s^{2n-1}|\varphi(s)-\alpha|\; ds &\ge \int\limits_0^{a^{m+1}} s^{2n-1}|\varphi(s)-\alpha|\; ds\\
&= \sum_{k=m+1}^{\infty}\int\limits_{a^{k+1}}^{a^k} s^{2n-1}|\varphi(s)-\alpha|\; ds
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=m+1}^{\infty} a^{2nk} \int_a^1 v^{2n-1} |\varphi(v) - \alpha| \, dv \\
&= \frac{a^{2n(m+1)}}{1 - a^{2n}} \int_a^1 v^{2n-1} |\varphi(v) - \alpha| \, dv \\
&\geq \frac{a^{2n} r^{2n}}{1 - a^{2n}} \int_a^1 v^{2n-1} |\varphi(v) - \alpha| \, dv \\
&= \frac{a^{2n} r^{2n}}{1 - a^{2n}} \int_a^1 v^{2n-1} |\eta(v) - \alpha| \, dv.
\end{aligned}
$$

Let $c = (\sqrt{2} - 1)^{n-2} \dfrac{a^{2n}}{1 - a^{2n}} \displaystyle\int_a^1 v^{2n-1} |\eta(v) - \alpha| \, dv$. We then have the inequality (4.1). □

For each function $f \in L^1(S^n)$, we use f to denote the harmonic extension of f in B^n. So for any $z \in B^n$,

$$
f(z) = \int_{S^n} f(w) P_z(w) \, d\sigma(w),
$$

where $P_z(w) = \dfrac{(1 - |z|^2)^n}{|1 - \langle w, z \rangle|^{2n}}$ is the Poisson's kernel.

For for $\zeta \in S^n$, any $f, g \in \mathrm{BMO}$ and $1 \leq p < \infty$, define

$$
\begin{aligned}
P_p(f,g)(\zeta) = \lim_{\delta \downarrow 0} \sup \Big\{ \big(\int_{S^n} |f - f(z)|^p P_z \, d\sigma \int_{S^n} |g - g(z)|^p P_z \, d\sigma \big)^{1/p} : \\
|z| < 1, d(z, \zeta) < \delta \Big\},
\end{aligned}
$$

$$
\begin{aligned}
M_p(f,g)(\zeta) = \lim_{\delta \downarrow 0} \sup \Big\{ \big(&\frac{1}{\sigma(Q(z,r))} \int_{Q(z,r)} |f - f_{Q(z,r)}|^p \, d\sigma \\
&\times \frac{1}{\sigma(Q(z,r))} \int_{Q(z,r)} |g - g_{Q(z,r)}|^p \, d\sigma \big)^{1/p} : Q(z,r) \subset Q(\zeta, \delta) \Big\}.
\end{aligned}
$$

Then $\mathrm{LMO}(f,g)(\zeta) = M_1(f,g)(\zeta)$ is called the *joint local mean oscillation* of f and g at ζ.

It was proved in [12] that for any $f, g \in \mathrm{BMO}$, any $\zeta \in S^1$ and any $1 \le p < \infty$, $M_p(f,g)(\zeta) = 0$ if and only if $P_p(f,g)(\zeta) = 0$. The proof can be carried over to the case $n \ge 2$ with only some minor changes. In particular, for any $f, g \in L^\infty$ and $\zeta \in S^n$, $\mathrm{LMO}(f,g)(\zeta) = 0$ implies that $P_1(f,g)(\zeta) = 0$. It then gives $P_2(f,g)(\zeta) = 0$ because f, g are bounded.

For $f \in L^2$ and $z \in B^n$, define

$$\mathrm{Var}(f;z) = \int\limits_{S^n} |f - \int\limits_{S^n} f P_z \, d\sigma|^2 P_z \, d\sigma = \int\limits_{S^n} |f - f(z)|^2 P_z \, d\sigma.$$

Then for any $f, g \in L^\infty$ and any $\zeta \in S^n$, we have

$$\begin{aligned} \big(P_2(f,g)(\zeta)\big)^2 &= \limsup_{\delta \downarrow 0} \big\{ \mathrm{Var}(f;z)\,\mathrm{Var}(g;z) : |z| < 1, d(z,\zeta) < \delta \big\} \\ &= \limsup_{\substack{|z|<1 \\ z \to \zeta}} \ \mathrm{Var}(f;z)\,\mathrm{Var}(g;z) \\ &\ge \limsup_{\substack{|z|<1 \\ z \to \zeta}} \ \|H_f k_z\|^2 \|H_{\overline{g}} k_z\|^2, \end{aligned} \tag{4.2}$$

since $\|H_f k_z\|^2 \le \mathrm{Var}(f;z)$ and $\|H_{\overline{g}} k_z\|^2 \le \mathrm{Var}(g;z)$, see [10, Inequality 6.4].

Inequality (4.2) the above remark imply that for any $f, g \in L^\infty$ and any $\zeta \in S^n$, if $\mathrm{LMO}(f,g)(\zeta) = 0$ then $\lim\limits_{\substack{|z|<1 \\ z \to \zeta}} \|H_f k_z\| \|H_{\overline{g}} k_z\| = 0$.

Now fix $0 < a < 1$. Choose a function η as in the hypothesis of Lemma 4.1 so that $\int\limits_a^1 t^{2n-1} |\eta(t) - \alpha| \, dt \ge c_0 > 0$ for all α, where c_0 is a constant independent of α. Any continuous function η on $[a,1]$ with $\eta(a) = \eta(1) = 0$, $\eta(t) = 1$ if $a_1 < t < a_2$, $\eta(t) = 0$ if $a_3 < t < a_4$, where $a < a_1 < a_2 < a_3 < a_4$ will satisfy the requirements. Let φ be as in Lemma 4.1. Let ζ be a point on S^n. Define $f(w) = \varphi(d(w,\zeta))$ for all $w \in S^n$. Then f is a continuous function on $S^n \backslash \{\zeta\}$. The following proposition gives more properties of the function f.

Proposition 4.1. *Let f be as in the preceding paragraph. Then the followings hold true.*

(a) $\mathrm{LMO}(f)(\zeta) > 0$.

(b) *For any function $g \in L^\infty$ so that ζ is a Lebesgue point of g, we have* $\mathrm{LMO}(f,g)(\zeta) = 0$. *Therefore,* $\lim_{\substack{|z|<1 \\ z\to\zeta}} \|H_f k_z\|\|H_{\overline{g}} k_z\| = 0$.

Proof. (a) Let $0 < r < 1$ be given. Put $\alpha_r = \frac{1}{\sigma(Q(\zeta,r))} \int_{Q(\zeta,r)} f \, d\sigma$.

We have

$$\begin{aligned}
&\int_{Q(\zeta,r)} |f - f_{Q(\zeta,r)}| \, d\sigma \\
&= \int_{Q(\zeta,r)} \left|\varphi(|1 - \langle w,\zeta\rangle|^{1/2}) - \alpha_r\right| \, d\sigma(w) \\
&= (n-1) \int_{\substack{|u|\le 1 \\ |1-u|^{1/2}\le r}} (1-|u|^2)^{n-2} |\varphi(|1-u|^{1/2}) - \alpha_r| \, dA(u) \text{ (see [9])} \\
&\ge (n-1)(\sqrt{2}-1)^{n-2} \frac{a^{2n}}{1-a^{2n}} c_0 r^{2n} \text{ (from Lemma 4.1)}
\end{aligned}$$

Recall that there is a constant A_0 so that $\sigma(Q(\zeta,r)) \le A_0 r^{2n}$. Hence for all $0 < r < 1$,

$$\frac{1}{\sigma(Q(\zeta,r))} \int_{Q(\zeta,r)} |f - f_{Q(\zeta,r)}| \, d\sigma \ge (n-1)(\sqrt{2}-1)^{n-2} \frac{a^{2n}}{1-a^{2n}} \frac{c_0}{A_0}.$$

This implies that $\mathrm{LMO}(f)(\zeta) > 0$.

(b) Now let g be a function in L^∞ so that τ is a Lebesgue point of g.

Let $\epsilon > 0$ be given. By Lemma 4.1(b), there is an $M > 0$ so that $|\varphi(s) - \varphi(t)| < \epsilon$ for all $s,t \in (0,1]$ with $|1 - s^{-1}t| < M^{-1}$.

Since ζ is a Lebesgue point of g, there is a $\delta_0 > 0$ so that for all $0 < r < \delta_0$,

$$\frac{1}{\sigma(Q(\zeta,r))} \int_{Q(\zeta,r)} |g - g(\zeta)| \, d\sigma < (M+2)^{-2n}\epsilon. \tag{4.3}$$

Now for all $0 < r < (M+2)^{-1}\delta_0$, we have

$$\frac{1}{\sigma(Q(\zeta,r))} \int\limits_{Q(\zeta,(M+2)r)} |g - g(\zeta)|\, d\sigma \leq \frac{\sigma(Q(\zeta,(M+2)r))}{\sigma(Q(\zeta,r))}(M+2)^{-2n}\epsilon.$$

Since $\sigma(Q(\zeta,r)) \geq 2^{-n}r^{2n}$ and $\sigma(Q(\zeta,(M+2)r)) \leq A_0(M+2)^{2n}r^{2n}$, the above inequality implies

$$\frac{1}{\sigma(Q(\zeta,r))} \int\limits_{Q(\zeta,(M+2)r)} |g - g(\zeta)|\, d\sigma \leq 2^n A_0 \epsilon. \tag{4.4}$$

Let $Q(z,r)$ be any ball of radius r centered at z that is contained in $Q(\zeta,(M+2)^{-1}\delta_0)$. Then $r \leq (M+2)^{-1}\delta_0$.

There are two cases. First, suppose $Q(z,r) \cap Q(\zeta, Mr) \neq \emptyset$. It then follows that $Q(z,r) \subset Q(\zeta,(M+2)r)$. Therefore,

$$\begin{aligned}\frac{1}{\sigma(Q(z,r))} \int\limits_{Q(z,r)} |g - g_{Q(z,r)}|\, d\sigma &\leq \frac{2}{\sigma(Q(z,r))} \int\limits_{Q(z,r)} |g - g(\zeta)|\, d\sigma \\ &\leq \frac{2}{\sigma(Q(z,r))} \int\limits_{Q(\zeta,(M+2)r)} |g - g(\zeta)|\, d\sigma \\ &\leq 2^{n+1}A_0\epsilon \qquad \text{(because of (4.4))}.\end{aligned}$$

Here we have used $\dfrac{1}{\sigma(E)} \displaystyle\int\limits_E |g - g_E|\, d\sigma \leq \dfrac{2}{\sigma(E)} \displaystyle\int\limits_E |g - \alpha|\, d\sigma$ for all α, in the first inequality.

Hence

$$\frac{1}{\sigma(Q(z,r))} \int\limits_{Q(z,r)} |f - f_{Q(z,r)}|\, d\sigma \frac{1}{\sigma(Q(z,r))} \int\limits_{Q(z,r)} |g - g_{Q(z,r)}|\, d\sigma \leq \|f\|_\infty 2^n A_0 \epsilon. \tag{4.5}$$

Second, suppose $Q(z,r) \cap Q(\zeta, Mr) = \emptyset$. For any $w \in Q(z,r)$ we have $d(w,\zeta) \geq Mr$. This gives

$$\begin{aligned} \frac{|d(w,\zeta) - d(z,\zeta)|}{d(w,\zeta)} &\leq \frac{d(w,z)}{d(w\zeta)} \\ &< r(Mr)^{-1} \\ &= M^{-1}. \end{aligned}$$

By our choice of M, we then have

$$|f(w) - f(z)| = |\varphi(d(w,\zeta)) - \varphi(d(z,\zeta))| < \epsilon.$$

Therefore

$$\begin{aligned} \frac{1}{\sigma(Q(z,r))} \int\limits_{Q(z,r)} |f - f_{Q(z,r)}| \, d\sigma &\leq \frac{2}{\sigma(Q(z,r))} \int\limits_{Q(z,r)} |f - f(z)| \, d\sigma \\ &< 2\epsilon. \end{aligned}$$

Hence for this case,

(4.6)
$$\frac{1}{\sigma(Q(z,r))} \int\limits_{Q(z,r)} |f - f_{Q(z,r)}| \, d\sigma \frac{1}{\sigma(Q(z,r))} \int\limits_{Q(z,r)} |g - g_{Q(z,r)}| \, d\sigma \leq 2\|g\|_\infty \epsilon.$$

Inequalities (4.5) and (4.6) give

$$\begin{aligned} \frac{1}{\sigma(Q(z,r))} \int\limits_{Q(z,r)} |f - f_{Q(z,r)}| \, d\sigma \frac{1}{\sigma(Q(z,r))} \int\limits_{Q(z,r)} |g - g_{Q(z,r)}| \, d\sigma & \\ \leq \max\{2^{n+1} A_0 \|f\|_\infty, 2\|g\|_\infty\}\epsilon&, \end{aligned}$$

for any $Q(z,r) \subset Q(\zeta, (M+2)^{-1}\delta_0)$.

Thus, $\mathrm{LMO}(f,g)(\zeta) = 0$. □

Proof of Theorem 1.5. The separability of $\mathcal{S}$ implies that $\mathcal{S}$ is in the operator norm closure of a countable subset $\{A_1, A_2, \ldots\}$ of $\mathfrak{T}$. Now each A_j is the limit in the operator norm of a sequence of operators of the form $\sum_{k=1}^{M} T_{g_{k1}} \cdots T_{g_{kM}}$, where $g_{kl} \in L^\infty$. Hence we can find a countable set $G = \{g_1, g_2, \cdots\}$ of real-valued functions in L^∞ so that $\mathcal{S}$ is contained in $\mathfrak{T}(G)$.

For each g_j, all most every point of S^n is a Lebesgue point. Therefore there is a $\zeta \in S^n$ which is a Lebesgue point for *all* functions g_j, for $j = 1, 2 \ldots$. For this ζ, let f be the function in Proposition (4.1). Since f is continuous on $S^n \backslash \{\zeta\}$, $\mathrm{LMO}(f, g_j)(w) = 0$ for all $w \in S^n \backslash \{\zeta\}$. So

$\lim\limits_{\substack{|z|<1 \\ z\to w}} \|H_f k_z\| \|H_{\overline{g}_j} k_z\| = 0$ for all $w \in S^n \backslash \{\zeta\}$. Proposition 4.1 also gives $\lim\limits_{\substack{|z|<1 \\ z\to \zeta}} \|H_f k_z\| \|H_{\overline{g}_j} k_z\| = 0$. By Theorem 1.4 (b), $T_{g_j f} - T_{g_j} T_f \in K_w$ for all $w \in S^n$. Thus, $T_{g_j f} - T_{g_j} T_f$ is compact. Since both f and g_j are real valued-functions, it then follows that $[T_f, T_{g_j}]$ is compact for all j. Hence $[T_f, S]$ is compact for all $S \in \mathfrak{T}(G)$. Proposition 4.1 also yields that $\mathrm{LMO}(f)(\zeta) > 0$, hence $f \notin \mathrm{VMO}$. □

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