FINITE-RANK PRODUCTS OF TOEPLITZ OPERATORS IN SEVERAL COMPLEX VARIABLES

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ABSTRACT. For any $\alpha > -1$, let A^2_α be the weighted Bergman space on the unit ball corresponding to the weight $(1-|\mathbf{z}|^2)^\alpha$. We show that if all except possibly one of the Toeplitz operators $T_{f_1}, \ldots, T_{f_r}$ are diagonal with respect to the standard orthonormal basis of A^2_α and $T_{f_1} \cdots T_{f_r}$ has finite rank then one of the functions $f_1, \ldots, f_r$ must be the zero function.

1. INTRODUCTION

As usual, let $\mathbb{B}_n$ denote the open unit ball in $\mathbb{C}^n$. Let ν denote the Lebesgue measure on $\mathbb{B}_n$ normalized so that $\nu(\mathbb{B}_n) = 1$. Fix a real number $\alpha > -1$. The weighted Lebesgue measure ν_α on $\mathbb{B}_n$ is defined by $\mathrm{d}\nu_\alpha(\mathbf{z}) = c_\alpha(1-|\mathbf{z}|^2)^\alpha \mathrm{d}\nu(\mathbf{z})$, where c_α is a normalizing constant so that $\nu_\alpha(\mathbb{B}_n) = 1$. A direct computation shows that $c_\alpha = \dfrac{\Gamma(n+\alpha+1)}{\Gamma(n+1)\Gamma(\alpha+1)}$. Let L^2_α denote $L^2(\mathbb{B}_n, \mathrm{d}\nu_\alpha)$ and L^∞ denote $L^\infty(\mathbb{B}_n, \mathrm{d}\nu)$, which is the same as $L^\infty(\mathbb{B}_n, \mathrm{d}\nu_\alpha)$. We denote the inner product in L^2_α by $\langle\cdot,\cdot\rangle_\alpha$ and the corresponding norm by $\|\cdot\|_{2,\alpha}$.

The weighted Bergman space A^2_α consists of all functions in L^2_α which are holomorphic on $\mathbb{B}_n$. It is well-known that A^2_α is a closed subspace of L^2_α.

For any multi-index $\mathbf{m} = (m_1, \ldots, m_n) \in \mathbb{N}^n$ (here $\mathbb{N}$ denotes the set of all *non-negative* integers), we write $|\mathbf{m}| = m_1 + \cdots + m_n$ and $\mathbf{m}! = m_1! \cdots m_n!$. For any $\mathbf{z} = (z_1, \ldots, z_n) \in \mathbb{C}^n$, we write $\mathbf{z}^\mathbf{m} = z_1^{m_1} \cdots z_n^{m_n}$ and $\bar{\mathbf{z}}^\mathbf{m} = \bar{z}_1^{m_1} \cdots \bar{z}_n^{m_n}$. The standard orthonormal basis for A^2_α is $\{e_\mathbf{m} : \mathbf{m} \in \mathbb{N}^n\}$, where

$$e_\mathbf{m}(\mathbf{z}) = \Big[\frac{\Gamma(n+|\mathbf{m}|+\alpha+1)}{\mathbf{m}!\,\Gamma(n+\alpha+1)}\Big]^{1/2} \mathbf{z}^\mathbf{m},\ \ \mathbf{m} \in \mathbb{N}^n, \mathbf{z} \in \mathbb{B}_n.$$

For a more detailed discussion of A^2_α, see Chapter 2 in [8].

Since A^2_α is a closed subspace of the Hilbert space L^2_α, there is an orthogonal projection P_α from L^2_α onto A^2_α. For any function $f \in L^2_\alpha$ the Toeplitz operator with symbol f is denoted by T_f, which is densely defined on A^2_α by $T_f\varphi = P_\alpha(f\varphi)$ for bounded holomorphic functions φ on $\mathbb{B}_n$. If f is a bounded function then T_f is a bounded operator on A^2_α with $\|T_f\| \leq \|f\|_\infty$

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and $(T_f)^* = T_{\bar{f}}$. However, there are unbounded functions f that give rise to bounded operators T_f.

Let $\mathcal{P}$ be the space of holomorphic polynomials in the variable $\mathbf{z} = (z_1, \ldots, z_n)$ in $\mathbb{C}^n$. For any $f \in L^2_\alpha$ and holomorphic polynomials $p, q \in \mathcal{P}$ we have $\langle T_f p, q\rangle_\alpha = \int_{\mathbb{B}_n} p\bar{q}f\,\mathrm{d}\nu_\alpha$. This shows that T_f can be viewed as an operator from $\mathcal{P}$ into the space $L^*(\mathcal{P}, \mathbb{C})$ of conjugate-linear functionals on $\mathcal{P}$. More generally, for any compactly supported regular Borel measure μ on $\mathbb{C}^n$, we define $L_\mu : \mathcal{P} \longrightarrow L^*(\mathcal{P}, \mathbb{C})$ by the formula $L_\mu p(q) = \int_{\mathbb{C}^n} p\bar{q}\,\mathrm{d}\mu$, for $p, q \in \mathcal{P}$. For $f \in L^2_\alpha$ if we let $\mathrm{d}\mu = f\,\mathrm{d}\nu_\alpha$ then $T_f = L_\mu$ on $\mathcal{P}$. It follows from Stone-Weierstrass's Theorem that if $L_\mu = 0$ then $\mu = 0$. It is also immediate that if μ is a linear combination of point masses then L_μ has finite rank. That the converse is also true is the content of the following theorem, which had been an open conjecture for about twenty years. See [1, 6, 7].

Theorem 1.1. *L_μ has finite rank if and only if μ is a (finite) linear combination of point masses.*

Theorem 1.1 for the case $n = 1$ was proved by D. Luecking in [6]. Using a refined version of Theorem 1.1 in this case, the current author was able to show that if $f_1, \ldots, f_r$ are bounded measurable functions on the disk, all but possibly one of them are radial functions and $T_1 \cdots T_{f_r}$ has finite rank then one of these functions is the zero function. See [5] for more detail.

To the best of the author's knowledge, Theorem 1.1 in high dimensions has been proved in at least two preprints. In [7], G. Rozenblum and N. Shirokov give a proof by induction on the dimension n. In the base case ($n = 1$), they use the above Luecking's result. In [1], B. Choe follows Luecking's scheme with modifications (to the setting of several variables) to prove Theorem 1.1 for all $n \geq 1$. In this note, we modify Choe's proof to obtain a refined version of Theorem 1.1. We then apply the refined theorem to solve the problem about finite-rank products of Toeplitz operators in all dimensions, when all but possibly one of the operators are (weighted) shifts. This result is Theorem 3.2, which is a generalization of the main result in [5].

2. A Refined Luecking's Theorem in High Dimensions

For any $1 \leq j \leq n$, let $\sigma_j : \mathbb{N} \times \mathbb{N}^{n-1} \longrightarrow \mathbb{N}^n$ be the map defined by the formula $\sigma_j(s, (r_1, \ldots, r_{n-1})) = (r_1, \ldots, r_{j-1}, s, r_j, \ldots, r_{n-1})$ for all $s \in \mathbb{N}$ and $(r_1, \ldots, r_{n-1}) \in \mathbb{N}^{n-1}$. If $\mathcal{S}$ is a subset of $\mathbb{N}^n$ and $1 \leq j \leq n$, we define

$$\widetilde{\mathcal{S}}_j = \Big\{\tilde{\mathbf{r}} = (r_1, \ldots, r_{n-1}) \in \mathbb{N}^{n-1} : \sum_{\substack{s \in \mathbb{N} \\ \sigma_j(s, \tilde{\mathbf{r}}) \in \mathcal{S}}} \frac{1}{s+1} = \infty\Big\}.$$

The following definition is given in [4]. For completeness, we recall it here.

Definition 2.1. *We say that $\mathcal{S}$ has property (P) if one of the following statements holds.*

(1) *$\mathcal{S}=\emptyset$, or*
(2) *$\mathcal{S}\neq\emptyset$, $n=1$ and $\sum\limits_{s\in\mathcal{S}}\dfrac{1}{s+1}<\infty$, or*
(3) *$\mathcal{S}\neq\emptyset$, $n\geq 2$ and for any $1\leq j\leq n$, the set $\widetilde{\mathcal{S}}_j$ has property (P) as a subset of $\mathbb{N}^{n-1}$.*

With the above definition, the following statements hold.

(1) If $\mathcal{S}\subset\mathbb{N}$ and $\mathcal{S}$ does not have property (P), then $\sum_{s\in\mathcal{S}}\frac{1}{s+1}=\infty$. If $\mathcal{S}\subset\mathbb{N}^n$ with $n\geq 2$ and $\mathcal{S}$ does not have property (P), then $\widetilde{\mathcal{S}}_j$ does not have property (P) as a subset of $\mathbb{N}^{n-1}$ for some $1\leq j\leq n$.
(2) If $\mathcal{S}_1$ and $\mathcal{S}_2$ are subsets of $\mathbb{N}^n$ that both have property (P) then $\mathcal{S}_1\cup\mathcal{S}_2$ also has property (P).
(3) If $\mathcal{S}\subset\mathbb{N}^n$ has property (P) and $l\in\mathbb{Z}^n$ then $(\mathcal{S}+l)\cap\mathbb{N}^n$ also has property (P). Here, $\mathcal{S}+l=\{m+l : m\in\mathcal{S}\}$.
(4) If $\mathcal{S}\subset\mathbb{N}^n$ has property (P) then $\mathbb{N}\times\mathcal{S}$ also has property (P) as a subset of $\mathbb{N}^{n+1}$. This follows by induction on n.
(5) The set $\mathbb{N}^n$ does not have property (P) for all $n\geq 1$. This together with (2) shows that if $\mathcal{S}\subset\mathbb{N}^n$ has property (P) then $\mathbb{N}^n\backslash\mathcal{S}$ does not have property (P).
(6) For any $\mathbf{m}=(m_1,\ldots,m_n)$ and $\mathbf{k}=(k_1,\ldots,k_n)$ in $\mathbb{N}$, we write $\mathbf{m}\succeq\mathbf{k}$ if $m_j\geq k_j$ for all $1\leq j\leq n$ and write $\mathbf{m}\not\succeq\mathbf{k}$ if otherwise. Then for any fixed $\mathbf{k}\in\mathbb{N}^n$, the set $\mathcal{S}=\{\mathbf{m}\in\mathbb{N}^n : \mathbf{m}\not\succeq\mathbf{k}\}$ has property (P). This follows from (2), (4) and the fact that
$$\mathcal{S}\subset\bigcup_{j=1}^{n}\mathbb{N}\times\cdots\times\mathbb{N}\times\{0,\ldots,k_j-1\}\times\mathbb{N}\times\cdots\times\mathbb{N}.$$

The following proposition shows that if the zero set of a holomorphic function (under certain additional assumptions) does not have property (P) then the function is identically zero. The proof is in Section 3 in [4].

Proposition 2.2 (Proposition 3.2 in [4])**.** *Let $\mathbb{K}$ denote the right half of the complex plane. Let $F:\mathbb{K}^n\to\mathbb{C}$ be a holomorphic function. Suppose there exists a polynomial p such that $|F(\mathbf{z})|\leq p(|\mathbf{z}|)$ for all $\mathbf{z}\in\mathbb{K}^n$. Put $Z(F)=\{\mathbf{r}\in\mathbb{N}^n : F(\mathbf{r})=0\}$. If $Z(F)$ does not have property (P), then F is identically zero in $\mathbb{K}^n$.*

We are now ready for the statement and proof of a refined version of Theorem 1.1.

Theorem 2.3. *Suppose $\mathcal{S}\subset\mathbb{N}^n$ is a set that has property (P). Let $\mathcal{N}$ be the linear subspace of $\mathcal{P}$ spanned by the monomials $\{\mathbf{z}^{\mathbf{m}} : \mathbf{m}\in\mathbb{N}^n\backslash\mathcal{S}\}$. Let $L^*(\mathcal{N},\mathbb{C})$ denote the space of all conjugate-linear functionals on $\mathcal{N}$. Suppose μ is a complex regular Borel measure on $\mathbb{C}^n$ with compact support. Let $L_\mu:\mathcal{N}\longrightarrow L^*(\mathcal{N},\mathbb{C})$ be the operator defined by $L_\mu f(g)=\int_{\mathbb{C}^n}f\bar{g}\mathrm{d}\mu$ for $f,g\in\mathcal{N}$. If L_μ has finite rank, then $\tilde{\mu}$ is a linear combination of point masses, where $\mathrm{d}\tilde{\mu}(\mathbf{z})=|z_1|\cdots|z_n|\mathrm{d}\mu(\mathbf{z})$ for $\mathbf{z}\in\mathbb{C}^n$. As a consequence, if μ*

is absolutely continuous with respect to the Lebesgue measure on $\mathbb{C}^n$*, then* μ *is the zero measure.*

Proof. Suppose L_μ has rank strictly less than N, where $N \geq 1$. Arguing as in pages 2 and 3 in [1], for any polynomials $f_1, \ldots, f_N$ and $g_1, \ldots, g_N$ in $\mathcal{N}$, we have

$$\int_{\mathbb{C}^{n\times N}} \Big(\prod_{j=1}^{N} f_j(\mathbf{z}_j)\Big) \det(\bar{g}_i(\mathbf{z}_j)) \mathrm{d}\mu^N(\mathbf{z}_1, \ldots, \mathbf{z}_N) = 0, \tag{1}$$

where μ^N is the product of N copies of μ on $\mathbb{C}^{n\times N}$.

Let $\mathbf{m}_1, \ldots, \mathbf{m}_N$ and $\mathbf{k}_1, \ldots, \mathbf{k}_N$ be multi-indices in $\mathbb{N}^n$. Let

$$\begin{aligned} L &= \{\mathbf{l} \in \mathbb{N}^n : \mathbf{l} + \mathbf{m}_j \notin \mathcal{S} \text{ and } \mathbf{l} + \mathbf{k}_j \notin \mathcal{S} \text{ for all } 1 \leq j \leq N\} \\ &= \mathbb{N}^n \backslash \Big((\bigcup_{j=1}^{N} (\mathcal{S} - \mathbf{m}_j)) \bigcup (\bigcup_{j=1}^{N} (\mathcal{S} - \mathbf{k}_j)) \Big). \end{aligned}$$

Since $\mathcal{S}$ has property (P) we see that $\mathbb{N}^n \backslash L$ has property (P). This implies that L does not have property (P). For any $\mathbf{l} \in L$, the monomials $f_j(\mathbf{z}) = \mathbf{z}^{\mathbf{m}_j + \mathbf{l}}$ and $g_j(\mathbf{z}) = \mathbf{z}^{\mathbf{k}_j + \mathbf{l}}$ are in $\mathcal{N}$ for $j = 1, \ldots, N$. Equation (1) then implies that

$$\begin{aligned} 0 &= \int_{\mathbb{C}^{n\times N}} \Big(\prod_{j=1}^{N} \mathbf{z}_j^{\mathbf{m}_j + \mathbf{l}}\Big) \det((\bar{\mathbf{z}}_j^{\mathbf{k}_i + \mathbf{l}})) \mathrm{d}\mu^N(\mathbf{z}_1, \ldots, \mathbf{z}_N) \\ &= \int_{\mathbb{C}^{n\times N}} \Big(\prod_{j=1}^{N} \mathbf{z}_j^{\mathbf{m}_j}\Big) \det((\bar{\mathbf{z}}_j^{\mathbf{k}_i})) \Big(\prod_{j=1}^{N} \mathbf{z}_j^{\mathbf{l}} \bar{\mathbf{z}}_j^{\mathbf{l}}\Big) \mathrm{d}\mu^N(\mathbf{z}_1, \ldots, \mathbf{z}_N) \\ &= \int_{\mathbb{C}^{n\times N}} \Big(\prod_{j=1}^{N} \mathbf{z}_j^{\mathbf{m}_j}\Big) \det((\bar{\mathbf{z}}_j^{\mathbf{k}_i})) \Big(\prod_{j=1}^{N} \prod_{s=1}^{n} |z_{j,s}|^{2l_s}\Big) \mathrm{d}\mu^N(\mathbf{z}_1, \ldots, \mathbf{z}_N) \\ &= \int_{\mathbb{C}^{n\times N}} \Big(\prod_{j=1}^{N} \mathbf{z}_j^{\mathbf{m}_j}\Big) \det((\bar{\mathbf{z}}_j^{\mathbf{k}_i})) \Big(\prod_{s=1}^{n} (\prod_{j=1}^{N} |z_{j,s}|)^{2l_s}\Big) \mathrm{d}\mu^N(\mathbf{z}_1, \ldots, \mathbf{z}_N), \end{aligned}$$

where $\mathbf{l} = (l_1, \ldots, l_n)$ and $\mathbf{z}_j = (z_{j,1}, \ldots, z_{j,n})$ for $1 \leq j \leq N$.

Suppose that μ is supported in the ball $\mathbb{B}(0, R)$ of radius R centered at 0 in $\mathbb{C}^n$. Then μ^N is supported in the product of N copies of $\mathbb{B}(0, R)$ in $\mathbb{C}^{n\times N}$. For any $\zeta = (\zeta_1, \ldots, \zeta_n) \in \mathbb{C}^n$ with $\Re(\zeta_1), \ldots, \Re(\zeta_n) > 0$, define

$$\begin{aligned} F(\zeta) &= \int_{\mathbb{C}^{n\times N}} \Big(\prod_{j=1}^{N} \mathbf{z}_j^{\mathbf{m}_j}\Big) \det((\bar{\mathbf{z}}_j^{\mathbf{k}_i})) \Big(\prod_{s=1}^{n} (\prod_{j=1}^{N} |z_{j,s}| R^{-1})^{2\zeta_s}\Big) \mathrm{d}\mu^N(\mathbf{z}_1, \ldots, \mathbf{z}_N) \\ &= \int_{(\mathbb{B}(0,R))^N} \Big(\prod_{j=1}^{N} \mathbf{z}_j^{\mathbf{m}_j}\Big) \det((\bar{\mathbf{z}}_j^{\mathbf{k}_i})) \Big(\prod_{s=1}^{n} (\prod_{j=1}^{N} |z_{j,s}| R^{-1})^{2\zeta_s}\Big) \mathrm{d}\mu^N(\mathbf{z}_1, \ldots, \mathbf{z}_N). \end{aligned}$$

Then F is holomorphic and bounded on its defining domain and $F(\mathbf{l}) = 0$ for all $\mathbf{l}$ in L. Since L does not have property (P), Proposition 2.2 implies that $F(\zeta) = 0$ for all $\zeta = (\zeta_1, \ldots, \zeta_n) \in \mathbb{C}^n$ with $\Re(\zeta_1), \ldots, \Re(\zeta_n) > 0$. In particular, we have $F(\frac{1}{2}, \ldots, \frac{1}{2}) = 0$. This shows that

$$\begin{aligned} 0 &= \int\limits_{\mathbb{C}^{n\times N}} \big(\prod_{j=1}^{N} \mathbf{z}_j^{\mathbf{m}_j}\big) \det((\bar{\mathbf{z}}_j^{\mathbf{k}_i}))\big(\prod_{s=1}^{n}(\prod_{j=1}^{N}|z_{j,s}|)\big)\mathrm{d}\mu^N(\mathbf{z}_1, \ldots, \mathbf{z}_N) \\ &= \int\limits_{\mathbb{C}^{n\times N}} \big(\prod_{j=1}^{N} \mathbf{z}_j^{\mathbf{m}_j}\big) \det((\bar{\mathbf{z}}_j^{\mathbf{k}_i}))\mathrm{d}\tilde{\mu}^N(\mathbf{z}_1, \ldots, \mathbf{z}_N), \end{aligned}$$

where $\tilde{\mu}^N$ is the product of N copies of $\tilde{\mu}$. Since $\mathbf{m}_1, \ldots, \mathbf{m}_N$ and $\mathbf{k}_1, \ldots, \mathbf{k}_N$ were arbitrary, by taking finite sums, we conclude that

$$\int\limits_{\mathbb{C}^{n\times N}} \big(\prod_{j=1}^{N} f_j(\mathbf{z}_j)\big) \det(\bar{g}_i(\mathbf{z}_j))\mathrm{d}\tilde{\mu}^N(\mathbf{z}_1, \ldots, \mathbf{z}_N) = 0, \tag{2}$$

where $f_1, \ldots, f_N$ and $g_1, \ldots, g_N$ are in $\mathcal{P}$. Now following Choe's proof on pages 3–6 in [1], we see that $\tilde{\mu}$ is supported in a set of less than N points. □

Remark 2.4. *Suppose $n = 1$. Then $\tilde{\mu}$ is a linear combination of point masses implies that μ is also a linear combination of point masses.*

Remark 2.5. *Suppose $n \geq 2$. Let $\mathcal{S} = \{\mathbf{m} = (m_1, \ldots, m_n) \in \mathbf{N}^n : m_1 \cdots m_n = 0\}$. Then $\mathcal{S}$ has property (P). Let $W = \{\mathbf{z} = (z_1, \ldots, z_n) \in \mathbb{C}^n : z_1 \cdots z_n = 0\}$. If μ is any complex regular Borel measure supported on W then for any f, g in $\mathcal{N}$ (recall that $\mathcal{N} = \mathrm{Span}\{e_{\mathbf{m}} : \mathbf{m} \in \mathbb{N}^n \backslash \mathcal{S}\}$), we have*

$$\int\limits_{\mathbb{C}^n} f\bar{g}\mathrm{d}\mu = \int\limits_{W} f\bar{g}\mathrm{d}\mu = 0,$$

because f and g vanish on W. This shows that L_μ is the zero operator from $\mathcal{N}$ into $L^*(\mathcal{N}, \mathbb{C})$. However, since W is an infinite set, μ may not be a inear combination of point masses.

3. Finite-rank Toeplitz Products

In the first part of this section, we use Theorem 2.3 to show that under certain conditions on the bounded operators S_1 and S_2 on A^2_α, if $f \in L^2_\alpha$ so that $S_2 T_f S_1$ is a finite-rank operator, then f must be zero almost everywhere on $\mathbb{B}_n$.

Theorem 3.1. *Let S_1, S_2 be two bounded operators on A^2_α. Suppose there is a set $\mathcal{S} \subset \mathbb{N}^n$ which has property (P) such that $\ker(S_2) \subset \bar{\mathcal{M}}$ and $\mathcal{N} \subset \mathrm{ran}(S_1)$. Here $\mathcal{M}$ (respectively, $\mathcal{N}$) is the linear subspace of A^2_α spanned by $\{\mathbf{z}^{\mathbf{m}} : \mathbf{m} \in \mathcal{S}\}$ (respectively, $\{\mathbf{z}^{\mathbf{m}} : \mathbf{m} \in \mathbb{N}^n \backslash \mathcal{S}\}$). Suppose $f \in L^2_\alpha$ so that the operator $S_2 T_f S_1$ has finite rank, then f is the zero function.*

Proof. Since $S_2T_fS_1$ has finite rank and $\mathcal{N}\subset S_1(A^2_\alpha)$, we see that $S_2T_f(\mathcal{N})$ is a finite-dimensional linear subspace of A^2_α. Let $\{u_1,\ldots,u_N\}$ be a basis for this space. Let $v_j\in A^2_\alpha$ such that $S_2v_j=u_j$ for $1\leq j\leq N$. It then follows that $T_f(\mathcal{N})$ is contained in $\mathrm{Span}(\ker(S_2)\cup\{v_1,\ldots,v_N\})$, which is a subspace of $\mathrm{Span}(\bar{\mathcal{M}}\cup\{v_1,\ldots,v_N\})$. Let $P_{\bar{\mathcal{M}}}$ denote the orthogonal projection from A^2_α onto $\bar{\mathcal{M}}$. Replacing v_j by $v_j-P_{\bar{\mathcal{M}}}v_j$ if necessary, we may assume that $v_j\perp\mathcal{M}$ for $1\leq j\leq N$. Using the Gram-Schmidt process if necessary, we may assume that $\{v_1,\ldots,v_N\}$ is an orthonormal set in A^2_α (we may have fewer vectors after using Gram-Schmidt process but let us still denote by N the total number of vectors).

For any p in $\mathcal{N}$ we have

$$T_fp=P_{\bar{\mathcal{M}}}T_fp+\sum_{j=1}^{N}\langle T_fp,v_j\rangle_\alpha v_j=P_{\bar{\mathcal{M}}}T_fp+\sum_{j=1}^{N}\langle fp,v_j\rangle_\alpha v_j.$$

Now for any q in $\mathcal{N}$, since $q\perp\mathcal{M}$, we have

$$\begin{aligned}\int_{\mathbb{B}_n} fp\bar{q}\,\mathrm{d}\nu_\alpha&=\langle T_fp,q\rangle_\alpha\\&=\langle P_{\bar{\mathcal{M}}}T_fp,q\rangle_\alpha+\sum_{j=1}^{N}\langle fp,v_j\rangle_\alpha\langle v_j,q\rangle_\alpha\\&=\sum_{j=1}^{N}\langle fp,v_j\rangle_\alpha\langle v_j,q\rangle_\alpha.\end{aligned}$$

Let $\mathrm{d}\mu=f\mathrm{d}\nu_\alpha$. Then the map L_μ from $\mathcal{N}$ into the space of all conjugate-linear functionals on $\mathcal{N}$ defined by $L_\mu p(q)=\int_{\mathbb{B}_n}p\bar{q}\mathrm{d}\mu=\int_{\mathbb{B}_n}p\bar{q}f\mathrm{d}\nu_\alpha$ has finite rank. Theorem 2.3 then implies that μ is the zero measure. Thus f is zero almost everywhere on $\mathbb{B}_n$. □

Suppose $\tilde{f}\in L^\infty$ such that

$$\tilde{f}(z_1,\ldots,z_n)=\tilde{f}(|z_1|,\ldots,|z_n|)\text{ for almost all }\mathbf{z}\in\mathbb{B}_n. \tag{3}$$

Then for any $\mathbf{m},\mathbf{k}$ in $\mathbb{N}^n$, we have

$$\begin{aligned}\langle T_{\tilde{f}}e_{\mathbf{m}},e_{\mathbf{k}}\rangle_\alpha&=\int_{\mathbb{B}_n}\tilde{f}(\mathbf{z})e_{\mathbf{m}}(\mathbf{z})\bar{e}_{\mathbf{k}}(\mathbf{z})\mathrm{d}\nu_\alpha(\mathbf{z})\\&=\int_{\mathbb{B}_n}\tilde{f}(|z_1|,\ldots,|z_n|)e_{\mathbf{m}}(\mathbf{z})\bar{e}_{\mathbf{k}}(\mathbf{z})\mathrm{d}\nu_\alpha(\mathbf{z})\\&=C(\mathbf{m},\mathbf{k},n,\alpha)\int_{\mathbb{B}_n}\tilde{f}(|z_1|,\ldots,|z_n|)\mathbf{z}^{\mathbf{m}}\bar{\mathbf{z}}^{\mathbf{m}}\mathrm{d}\nu_\alpha(\mathbf{z})\end{aligned}$$

$$= \begin{cases} 0 & \text{if } \mathbf{m} \neq \mathbf{k}, \\ C(\mathbf{m}, n, \alpha) \int\limits_{\mathbb{B}_n} \tilde{f}(|z_1|, \ldots, |z_n|) \mathbf{z}^{\mathbf{m}} \bar{\mathbf{z}}^{\mathbf{m}} \mathrm{d}\nu_\alpha(\mathbf{z}) & \text{if } \mathbf{m} = \mathbf{k}. \end{cases}$$

The last equality follows from the invariance of the measure ν_α under the action of the n-torus (see also Corollary 3.5 in [4] for more detail). This implies that $T_{\tilde{f}}$ is a diagonal operator with respect to the standard orthonormal basis of A^2_α. The eigenvalues of $T_{\tilde{f}}$ are given by

$$\omega_\alpha(\tilde{f}, \mathbf{m}) = \langle T_{\tilde{f}} e_{\mathbf{m}}, e_{\mathbf{m}} \rangle_\alpha, \quad \mathbf{m} \in \mathbb{N}^n. \tag{4}$$

If the set $Z(\tilde{f}) = \{\mathbf{m} \in \mathbb{N}^n : \omega_\alpha(\tilde{f}, \mathbf{m}) = 0\}$ does not have property (P) then by Lemma 3.3 in [5], it is all of $\mathbb{N}^n$. This implies that $T_{\tilde{f}}$ is the zero operator, hence $\tilde{f}$ is the zero function on $\mathbb{B}_n$. Therefore, if $\tilde{f}$ is not the zero function on $\mathbb{B}_n$ then $Z(\tilde{f})$ has property (P).

Now suppose $\mathbf{s}, \mathbf{t}$ are in $\mathbb{N}^n$. Let $f(\mathbf{z}) = \mathbf{z}^{\mathbf{s}} \bar{\mathbf{z}}^{\mathbf{t}} \tilde{f}(\mathbf{z})$ for $\mathbf{z} \in \mathbb{B}_n$. For $\mathbf{m}, \mathbf{k}$ in $\mathbb{N}^n$, we have

$$\begin{aligned} \langle T_f e_{\mathbf{m}}, e_{\mathbf{k}} \rangle_\alpha &= C(\mathbf{m}, \mathbf{s}, \mathbf{k}, \mathbf{t}, n, \alpha) \langle \tilde{f} e_{\mathbf{m}+\mathbf{s}}, e_{\mathbf{k}+\mathbf{t}} \rangle_\alpha \\ &= \begin{cases} 0 & \text{if } \mathbf{m}+\mathbf{s} \neq \mathbf{k}+\mathbf{t}, \\ C(\mathbf{m}, \mathbf{s}, \mathbf{k}, \mathbf{t}, n, \alpha) \omega_\alpha(\tilde{f}, \mathbf{m}+\mathbf{s}) & \text{if } \mathbf{m}+\mathbf{s} = \mathbf{k}+\mathbf{t}. \end{cases} \end{aligned}$$

This shows that

$$T_f e_{\mathbf{m}} = \begin{cases} 0 & \text{if } \mathbf{m} \not\succeq \mathbf{t}-\mathbf{s}, \\ C(\mathbf{m}, \mathbf{s}, \mathbf{m}+\mathbf{s}-\mathbf{t}, \mathbf{t}, n, \alpha) \omega_\alpha(\tilde{f}, \mathbf{m}+\mathbf{s}) e_{\mathbf{m}+\mathbf{s}-\mathbf{t}} & \text{if } \mathbf{m} \succeq \mathbf{t}-\mathbf{s}. \end{cases} \tag{5}$$

Note that the constant $C(\mathbf{m}, \mathbf{s}, \mathbf{m}+\mathbf{s}-\mathbf{t}, \mathbf{t}, n, \alpha)$ is positive.

Now suppose $\tilde{f}_1, \ldots, \tilde{f}_r$ be functions in L^∞ satisfying (3), none of which is the zero function. Let $\mathbf{s}_1, \ldots, \mathbf{s}_r$ and $\mathbf{t}_1, \ldots, \mathbf{t}_r$ be multi-indices in $\mathbb{N}^n$. For each $1 \leq j \leq r$, let $f_j(\mathbf{z}) = \mathbf{z}^{\mathbf{s}_j} \bar{\mathbf{z}}^{\mathbf{t}_j} \tilde{f}_j(\mathbf{z})$ for $\mathbf{z} \in \mathbb{B}_n$. Let $S = T_{f_r} \cdots T_{f_1}$. For any multi-index $\mathbf{m} \succeq \sum\limits_{j=1}^{r} (\mathbf{s}_j + \mathbf{t}_j)$, using (5), we see that there is a positive constant C depending on $\mathbf{m}, \mathbf{s}_1, \ldots, \mathbf{s}_r, \mathbf{t}_1, \ldots, \mathbf{t}_r, n$ and α so that

$$S e_{\mathbf{m}} = C \cdot \left(\prod_{j=1}^{r} \omega_\alpha\Big(\tilde{f}_j, \mathbf{m} + \sum_{i=1}^{j-1} (\mathbf{s}_i - \mathbf{t}_i) + \mathbf{s}_j\Big) \right) e_{\mathbf{m}+\sum_{j=1}^{r}(\mathbf{s}_j - \mathbf{t}_j)}. \tag{6}$$

Define

$$\begin{aligned}\mathcal{J} &= \Big\{\mathbf{m} : \mathbf{m} \not\geq \sum_{j=1}^{r}(\mathbf{s}_j + \mathbf{t}_j)\Big\}\\ &\qquad \bigcup \Big\{\mathbf{m} : \prod_{j=1}^{r}\omega_\alpha\Big(\tilde{f}_j, \mathbf{m} + \sum_{i=1}^{j-1}(\mathbf{s}_i - \mathbf{t}_i) + \mathbf{s}_j\Big) = 0\Big\}\\ &= \Big\{\mathbf{m} : \mathbf{m} \not\geq \sum_{j=1}^{r}(\mathbf{s}_j + \mathbf{t}_j)\Big\} \bigcup \Big(\bigcup_{j=1}^{r}\Big(Z(\tilde{f}_j) - (\sum_{i=1}^{j-1}(\mathbf{s}_i - \mathbf{t}_i) + \mathbf{s}_j)\Big)\Big).\end{aligned}$$

Since none of the functions $f_1, \ldots, f_r$ is the zero function, the set $\mathcal{J}$ has property (P). For $\mathbf{m} \in \mathbb{N}^n \backslash \mathcal{J}$, we see that $Se_{\mathbf{m}} \neq 0$ and $e_{\mathbf{m}+\sum_{j=1}^{r}(\mathbf{s}_j - \mathbf{t}_j)}$ is a multiple of $Se_{\mathbf{m}}$. Suppose $\varphi \in A^2_\alpha$ such that $S\varphi = 0$. Then we have

$$0 = S\varphi = S\Big(\sum_{\mathbf{m}\in\mathbb{N}^n}\langle\varphi, e_{\mathbf{m}}\rangle_\alpha e_{\mathbf{m}}\Big) = \sum_{\mathbf{m}\in\mathbb{N}^n}\langle\varphi, e_{\mathbf{m}}\rangle_\alpha Se_{\mathbf{m}}.$$

So (6) implies that for any $\mathbf{m} \in \mathbb{N}^n \backslash \mathcal{J}$, $\langle\varphi, e_{\mathbf{m}}\rangle_\alpha = 0$. Therefore $\ker(S)$ is contained in the closure in A^2_α of the linear span of $\{e_{\mathbf{m}} : \mathbf{m} \in \mathcal{J}\}$. Now put

$$\mathcal{I} = \Big\{\mathbf{k} \in \mathbb{N}^n : \mathbf{k} \not\geq \sum_{j=1}^{r}(\mathbf{s}_j - \mathbf{t}_j)\Big\} \bigcup \Big(\mathbb{N}^n \bigcap \Big(\mathcal{J} + \sum_{j=1}^{r}(\mathbf{s}_j - \mathbf{t}_j)\Big)\Big).$$

Then $\mathcal{I}$ has property (P) and for any $\mathbf{k} \in \mathbb{N}^n \backslash \mathcal{I}$, $\mathbf{m} = \mathbf{k} - \sum_{j=1}^{r}(\mathbf{s}_j - \mathbf{t}_j)$ belongs to $\mathbb{N}^n \backslash \mathcal{J}$. It then follows that $e_{\mathbf{k}} = e_{\mathbf{m}+\sum_{j=1}^{r}(\mathbf{s}_j - \mathbf{t}_j)}$ is a multiple of $Se_{\mathbf{m}}$. So the linear span of $\{e_{\mathbf{k}} : \mathbf{k} \in \mathbb{N}^n \backslash \mathcal{I}\}$ is contained in the range of S.

We now prove a result about products of Toeplitz operators on A^2_α.

Theorem 3.2. *Let r_1 and r_2 be two positive integers. Let $\tilde{f}_1, \ldots, \tilde{f}_{r_1+r_2}$ be functions in L^∞ satisfying* (3), *none of which is the zero function. Let $\mathbf{s}_1, \ldots, \mathbf{s}_{r_1+r_2}$ and $\mathbf{t}_1, \ldots, \mathbf{t}_{r_1+r_2}$ be multi-indices in $\mathbb{N}^n$. For each $1 \leq j \leq r_1 + r_2$, let $f_j(\mathbf{z}) = \mathbf{z}^{\mathbf{s}_j}\bar{\mathbf{z}}^{\mathbf{t}_j}\tilde{f}_j(\mathbf{z})$ for $\mathbf{z} \in \mathbb{B}_n$. If $f \in L^2_\alpha$ such that the operator $T_{f_{r_1+r_2}}\cdots T_{f_{r_1+1}}T_f T_{f_{r_1}}\cdots T_{f_1}$ (which is densely defined on A^2_α) has finite rank, then f is the zero function.*

Proof. Let $S_1 = T_{f_{r_1}}\cdots T_{f_1}$ and $S_2 = T_{f_{r_1+r_2}}\cdots T_{f_{r_1+1}}$. From the discussion preceding the theorem, there are subsets $\mathcal{J}$ and $\mathcal{I}$ of $\mathbb{N}^n$ that have property (P) such that $\ker(S_2)$ is contained in the closure in A^2_α of $\mathrm{Span}(\{e_{\mathbf{m}} : m \in \mathcal{J}\})$ and $\mathrm{Span}(\{e_{\mathbf{k}} : \mathbf{k} \in \mathbb{N}^n\backslash\mathcal{I}\})$ is a subspace of $S_1(A^2_\alpha)$. Let $\mathcal{S} = \mathcal{J} \cup \mathcal{I}$. Then $\mathcal{S}$ has property (P), $\ker(S_2) \subset \bar{\mathcal{M}}$ and $\mathcal{N} \subset S_1(A^2_\alpha)$, where $\mathcal{M}$ (respectively, $\mathcal{N}$) is the linear subspace of A^2_α spanned by $\{\mathbf{z}^{\mathbf{m}} : \mathbf{m} \in \mathcal{S}\}$ (respectively, $\{\mathbf{z}^{\mathbf{m}} : \mathbf{m} \in \mathbb{N}^n\backslash\mathcal{S}\}$). If $f \in L^2_\alpha$ such that $S_2 T_f S_1$ has finite rank, then Theorem 3.1 implies that f is the zero function. □

Remark 3.3. *Suppose $n = 1$. The functions f_j's in the hypothesis of Theorem 3.2 are called quasihomogeneous functions. Each T_{f_j} is a weighted forward or backward shift. It was showed in* [2] *(Theorem 2) that if the product* *of finitely many Toeplitz operators whose symbols are bounded quasihomogeneous functions is of finite rank then one of the functions must be zero. This result is a special case of our Theorem 3.2 above.*

In [3]*, K. Guo, S. Sun and D. Zheng showed that if f and g are bounded harmonic functions on the unit disk and $T_f T_g$ has finite rank then either $f = 0$ or $g = 0$. However, for arbitrary bounded measurable functions f and g, it is still not known whether $T_f T_g = 0$ implies one of these functions must be the zero function.*

References

[1] Boo Rim Choe, *On higher dimensional Luecking's theorem*, preprint, available at http://math.korea.ac.kr/∼choebr/recent.html.

[2] Željko Čučković and Issam Louhichi, *Finite rank commutators and semicommutators of quasihomogeneous toeplitz operators*, Complex Anal. Oper. Theory **2** (2008), no. 3, 429–439.

[3] Kunyu Guo, Shunhua Sun, and Dechao Zheng, *Finite rank commutators and semi-commutators of Toeplitz operators with harmonic symbols*, Illinois J. Math. **51** (2007), no. 2, 583–596 (electronic). MR 2342676

[4] Trieu Le, *The commutants of certain Toeplitz operators on weighted Bergman spaces*, preprint, available at http://www.math.uwaterloo.ca/∼t29le/Papers.htm.

[5] ———, *A refined Luecking's theorem and finite rank products of Toeplitz operators*, preprint, available at http://www.math.uwaterloo.ca/∼t29le/Papers.htm.

[6] Daniel H. Luecking, *Finite rank Toeplitz operators on the Bergman space*, Proc. Amer. Math. Soc. **136** (2008), no. 5, 1717–1723.

[7] Grigori Rozenblum and Nikolay Shirokov, *Finite rank Bergman-Toeplitz and Bargmann-Toeplitz operators in many dimensions*, preprint, available at http://front.math.ucdavis.edu/0802.0192.

[8] Kehe Zhu, *Spaces of holomorphic functions in the unit ball*, Graduate Texts in Mathematics, vol. 226, Springer-Verlag, New York, 2005. MR 2115155 (2006d:46035)

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