FINITE RANK TOEPLITZ OPERATORS

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Abstract. In this note we offer a modified proof of a theorem by Borichev and Rozenblum [2] on finite rank Toeplitz operators whose symbols may have unbounded supports.

For the background on the problem, the reader is referred to [1, 2]. It is now a well-known result of D. Luecking [4] that if μ is a complex Borel measure with a compact support such that the functional

$$(T_\mu p)(\overline{q}) = \int_{\mathbb{C}} p\overline{q}\, d\mu \quad \text{for } p, q \text{ analytic polynomials,}$$

has finite rank, then μ is a finite combination of point masses. As a consequence, if φ is a bounded function with a compact support and T_φ has finite rank on the Fock space $\mathcal{F}^2$, then $\varphi \equiv 0$.

Luecking's proof does not carry over to the case where the measure μ has an unbounded support. In fact, there are examples where $\mu \not\equiv 0$ but $T_\mu \equiv 0$. This was discovered by Grudsky and Vasilevski [3]. Concrete examples were presented in [1, Proposition 4.6].

In [5], Rozenblum obtained Luecking's Theorem for non-compactly supported measures with certain decay restrictions at infinity. Very recently, Borichev and Rozenblum [2] settled the finite rank problem, proving that if φ is bounded and T_φ has finite rank on $\mathcal{F}^2$, then $\varphi \equiv 0$. In this note we provide a simplification of their proof.

We first recall the following result from [1].

Lemma 1. *Let φ be a bounded measurable function. Suppose $f_1, \ldots, f_N$ and $g_1, \ldots, g_N$ are functions in $\mathcal{F}^2$ such that $T_\varphi = \sum_{j=1}^{N} \langle \cdot, f_j \rangle g_j$. Then the function $W(z) = \sum_{j=1}^{N} \overline{f_j(z)} g_j(-z)$ and all of its partial derivatives vanish at infinity.*

Furthermore, if $W \equiv 0$, then $\varphi = 0$ almost everywhere.

It was shown in [2] that such a function W in Lemma 1 must vanish identically on $\mathbb{C}$. The main purpose of this note is to provide a simplified proof of this result. The proof presented here essentially follows the arguments in [2]. My contribution is Lemma 3 below.

Theorem 2 (Borichev-Rozenblum). *Let $f_1, \ldots, f_N$ and $g_1, \ldots, g_N$ be entire functions. Put*

$$W(z) = f_1(z)\overline{g}_1(z) + \cdots + f_N(z)\overline{g}_N(z) \quad \text{for } z \in \mathbb{C}.$$

Suppose all partial derivatives $\partial_z^k \partial_{\overline{z}}^l W$ with $0 \le k, l \le N-1$ vanish at infinity. Then $W(z) = 0$ for all $z \in \mathbb{C}$.

We first prove an auxiliary result. We shall think of any vector in $\mathbb{C}^N$ as a column vector. For $\mathbf{v}_0, \ldots, \mathbf{v}_{N-1}$ in $\mathbb{C}^N$, we use $\det(\mathbf{v}_0, \ldots, \mathbf{v}_{N-1})$ to denote the determinant of the matrix whose jth column is the vector $\mathbf{v}_j$, for each $0 \leq j \leq N-1$.

Lemma 3. *Let $\mathbf{v}_0, \ldots, \mathbf{v}_{N-1}$ and $\mathbf{u}_0, \ldots, \mathbf{u}_{N-1}$ be vectors in $\mathbb{C}^N$. Suppose there is a number $\epsilon > 0$ such that $|\langle \mathbf{v}_k, \mathbf{u}_l \rangle| \leq \epsilon$ for all $0 \leq k, l \leq N-1$. Then*

$$|\det(\mathbf{v}_0, \ldots, \mathbf{v}_{N-1}) \det(\mathbf{u}_0, \ldots, \mathbf{u}_{N-1})| \leq (\epsilon\sqrt{N})^N.$$

Proof. Let A denote the matrix whose columns are the vectors $\mathbf{v}_0, \ldots, \mathbf{v}_{N-1}$ and B be the matrix whose columns are $\mathbf{u}_0, \ldots \mathbf{u}_{N-1}$. By assumption, the modulus of each entry of the product B^*A is at most ϵ. Hadamard's inequality gives $|\det(B^*A)| \leq (\epsilon\sqrt{N})^N$. Since

$$\begin{aligned} |\det(B^*A)| &= |\det(B^*)\det(A)| = |\det(B)\det(A)| \\ &= |\det(\mathbf{v}_0, \ldots, \mathbf{v}_{N-1}) \det(\mathbf{u}_0, \ldots, \mathbf{u}_{N-1})|, \end{aligned}$$

the conclusion of the lemma follows. □

Proof of Theorem 2. For the purpose of obtaining a contradiction, suppose W were not identically zero on $\mathbb{C}$. By combining the functions if necessary, we may assume that the functions $f_1, \ldots, f_N$ are linearly independent and $g_1, \ldots, g_N$ are also linearly independent, where $N \geq 1$.

For $0 \leq j \leq N-1$, let $\mathbf{v}_j$ (respectively, $\mathbf{u}_j$) be a column vector whose components are the derivatives $f_1^{(j)}, \ldots, f_N^{(j)}$ (respectively, $g_1^{(j)}, \ldots, g_N^{(j)}$). Let F (respectively, G) denote the Wronskian of the functions $f_1, \ldots, f_N$ (respectively, $g_1, \ldots, g_N$). We then have $F(z) = \det(\mathbf{v}_0(z), \ldots, \mathbf{v}_{N-1}(z))$ and $G(z) = \det(\mathbf{u}_0(z), \ldots, \mathbf{u}_{N-1}(z))$.

Let $\epsilon > 0$ be given. By the hypothesis, there is a number $R_\epsilon > 0$ such that

$$|\langle \mathbf{v}_k(z), \mathbf{u}_l(z) \rangle| = |f_1^{(k)}(z)\overline{g}_1^{(l)}(z) + \cdots + f_N^{(k)}(z)\overline{g}_N^{(l)}(z)| \leq \epsilon,$$

for $|z| > R_\epsilon$ and all $0 \leq k, l \leq N-1$. Using Lemma 3, we conclude that $|F(z)G(z)| \leq (\epsilon\sqrt{N})^N$ for all such z. This implies that the entire function $F \cdot G$ vanishes at infinity. It follows that either $F \equiv 0$ or $G \equiv 0$. Without loss of generality, we may assume that $F \equiv 0$, which implies that the functions $f_1, \ldots, f_N$ are linearly dependent since they are entire functions. (Note that without certain additional assumptions, the vanishing of the Wronskian does not imply linear dependence.) We have now reached a contraction. □

Combining Theorem 2 and Lemma 1 we conclude

Theorem 4. *Let φ be a bounded function on $\mathbb{C}$. If T_φ has finite rank on $\mathcal{F}^2$, then $\varphi = 0$ almost everywhere.*

References

[1] Wolfram Bauer and Trieu Le, *Algebraic properties and the finite rank problem for Toeplitz operators on the Segal-Bargmann space*, J. Funct. Anal. **261** (2011), 2617–2640.

[2] Alexander Borichev and Grigori Rozenblum, *The finite rank theorem for Toeplitz operators on the Fock space*, Journal of Geometric Analysis (2013), 1–10.

[3] S. M. Grudsky and N. L. Vasilevski, *Toeplitz operators on the Fock space: radial component effects*, Integral Equations Operator Theory **44** (2002), no. 1, 10–37. MR 1913421 (2003g:47049)

[4] Daniel H. Luecking, *Finite rank Toeplitz operators on the Bergman space*, Proc. Amer. Math. Soc. **136** (2008), no. 5, 1717–1723.

[5] Grigori Rozenblum, *Finite rank Bargmann-Toeplitz operators with non-compactly supported symbols*, Bull. Math. Sci. **2** (2012), no. 2, 331–341. MR 2994207

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