Hilbert–Schmidt Hankel operators over complete Reinhardt domains

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Abstract. Let $\mathcal{R}$ be an arbitrary bounded complete Reinhardt domain in $\mathbb{C}^n$. We show that for $n \geq 2$, if a Hankel operator with an antiholomorphic symbol is Hilbert–Schmidt on the Bergman space $A^2(\mathcal{R})$, then it must equal zero. This fact has previously been proved only for strongly pseudoconvex domains or for a certain class of pseudoconvex domains.

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1. Introduction

We denote by dV the Lebesgue volume measure on $\mathbb{C}^n$. Let $\Omega \subseteq \mathbb{C}^n$ be a bounded domain. The Bergman space $A^2(\Omega)$ consists of all holomorphic functions on Ω that are square integrable with respect to dV. It is well known that $A^2(\Omega)$ is a closed subspace of $L^2(\Omega) = L^2(\Omega, dV)$. Let P denote the orthogonal projection from $L^2(\Omega)$ onto $A^2(\Omega)$. For any function $\varphi \in L^2(\Omega)$, the Hankel operator $H_\varphi : A^2(\Omega) \to L^2(\Omega) \ominus A^2(\Omega)$ is defined by $H_\varphi h = (I - P)(\varphi h)$. When φ is bounded, it is clear that H_φ is a bounded operator. If φ is not bounded, H_φ is a densely defined operator and it may be unbounded. Hankel operators have been of interest to researchers in several complex variables due to their connection with the $\bar{\partial}$-Neumann operator. When Ω is a bounded pseudoconvex domain, Kohn's formula [5, Theorem 4.4.5] gives $I - P = \bar{\partial}^* N_{(0,1)} \bar{\partial}$, where $N_{(0,1)}$ is the $\bar{\partial}$-Neumann operator on $(0,1)$-forms, which is the bounded inverse of the complex Laplacian $\bar{\partial}\,\bar{\partial}^* + \bar{\partial}^* \bar{\partial}$ (see [5, Chapter 4] for more details). This implies that when φ is C^1 up to the boundary, we have $H_\varphi h = \bar{\partial}^* N_{(0,1)}(h\bar{\partial}\varphi)$ for all $h \in A^2(\Omega)$. In this paper, we are particularly interested in Hankel operators with antiholomorphic symbols, that is, $\varphi = \bar{f}$ for some f in $A^2(\Omega)$.

There is a vast literature on the studies of boundedness, compactness, and Schatten-class membership of Hankel operators on various domains in $\mathbb{C}^n$. In this paper, we would like to investigate Hilbert–Schmidt Hankel operators with anti-holomorphic symbols.

Recall that a linear operator $S : \mathcal{H}_1 \to \mathcal{H}_2$ between two Hilbert spaces is called a Hilbert–Schmidt operator if

$$\|S\|_{\mathrm{HS}}^2 = \sum_m \|Se_m\|^2 < \infty$$

for some orthonormal basis $\{e_m\}$ of $\mathcal{H}_1$. It is well known that the quantity $\|S\|_{\mathrm{HS}}^2$ defined above is independent of the choice of the orthonormal basis.

In the case of dimension one, Arazy et. al. [2] proved the following remarkable formula:

$$\|H_{\bar{f}}\|_{\mathrm{HS}}^2 = \frac{1}{\pi}\int_{\Omega} |f'(z)|^2 dV(z).$$

This shows that $H_{\bar{f}}$ is Hilbert–Schmidt if and only if f belongs to the Dirichlet space over Ω. The formula remains valid in more general settings and we refer the interested reader to [2] for more details.

In higher dimensions ($n \geq 2$), the situation is quite different. Zhu [11] showed that when $\Omega = \mathbb{B}_n$, the unit ball, $H_{\bar{f}}$ is a Hilbert–Schmidt operator if and only if f is a constant function. This result and more generally, the characterization of Schatten-class membership of $H_{\bar{f}}$ on various domains have been investigated by many researchers. Most results were first obtained for the unit ball in $\mathbb{C}^n$, then generalized to strongly pseudoconvex domains. Smoothness of the boundaries is usually assumed in the hypotheses. See, just to list a few, [1, 6, 7, 8, 10, 12] and the references therein. In [6], the authors considered smoothly pseudoconvex domains of finite type in $\mathbb{C}^2$. Most of the above papers used deep results and sophisticated techniques in the theory of several complex variables. In [9], Schneider reproved Zhu's aforementioned result on Hilbert–Schmidt Hankel operators using a more elementary approach via direct computations. Because of the symmetry of the ball, such computations can be carried out explicitly. In their recent paper [4], Çelik and Zeytuncu investigated the same problem when Ω is a complex ellipsoid in $\mathbb{C}^n$ given by

$$\Omega = \{(z_1, \ldots, z_n) \in \mathbb{C}^n : |z_1|^{r_1} + \cdots + |z_n|^{r_n} < 1\},$$

where $r_1, \ldots, r_n$ are positive real numbers. Motivated by their work, in this paper we generalize their result to bounded complete Reinhardt domains in $\mathbb{C}^n$. We recall the definition of complete Reinhardt domains.

Definition 1.1. A domain Ω is a complete Reinhardt domain if for any point $(w_1, \ldots, w_n)$ in Ω and any complex numbers $\zeta_1, \ldots, \zeta_n$ in the closed the unit disk, the point $(\zeta_1 w_1, \ldots, \zeta_n w_n)$ belongs to Ω.

The radial image (also called the base) D of Ω is a subset of $\mathbb{R}_+^n$ (here $\mathbb{R}_+ = [0, \infty)$) defined by

$$D = \{(|w_1|, \ldots, |w_n|) \in \mathbb{R}_+^n : (w_1, \ldots, w_n) \text{ belongs to } \Omega)\}.$$

We now state the main result in this paper.

Theorem 1.2. *Let Ω be a bounded complete Reinhardt domain in $\mathbb{C}^n$ with $n \geq 2$ and let f be in $A^2(\Omega)$. If $H_{\bar{f}}$ is a Hilbert–Schmidt operator on $A^2(\Omega)$, then f must be a constant function.*

There are several advantages of our approach. Firstly, it is quite elementary. We do not use any deep results in the theory of several complex variables. The key step in the proof involves only real analysis. Secondly, our approach works for any bounded (not necessarily complete) Reinhardt domain provided that the set of all monomials forms a complete orthogonal set in the Bergman space. The convexity of the domain or regularity of the boundary does not play any role in our analysis. Thirdly, our result remains valid when the Lebesgue measure dV is replaced by a positive Borel measure on Ω, again provided that the monomials still form a complete orthogonal set in the corresponding Bergman space. On the other hand, there are obvious drawbacks of our method. Since we make heavy use of the orthogonality and density of the monomials, we are not able to carry out our work on non-Reinhardt domains. In addition, our approach does not seem to work in the study of Schatten-class membership of Hankel operators.

Remark 1.3. The author has recently learned that in their latest preprint [3], Çelik and Zeytuncu obtained Theorem 1.2 under an additional assumption that Ω be pseudoconvex in $\mathbb{C}^2$.

Remark 1.4. Our approach relies heavily on the boundedness of the domains. In fact, Theorem 1.2 may fail if Ω is unbounded. Çelik and Zeytuncu [3] gave an example of an unbounded complete Reinhardt domain $\Omega \subset \mathbb{C}^2$ for which $A^2(\Omega)$ has infinite dimension and $H_{\overline{z_1 z_2}}$ is a Hilbert–Schmidt operator.

2. Hankel operators with anti-holomorphic symbols

Throughout this section, we assume that Ω is a bounded complete Reinhardt domain. We denote by $\mathbb{Z}_+$ the set of non-negative integers $\{0, 1, \ldots\}$. For any index $\gamma = (\gamma_1, \ldots, \gamma_n) \in \mathbb{Z}_+^n$ and any $z = (z_1, \ldots, z_n)$ we write $|\gamma| = \gamma_1 + \cdots + \gamma_n$ and $z^\gamma = z_1^{\gamma_1} \cdots z_n^{\gamma_n}$ (with the convention $0^0 = 1$).

Define $c_\gamma = \|z^\gamma\|_{A^2(\Omega)} = (\int_\Omega |z^\gamma|^2 \, dV(z))^{1/2}$. Since Ω is a bounded complete Reinhardt domain, it is known that the set $\left\{e_\gamma(z) = \dfrac{z^\gamma}{c_\gamma} : \gamma \in \mathbb{Z}_+^n\right\}$ is an orthonormal basis for $A^2(\Omega)$.

Now write $\Omega = \mathbb{T}^n \times D$, where D is the radial image of Ω. Using polar coordinates $z_j = t_j e^{i\theta_j}$ with $(\theta_1, \ldots, \theta_n) \in [0, 2\pi)^n$ and $(t_1, \ldots, t_n) \in D$, we obtain

$$c_\gamma^2 = \int_\Omega |z^\gamma|^2 \, dV(z) = \int_D t_1^{2\gamma_1} \cdots t_n^{2\gamma_n} \, d\nu(t) = \int_D t^{2\gamma} \, d\nu(t).$$

Here $d\nu(t) = (2\pi)^n t_1 \cdots t_n \, dt_1 \cdots dt_n$.

To simplify the notation, we set $c_\beta = \infty$ for any $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{Z}^n$ that contains a negative component. We shall use $\sum_\alpha$ to denote the infinite series with α running over $\mathbb{Z}^n_+$.

Lemma 2.1. *Let $f(z) = \sum_\alpha f_\alpha z^\alpha$ be a function in $A^2(\Omega)$. For any $\gamma \in \mathbb{Z}^n_+$ we have*

$$\|H_{\bar{f}} e_\gamma\|^2 = \sum_\alpha |f_\alpha|^2 \Big(\frac{c^2_{\alpha+\gamma}}{c^2_\alpha} - \frac{c^2_\alpha}{c^2_{\gamma-\alpha}} \Big) = \sum_\alpha |f_\alpha|^2 \, \|H_{\bar{z}^\alpha} e_\gamma\|^2.$$

As a result, we obtain

$$\|H_{\bar{f}}\|^2_{\mathrm{HS}} = \sum_\alpha |f_\alpha|^2 \Big\{ \sum_\gamma \Big(\frac{c^2_{\alpha+\gamma}}{c^2_\alpha} - \frac{c^2_\alpha}{c^2_{\gamma-\alpha}} \Big) \Big\} = \sum_\alpha |f_\alpha|^2 \, \|H_{\bar{z}^\alpha}\|^2_{\mathrm{HS}}.$$

Note that we allow the case where one of the sides, and hence both sides, are infinite.

Proof. The lemma is well known but for completeness we provide here the calculations. Note that for any α and β in $\mathbb{Z}^n_+$, we have

$$z^\alpha e_\beta(z) = c_\beta^{-1} \, z^\alpha z^\beta = c_\beta^{-1} \, z^{\beta+\alpha} = (c_{\beta+\alpha}/c_\beta) \, e_{\beta+\alpha}(z).$$

It then follows that

$$\begin{aligned} P(\bar{z}^\alpha e_\gamma) &= \sum_\beta \langle P(\bar{z}^\alpha e_\gamma) \, e_\beta \rangle e_\beta = \sum_\beta \langle e_\gamma, z^\alpha e_\beta \rangle \, e_\beta \\ &= \sum_\beta (c_{\beta+\alpha}/c_\beta) \langle e_\gamma, e_{\alpha+\beta} \rangle \, e_\beta = \begin{cases} 0 & \text{if } \alpha \not\preceq \gamma \\ (c_\gamma / c_{\gamma-\alpha}) \, e_{\gamma-\alpha} & \text{if } \alpha \preceq \gamma \end{cases}. \end{aligned}$$

Here we write $\alpha \preceq \gamma$ if $\alpha_j \le \gamma_j$ for all $j = 1, \dots, n$. Now we have

$$\begin{aligned} \|H_{\bar{f}} e_\gamma\|^2 &= \|(I-P)(\bar{f} e_\gamma)\|^2 = \|\bar{f} e_\gamma\|^2 - \|P(\bar{f} e_\gamma)\|^2 \\ &= \|f e_\gamma\|^2 - \|P(\bar{f} e_\gamma)\|^2 = \Big\| \sum_\alpha f_\alpha \, (z^\alpha e_\gamma) \Big\|^2 - \Big\| \sum_\alpha \bar{f}_\alpha P(\bar{z}^\alpha e_\gamma) \Big\|^2 \\ &= \Big\| \sum_\alpha (f_\alpha c_{\alpha+\gamma}/c_\alpha) \, e_{\alpha+\gamma} \Big\|^2 - \Big\| \sum_{\alpha \preceq \gamma} (\bar{f}_\alpha c_\alpha / c_{\gamma-\alpha}) \, e_{\gamma-\alpha} \Big\|^2 \\ &= \sum_\alpha |f_\alpha|^2 c^2_{\alpha+\gamma}/c^2_\alpha - \sum_{\alpha \preceq \gamma} |f_\alpha|^2 c^2_\alpha / c^2_{\gamma-\alpha} \\ &= \sum_\alpha |f_\alpha|^2 \Big(\frac{c^2_{\alpha+\gamma}}{c^2_\alpha} - \frac{c^2_\alpha}{c^2_{\gamma-\alpha}} \Big) = \sum_\alpha |f_\alpha|^2 \, \|H_{\bar{z}^\alpha} e_\gamma\|^2. \end{aligned}$$

Recall that $c_{\gamma-\alpha} = \infty$ if $\gamma - \alpha$ contains a negative component. Now summing over all multi-indices γ in $\mathbb{Z}^n_+$ and changing the order of summation, we obtain the required formula for the Hilbert–Schmidt norms. □

Remark 2.2. The quantity $\|H_{\bar{z}^\alpha}\|^2_{\mathrm{HS}}$ was denoted by S_α in [4].

The following lemma provides a lower estimate for the Hilbert–Schmidt norm of the Hankel operator $H_{\bar{z}^\alpha}$.

Lemma 2.3. *Let α be in $\mathbb{Z}_+^n$ with $|\alpha| > 0$. For any positive integer K, we have*

$$\|H_{\bar{z}^\alpha}\|_{\mathrm{HS}}^2 \geq \sum_{|\gamma|=K} \liminf_{s\to\infty} \frac{c_{s\gamma+\alpha}^2}{c_{s\gamma}^2}.$$

Proof. For any integer $N \geq 1$, using Lemma 2.1, we have the estimate

$$\begin{aligned}
\|H_{\bar{z}^\alpha}\|_{\mathrm{HS}}^2 &\geq \sum_{|\gamma|\leq N} \|H_{\bar{z}^\alpha} e_\gamma\|^2 = \sum_{|\gamma|\leq N} \Big(\frac{c_{\gamma+\alpha}^2}{c_\gamma^2} - \frac{c_\gamma^2}{c_{\gamma-\alpha}^2} \Big) \\
&= \sum_{|\gamma|\leq N} \frac{c_{\gamma+\alpha}^2}{c_\gamma^2} - \sum_{|\beta|+|\alpha|\leq N} \frac{c_{\beta+\alpha}^2}{c_\beta^2} \\
&\qquad \text{(by the change of index } \gamma = \beta + \alpha) \\
&= \sum_{N-|\alpha|<|\gamma|\leq N} \frac{c_{\gamma+\alpha}^2}{c_\gamma^2} \geq \sum_{|\gamma|=N} \frac{c_{\gamma+\alpha}^2}{c_\gamma^2}. \qquad (2.1)
\end{aligned}$$

Let s be a positive integer. Setting $N = sK$, we have

$$\|H_{\bar{z}^\alpha}\|_{\mathrm{HS}}^2 \geq \sum_{|\gamma|=sK} \frac{c_{\gamma+\alpha}^2}{c_\gamma^2} \geq \sum_{|\gamma|=K} \frac{c_{s\gamma+\alpha}^2}{c_{s\gamma}^2}.$$

Taking limit as $s \to \infty$, we obtain the required inequality. □

Remark 2.4. Inequality (2.1) was proved in [4]. For the reader's convenience, we have provided a short proof.

The following result is the most crucial estimate in our approach. It gives a lower bound for the limits that appeared in Lemma 2.3.

Proposition 2.5. *Let D be a bounded open subset in $\mathbb{R}_+^n$. Suppose μ is a positive Borel measure on D such that for any $c > 0$, if $D \cap (c,\infty)^n$ is nonempty, then $\mu(D \cap (c,\infty)^n) > 0$.*

Let $0 < r < 1$ be given. Then there exists $d > 0$ such that for any $\gamma \in \mathbb{Z}_+^n \backslash \{0\}$ satisfying

$$\frac{\gamma_j}{|\gamma|} \geq r \text{ for all } j = 1, \ldots, n, \qquad (2.2)$$

and any $\alpha \in \mathbb{Z}_+^n$ we have the inequality

$$\liminf_{s\to\infty} \frac{\int_D t^{s\gamma+\alpha}\, d\mu(t)}{\int_D t^{s\gamma}\, d\mu(t)} \geq d^{|\alpha|}.$$

Remark 2.6. If $d\mu(t) = g(t)\, dt_1 \cdots dt_n$, where g is positive on $D \cap (0,\infty)^n$, then μ satisfies the hypothesis of the proposition. In the proof of Theorem 1.2, we use $g(t) = (2\pi)^n \cdot t_1 \cdots t_n$.

Proof. Since D is bounded, WLOG, we may assume that $D \subseteq [0,1]^n$. Since D is relatively open in $\mathbb{R}_+^n$, there is a number $0 < b < 1$ such that the set $D \cap (b,\infty)^n$ is not empty. Choose any positive number d such that $d < b^{1/r}$.

We shall show that d satisfies the conclusion of the proposition. Note that d depends only on r and D.

Let α, γ be in $\mathbb{Z}_+^n$ such that γ satisfies the condition (2.2). Put $W = D \cap (d, \infty)^n$. For any positive integer s, we have

$$\begin{aligned}\int_D t^{s\gamma+\alpha}\, d\mu(t) \geq \int_W t^{s\gamma+\alpha}\, d\mu(t) &\geq d^{|\alpha|} \int_W t^{s\gamma}\, d\mu(t) \\ &= d^{|\alpha|}\Big(\int_D t^{s\gamma}\, d\mu(t) - \int_{D\backslash W} t^{s\gamma}\, d\mu(t)\Big).\end{aligned}$$

It follows that

$$\frac{\int_D t^{s\gamma+\alpha}\, d\mu(t)}{\int_D t^{s\gamma}\, d\mu(t)} \geq d^{|\alpha|}\Big(1 - \frac{\int_{D\backslash W} t^{s\gamma}\, d\mu(t)}{\int_D t^{s\gamma}\, d\mu(t)}\Big). \tag{2.3}$$

Now for $t \in D\backslash W \subseteq [0,1]^n \backslash (d, \infty)^n$, there is a j between 1 and n such that $t_j \leq d$. As a result,

$$t^\gamma = t_1^{\gamma_1} \cdots t_n^{\gamma_n} \leq d^{\gamma_j} \leq d^{r|\gamma|}.$$

The second inequality follows from the facts that $d < b^{1/r} < 1$ and that $\gamma_j \geq r|\gamma|$. We then have

$$\int_{D\backslash W} t^{s\gamma}\, d\mu(t) \leq (d^r)^{s|\gamma|} \mu(D\backslash W).$$

On the other hand, letting $V = D \cap (b, \infty)^n$, we obtain

$$\int_D t^{s\gamma}\, d\mu(t) \geq \int_V t^{s\gamma}\, d\mu(t) \geq b^{s|\gamma|}\mu(V).$$

Note that $\mu(V) > 0$ by the hypothesis on μ. Now inequality (2.3) implies

$$\begin{aligned}\frac{\int_D t^{s\gamma+\alpha}\, d\mu(t)}{\int_D t^{s\gamma}\, d\mu(t)} &\geq d^{|\alpha|}\Big(1 - \frac{(d^r)^{s|\gamma|}\mu(D\backslash W)}{b^{s|\gamma|}\mu(V)}\Big) \\ &= d^{|\alpha|}\Big\{1 - (d^r b^{-1})^{s|\gamma|} \cdot \frac{\mu(D\backslash W)}{\mu(V)}\Big\}.\end{aligned}$$

Since $d^r b^{-1} < 1$, taking limit as $s \to \infty$, we obtain the required inequality. $\square$

We are now ready to prove our main result, Theorem 1.2, which we restate here.

Theorem 2.7. *Let Ω be a bounded complete Reinhardt domain in $\mathbb{C}^n$ with $n \geq 2$ and let f be in $A^2(\Omega)$. If $H_{\bar{f}}$ is a Hilbert–Schmidt operator on $A^2(\Omega)$, then f must be a constant function.*

Proof. Write $f(z) = \sum_\alpha f_\alpha z^\alpha$. By Lemma 2.1, we have

$$\|H_{\bar{f}}\|_{\mathrm{HS}}^2 = \sum_\alpha |f_\alpha|^2 \|H_{\bar{z}^\alpha}\|_{\mathrm{HS}}^2.$$

If $\|H_{\bar{f}}\|_{\mathrm{HS}} < \infty$, then $|f_\alpha| \|H_{\bar{z}^\alpha}\|_{\mathrm{HS}} < \infty$ for all α. We shall show that whenever $|\alpha| > 0$, we have $\|H_{\bar{z}^\alpha}\|_{\mathrm{HS}} = \infty$, so as a result, $f_\alpha = 0$ for such α. This implies that f is a constant function.

Now let us fix an $\alpha \in \mathbb{Z}_+^n$ with $|\alpha| > 0$. For any positive integer M, Lemma 2.3 with $K = 2nM$ gives

$$\|H_{\bar{z}^\alpha}\|_{\mathrm{HS}}^2 \geq \sum_{|\gamma|=2nM} \liminf_{s\to\infty} \frac{c_{s\gamma+\alpha}^2}{c_{s\gamma}^2}. \tag{2.4}$$

For $j = 1, \ldots, n-1$, choose γ_j to be an any positive integer satisfying $M \leq \gamma_j \leq 2M$. Put $\gamma_n = 2nM - (\gamma_1 + \cdots + \gamma_{n-1})$. Then we have $|\gamma| = 2nM$ and

$$\gamma_n \geq 2nM - (n-1)(2M) = 2M > M.$$

It follows that $\gamma_j/|\gamma| \geq M/(2nM) = 1/(2n)$ for any $1 \leq j \leq n$. Note that there are M^{n-1} choices for such γ. Applying Proposition 2.5 with $r = 1/(2n)$ and $d\mu(t) = d\nu(t) = (2\pi)^n t_1 \cdots t_n \, dt_1 \cdots dt_n$, we obtain a constant $d > 0$ (dependent only on n and D) such that for any such γ,

$$\liminf_{s\to\infty} \frac{c_{s\gamma+\alpha}^2}{c_{s\gamma}^2} = \liminf_{s\to\infty} \frac{\int_D t^{2s\gamma+2\alpha} \, d\nu(t)}{\int_D t^{2s\gamma} \, d\nu(t)} \geq d^{2|\alpha|}.$$

Inequality (2.4) then implies

$$\|H_{\bar{z}^\alpha}\|_{\mathrm{HS}}^2 \geq M^{n-1} d^{2|\alpha|}.$$

Since $n > 1$ and M was arbitrary, we conclude that $\|H_{\bar{z}^\alpha}\|_{\mathrm{HS}} = \infty$, which is what we wished to show. $\square$

Remark 2.8. The only reason we have assumed Ω to be a complete Reinhardt domain is so that the set of monomials $\{z^\gamma/c_\gamma : \gamma \in \mathbb{Z}_+^n\}$ forms an orthonormal basis for $A^2(\Omega)$. This can occur even when Ω is just a bounded Reinhardt domain. In such a case, the conclusion of Theorem 2.7 remains valid. An example of such a domain is

$$\Omega = \big\{z \in \mathbb{C}^n : 1/2 < |z| < 1\big\}.$$

Since any function in $A^2(\Omega)$ extends to a holomorphic function on the entire open unit ball by Hartogs's extension theorem, it follows that holomorphic polynomials are dense in $A^2(\Omega)$.

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