ESSENTIAL NORMS OF WEIGHTED COMPOSITION OPERATORS ON $N_p$-SPACES IN THE BALL

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Abstract. We obtain the estimate for the essential norms of weighted composition operators acting from $N_p$-spaces into the Beurling/Bergman-type spaces $A^{-q}$ of holomorphic functions in the unit ball. Our results contain the results in the unit disk as particular cases.

1. Introduction

1.1. Notation and definitions. Throughout the paper, $n$ is a positive integer. If $z = (z_1, \ldots, z_n), \zeta = (\zeta_1, \ldots, \zeta_n) \in \mathbb{C}^n$, then we define the inner product $\langle z, \zeta \rangle = z_1\bar{\zeta}_1 + \cdots + z_n\bar{\zeta}_n$ and norm $|z| = (z_1\bar{z}_1 + \cdots + z_n\bar{z}_n)^{1/2}$. Let $B$ be the unit ball in $\mathbb{C}^n$ under this norm. Let $O(B)$ denote the space of holomorphic functions on $B$ with the compact-open topology, and $H^\infty(B)$ denote the Banach space of bounded holomorphic functions in $B$ with the norm $\|f\|_\infty = \sup_{z \in B} |f(z)|$.

If $X$ and $Y$ are two topological vector spaces, then the notation $X \hookrightarrow Y$ means the continuous embedding of $X$ into $Y$.

Let $p > 0$, the Beurling-type space (sometimes also called the Bergman-type space) $A^{-p}(B)$ in the unit ball is defined as

$$A^{-p} = A^{-p}(B) := \left\{ f \in O(B) : |f|_p = \sup_{z \in B} |f(z)|(1 - |z|^2)^p < \infty \right\}.$$

Moreover, the Bergman space $A^2(B)$ is defined as

$$A^2 = A^2(B) := \left\{ f \in O(B) : \|f\|_{A^2} = \left( \int_B |f(z)|^2 dV(z) \right)^{1/2} < \infty \right\},$$

where $dV$ is the normalized Lebesgue volume measure on $B$ so that $V(B) = 1$.

For a holomorphic self-mapping $\varphi$ of $B$ and a holomorphic function $u : B \to \mathbb{C}$, the linear operator $W_{u,\varphi} : O(B) \to O(B)$

$$W_{u,\varphi}(f)(z) = u(z) \cdot (f \circ \varphi(z)), f \in O(B), z \in B,$$

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is called the weighted composition operator with symbols \( u \) and \( \varphi \). Observe that \( W_{u,\varphi}(f) = M_u C_\varphi(f) \), where \( M_u(f) = uf \), is the multiplication operator with symbol \( u \), and \( C_\varphi(f) = f \circ \varphi \) is the composition operator with symbol \( \varphi \). If \( u \) is identically 1, then \( W_{u,\varphi} = C_\varphi \), and if \( \varphi \) is the identity, then \( W_{u,\varphi} = M_u \).

Composition operators and weighted composition operators acting on spaces of holomorphic functions in the unit disk \( \mathbb{D} \) of the complex plane have been studied quite well. We refer the readers to the monographs \([1, 8]\) for detailed information. Composition operators on \( A^{-p}(\mathbb{D}) \) have also been intensively studied (see, e.g., \([3, \text{Chapters 4 and 5}]\) and references therein).

1.2. \( N_p \)-spaces in the unit ball. Palmberg \([6]\) introduced \( N_p \)-spaces (for \( p > 0 \)) in the unit disk in the complex plane and studied their composition operators. In \([4]\), the first two authors defined and studied \( N_p \)-spaces in the ball \( \mathbb{B} \) together with weighted composition operators acting on them. We recall here the definition:

\[
N_p = N_p(\mathbb{B}) := \left\{ f \in \mathcal{O}(\mathbb{B}) : \| f \|_p = \sup_{a \in \mathbb{B}} \left( \int_{\mathbb{B}} |f(z)|^p (1 - |\Phi_a(z)|^2) dV(z) \right)^{1/2} < \infty \right\},
\]

where \( \Phi_a \in \text{Aut}(\mathbb{B}) \) is the automorphism of \( \mathbb{B} \) which permutes \( a \) and 0 (see, e.g., \([7, \text{Section 2.2}]\)). From this definition, it is clear that \( A^2 \subset N_p \) for all \( p > 0 \).

Several important properties of the \( N_p \)-spaces, and of the weighted composition operators from \( N_p \)-spaces to the spaces \( A^{-q} \) have been characterized in \([4]\). We list here the main results from \([4]\) for the reader’s convenience.

**Theorem 1.1.** The following statements hold:

(a) For \( p > q > 0 \), we have \( H^\infty \hookrightarrow N_q \hookrightarrow N_p \hookrightarrow A^{-\frac{n+1}{2}} \), where the last embedding is given by \( |f|_p^{n+1} \leq \frac{2p+1}{2p} \| f \|_p \), \( \forall f \in N_p \).

(b) For \( p > 0 \), if \( p > 2k - 1, k \in (0, \frac{n+1}{2}] \), then \( A^{-k} \hookrightarrow N_p \). In particular, when \( p > n \), all \( N_p = A^{-\frac{n+1}{2}} \).

(c) \( N_p \) is a Banach space with the norm \( \| \cdot \|_p \), and moreover, its norm topology is stronger than the compact-open topology.

(d) For \( 0 < p < \infty \), \( \mathcal{B} \hookrightarrow N_p \), where \( \mathcal{B} \) is the Bloch space in \( \mathbb{B} \).

**Theorem 1.2.** Let \( \varphi : \mathbb{B} \to \mathbb{B} \) be a holomorphic self-mapping, \( u : \mathbb{B} \to \mathbb{C} \) a holomorphic function, and \( p, q > 0 \). Then the weighted composition operator \( W_{u,\varphi} : N_p \to A^{-q} \)

(1) is bounded if and only if

\[
\sup_{z \in \mathbb{B}} \frac{|u(z)|(1 - |z|^2)^q}{(1 - |\varphi(z)|^2)^{\frac{n+1}{2}}} < \infty;
\]
(2) is compact if and only if
\[
\lim_{r \to 1^-} \sup_{|ϕ(z)| > r} \frac{|u(z)|(1 - |z|^2)^q}{(1 - |ϕ(z)|^2)^{n+1}} = 0.
\]

The aim of this paper is to obtain the estimate for the essential norms of weighted composition operators acting from \(N_p\)-spaces into the spaces \(A^{-q}\).

It should be noted that for the one-dimensional case, the essential norm of \(W_{u,ϕ}\) was considered in [9, Theorem 4]. However, it seems that there is a gap in the proof, namely, the author of [9] mistakenly used the Hahn-Banach Theorem to extend \(γ \in (N_p)'\) to \(Γ \in (H^\infty)'\). We solve this problem by a quite different approach by making use of weakly convergent sequences in \(A^2\). Our results fill the gap in [9] and moreover, contain the results in the unit disk as particular cases.

To formulate our results, we use the notation \(a \simeq b\) which means that there exist positive constants \(C_1\) and \(C_2\) independent of \(a\) and \(b\) such that \(C_1 b \leq a \leq C_2 b\).

2. Essential Norm

In this section, we study the essential norm of the weighted composition operator \(W_{u,ϕ}: N_p \to A^{-q}\). Let us denote by \(K := K(N_p, A^{-q})\) the set of all compact operators acting from \(N_p\) into \(A^{-q}\). Then the essential norm of \(W_{u,ϕ}\) is defined as follows:
\[
∥W_{u,ϕ}∥_e = \inf_{K \in K} \{∥W_{u,ϕ} - K∥\}.
\]

Obviously, the essential norm of a compact operator is zero.

Note that by using the standard argument, it can be shown that a composition operator \(C_ϕ: N_p \to N_p\) is compact if and only if for any bounded sequence \(\{f_m\} \subset N_p\) converging to zero uniformly on every compact subset of \(\mathbb{B}\), the sequence \(\{∥f_m ◦ ϕ∥_p\}\) converges to zero as \(m \to \infty\).

2.1. Upper bound of the essential norm. We need some auxiliary results.

Lemma 2.1. Suppose \(ϕ\) is a self-mapping of \(\mathbb{B}\) such that \(∥ϕ∥_\infty < 1\) and \(u \in N_p\). Then the weighted composition operator \(W_{u,ϕ}: N_p \to N_p\) is compact.

Proof. Let \(r = ∥ϕ∥_\infty\). Take an arbitrary \(f \in N_p\). We then have
\[
∥W_{u,ϕ}f∥_p = ∥u \cdot (f ◦ ϕ)∥_p \leq ∥f ◦ ϕ∥_\infty∥u∥_p \leq (\sup_{∥z:|z|\leq r} |f(z)|)∥u∥_p < ∞.
\]

This shows that \(W_{u,ϕ}\) maps \(N_p\) into itself.
Now suppose that \( \{f_m\} \) is a bounded sequence in \( \mathcal{N}_p \) that converges to zero uniformly on every compact subset of \( \mathbb{B} \). Applying the above estimate with \( f = f_m \), we have
\[
\|W_{u,\varphi} f_m\|_p \leq \left( \sup_{\{z : |z| \leq r\}} |f_m(z)| \right) \|u\|_p.
\]
Since the set \( \{z : |z| \leq r\} \) is compact, the right-hand side of the last quantity converges to 0 as \( m \to \infty \), hence so does the sequence \( \{\|W_{u,\varphi} f_m\|_p\} \). This means that \( W_{u,\varphi} \) is compact.

Recall that the pseudo-hyperbolic metric in the ball is defined as
\[
\rho(z, w) = |\Phi_w(z)|, \quad z, w \in \mathbb{B}.
\]
It is a true metric on \( \mathbb{B} \) (see, e.g., [2]). Also it is easy to verify, in particular, that \( \rho(0, w) = |w| \) and \( \rho(\Phi_w(z), w) = |z| \).

The following result of [5] will be used in the sequel.

**Lemma 2.2.** ([5, Lemma 2.3]) For \( f \in \mathcal{N}_p \) and \( z, w \in \mathbb{B} \), we have
\[
|f(z) - f(w)| \leq A\|f\|_p \max \left\{ \frac{1}{(1 - |z|^2)^{\frac{n+1}{2}}}, \frac{1}{(1 - |w|^2)^{\frac{n+1}{2}}} \right\} \rho(z, w).
\]
Here \( A = \frac{6^{n+1} \cdot 2^{p+1}(3 + 2\sqrt{3})}{3^{2n} \sqrt{n}} \).

In the case \( z \) and \( w \) are multiple of each other, the above estimate can be simplified as in the following lemma.

**Lemma 2.3.** Let \( f \) be in \( \mathcal{N}_p \). For any \( a \in \mathbb{B} \) and any \( \kappa \in (0, 1) \), we have
\[
|f(a) - f(\kappa a)| \leq A(1 - \kappa)\|f\|_p \max \left\{ \frac{1}{(1 - |a|^2)^{\frac{n+1}{2}}}, \frac{1}{(1 - |\kappa a|^2)^{\frac{n+1}{2}}} \right\} \rho(a, \kappa a).
\]
Consequently, for any \( 0 < r < 1 \), we have
\[
\sup_{\|a\| \leq r} |f(a) - f(\kappa a)| \leq \frac{Ar(1 - \kappa)\|f\|_p}{(1 - r^2)^{\frac{n+1}{2}}}.
\]
Here \( A \) is the constant from Lemma 2.2.

**Proof.** Lemma 2.2 shows that
\[
|f(a) - f(\kappa a)| \leq A\|f\|_p \max \left\{ \frac{1}{(1 - |a|^2)^{\frac{n+1}{2}}}, \frac{1}{(1 - |\kappa a|^2)^{\frac{n+1}{2}}} \right\} \rho(a, \kappa a).
\]
The well-known formula
\[
1 - |\rho(a, \kappa a)|^2 = 1 - |\Phi_a(\kappa a)|^2 = \frac{(1 - |a|^2)(1 - |\kappa a|^2)}{(1 - \langle a, \kappa a \rangle)^2}
\]

together with simple calculations gives
\[
\rho(a, \kappa a) = \frac{(1 - \kappa)\|a\|}{1 - \kappa\|a\|^2} \leq 1.
\]
On the other hand, \((1 - |κa|^2)^{-(n+1)/2} \leq (1 - |a|^2)^{-(n+1)/2}\). The inequalities in (2.1) now follow.

If \(|a| \leq r\), then \(1 - κ|a|^2 ≥ 1 - r^2\). Taking supremum of (2.1) in \(a\) yields (2.2). \(\square\)

Now we formulate and prove an estimate for the upper bound of the essential norm of \(W_{u,φ}\).

**Theorem 2.4.** Let \(p\) and \(q\) be two positive numbers. Let \(φ: \mathbb{B} \to \mathbb{B}\) be a holomorphic self-mapping and \(u: \mathbb{B} \to \mathbb{C}\) be a holomorphic function. Suppose that \(W_{u,φ}\) is a bounded operator acting from \(N_p\) to \(A^{-q}\). Then

\[
||W_{u,φ}||_e \leq A \lim_{r \to 1^-} \sup_{|φ(z)| > r} \frac{|u(z)|(1 - |z|^2)^q}{(1 - |φ(z)|^2)^{\frac{n+1}{2}}},
\]

where \(A\) is the constant from Lemma 2.2.

**Proof.** Since \(W_{u,φ}\) is bounded, we see that \(u\) belongs to \(A^{-q}\) and Theorem 1.2 shows that \(\lim_{r \to 1^-} \sup_{|φ(z)| > r} \frac{|u(z)|(1 - |z|^2)^q}{(1 - |φ(z)|^2)^{\frac{n+1}{2}}} \) exists and is a real number.

First we prove that for any \(r \in [0, 1),\)

\[
(2.3) \quad ||W_{u,φ}||_e \leq A \sup_{|φ(z)| > r} \frac{|u(z)|(1 - |z|^2)^q}{(1 - |φ(z)|^2)^{\frac{n+1}{2}}}.
\]

For each \(k \in \mathbb{N},\) set \(φ_k(z) = \frac{k}{k+1}z\) for all \(z \in \mathbb{B}\). By Lemma 2.1, \(C_{φ_k}\) is compact on \(N_p\), and hence \(W_{u,φ} \circ C_{φ_k}\) is compact acting from \(N_p\) into \(A^{-q}\). We then have for any \(k \in \mathbb{N},\)

\[
||W_{u,φ}||_e \leq ||W_{u,φ} - W_{u,φ} \circ C_{φ_k}|| = \sup_{||f||_p \leq 1} |(W_{u,φ} - W_{u,φ} \circ C_{φ_k})(f)|_q,
\]

which implies that

\[
(2.4) \quad ||W_{u,φ}||_e \leq \inf_{k \in \mathbb{N}} \left\{ \sup_{||f||_p \leq 1} |(W_{u,φ} - W_{u,φ} \circ C_{φ_k})(f)|_q \right\}.
\]

For \(f \in N_p\), we estimate

\[
| (W_{u,φ} - W_{u,φ} \circ C_{φ_k})(f) |_q
\]

\[
= \sup_{z \in \mathbb{B}} \left\{ |u(z)| \left| f(φ(z)) - f\left(\frac{k}{k+1}φ(z)\right) \right| (1 - |z|^2)^q \right\}
\]

\[
\leq \sup_{|φ(z)| > r} \left\{ |u(z)| \left| f(φ(z)) - f\left(\frac{k}{k+1}φ(z)\right) \right| (1 - |z|^2)^q \right\}
\]

\[
+ \sup_{|φ(z)| \leq r} \left\{ |u(z)| \left| f(φ(z)) - f\left(\frac{k}{k+1}φ(z)\right) \right| (1 - |z|^2)^q \right\}.
\]
On one hand, by (2.1),
\[
\sup_{|\varphi(z)| > r} \left\{ |u(z)| \left| f(\varphi(z)) - f\left(\frac{k}{k+1} \varphi(z)\right) \right| \right\} (1 - |z|^2)^q \\
\leq \left( \sup_{|\varphi(z)| > r} \frac{A|u(z)|(1 - |z|^2)^q}{(1 - |\varphi(z)|^2)^{\frac{p+1}{2}}} \right) \|f\|_p.
\]

On the other hand, by (2.2),
\[
\sup_{|\varphi(z)| \leq r} \left\{ |u(z)| \left| f(\varphi(z)) - f\left(\frac{k}{k+1} \varphi(z)\right) \right| \right\} (1 - |z|^2)^q \\
\leq \sup_{|\varphi(z)| \leq r} f(\varphi(z)) - f\left(\frac{k}{k+1} \varphi(z)\right) \cdot \sup_{z \in \mathbb{B}} \{ |u(z)|(1 - |z|^2)^q \} \\
\leq \left( \frac{Ar|u_q|}{(k+1)(1 - r^2)^{\frac{p+1}{2}}} \right) \|f\|_p.
\]

Therefore, if \(\|f\|_p \leq 1\), then
\[
| (W_{u,\varphi} - W_{u,\varphi} \circ C_{\varphi_k}) (f) |_q \\
\leq A \left( \sup_{|\varphi(z)| > r} \frac{|u(z)|(1 - |z|^2)^q}{(1 - |\varphi(z)|^2)^{\frac{p+1}{2}}} \right) + \frac{Ar|u_q|}{(k+1)(1 - r^2)^{\frac{p+1}{2}}}.
\]

It then follows that
\[
\inf_{k \in \mathbb{N}} \left\{ \sup_{\|f\|_p \leq 1} \| (W_{u,\varphi} - W_{u,\varphi} \circ C_{\varphi_k}) (f) \|_q \right\} \\
\leq \inf_{k \in \mathbb{N}} \left\{ A \left( \sup_{|\varphi(z)| > r} \frac{|u(z)|(1 - |z|^2)^q}{(1 - |\varphi(z)|^2)^{\frac{p+1}{2}}} \right) + \frac{Ar|u_q|}{(k+1)(1 - r^2)^{\frac{p+1}{2}}} \right\} \\
= A \sup_{|\varphi(z)| > r} \frac{|u(z)|(1 - |z|^2)^q}{(1 - |\varphi(z)|^2)^{\frac{p+1}{2}}}.
\]

Combining this and (2.4) we obtain (2.3).

Now letting \(r \to 1^-\) in (2.3), we arrive at the desired inequality
\[
\|W_{u,\varphi}\|_e \leq A \lim_{r \to 1^-} \sup_{|\varphi(z)| > r} \frac{|u(z)|(1 - |z|^2)^q}{(1 - |\varphi(z)|^2)^{\frac{p+1}{2}}}.
\]

2.2. Lower bound of the essential norm. We now discuss the estimation for the lower bound of \(\|W_{u,\varphi}\|_e\). We will make use of weakly convergent sequences in the Bergman space \(A^2\). The following lemma plays an important role.

**Lemma 2.5.** Suppose \( \{f_m\}_{m \geq 1} \subset A^2 \) is a sequence that converges weakly to zero in \(A^2\). Then \( \{f_m\}_{m \geq 1} \) converges weakly to zero in \(N_p\) as well.
Proof. Let $\Gamma \in (\mathcal{N}_p)'$ be a bounded linear functional on $\mathcal{N}_p$. Then
\[
\|\Gamma\|_{(A^2)'} = \sup_{f \in A^2} \frac{|\Gamma(f)|}{\|f\|_{A^2}} \leq \sup_{f \in A^2} \frac{|\Gamma(f)|}{\|f\|_p} \leq \sup_{f \in \mathcal{N}_p} \frac{|\Gamma(f)|}{\|f\|_p} = \|\Gamma\|_{(\mathcal{N}_p)'} ,
\]
which implies $\Gamma$ is also a bounded linear functional on $A^2$. The second and third inequalities follow from the fact that $\|f\|_p \leq \|f\|_{A^2}$ for any $f \in A^2$. Since $f_m \to 0$ weakly in $A^2$, we conclude that $\Gamma(f_m) \to 0$. Therefore, $f_m \to 0$ weakly in $\mathcal{N}_p$ as well. \hfill \square

For each $w \in \mathbb{B}$, set
\[
(2.5) \quad k_w(z) = \left( \frac{1 - |w|^2}{(1 - \langle z, w \rangle)^2} \right)^{\frac{n+1}{2}}, \quad z \in \mathbb{B}.
\]
Such $k_w$ is a normalized reproducing kernel function in the Bergman space $A^2$. By [4, Lemma 3.1], we have $k_w \in \mathcal{N}_p$ and $\sup_{w \in \mathbb{B}} \|k_w\|_p \leq 1$.

Also note that $k_w(w) = \left( \frac{1}{1 - |w|^2} \right)^{\frac{n+1}{2}}$, $\forall w \in \mathbb{B}$.

Corollary 2.6. Let $\{w_m\}_{m \in \mathbb{N}} \subset \mathbb{B}$ and $|w_m| \to 1$ as $m \to \infty$, then $\{k_{w_m}\}$ converges weakly to zero in $\mathcal{N}_p$.

Proof. It is well known that $k_{w_m} \to 0$ weakly in $A^2$ as $m \to \infty$. Indeed, for any $f \in A^2$, using the reproducing property, we have
\[
\langle f, k_{w_m} \rangle = (1 - |w_m|^2)^{(n+1)/2} f(w_m),
\]
which converges to zero as $m \to \infty$ by the remark after [10, Theorem 2.1]. The conclusion of the corollary follows immediately from Lemma 2.5. \hfill \square

Theorem 2.7. Let $p$ and $q$ be two positive numbers. Let $\varphi : \mathbb{B} \to \mathbb{B}$ be a holomorphic self-mapping and $u : \mathbb{B} \to \mathbb{C}$ be a holomorphic function. Suppose that $W_{u, \varphi}$ is a bounded operator acting from $\mathcal{N}_p$ to $A^{-q}$. Then
\[
\|W_{u, \varphi}\|_e \geq \lim_{r \to 1} \sup_{r < |\varphi(z)|} \frac{|u(z)|(1 - |z|^2)^q}{(1 - |\varphi(z)|^2)^{\frac{n+1}{2}}}.
\]

Proof. The case $\|\varphi\|_\infty < 1$ is obvious since the right hand side is zero. Now assume that $\|\varphi\|_\infty = 1$. For any $r \in (0, 1)$, the set $S_r := \{z \in \mathbb{B} : |\varphi(z)| > r\}$ is not empty. For each $z \in \mathbb{B}$, consider the probe function $k_{\varphi(z)}$ in (2.5). Then for any compact operator $Q \in \mathcal{K}$, we have
\[
\|W_{u, \varphi} - Q\| = \sup_{\|f\|_p \leq 1} |(W_{u, \varphi} - Q)(f)|_q \geq |(W_{u, \varphi} - Q)(k_{\varphi(z)})|_q \geq |W_{u, \varphi}(k_{\varphi(z)})|_q - |Q(k_{\varphi(z)})|_q \geq |u(z)|(1 - |z|^2)^q \left( \frac{1}{(1 - |\varphi(z)|^2)^{\frac{n+1}{2}}} - |Q(k_{\varphi(z)})|_q \right).
\]
which is equivalent to
\begin{equation}
\|W_{u,\varphi} - Q\| + |Q(k_{\varphi}(z))|_q \geq \frac{|u(z)|(1 - |z|^2)^q}{(1 - |\varphi(z)|^2)^{\frac{q+1}{2}}},
\end{equation}
Taking the supremum on $z$ over the set $S_r$ on both sides of (2.6) yields
\begin{equation}
\|W_{u,\varphi} - Q\| + \sup_{z \in S_r} |Q(k_{\varphi}(z))|_q \geq \sup_{z \in S_r} \frac{|u(z)|(1 - |z|^2)^q}{(1 - |\varphi(z)|^2)^{\frac{q+1}{2}}},
\end{equation}
which is
\begin{equation}
\|W_{u,\varphi} - Q\| + \sup_{|\varphi(z)| > r} |Q(k_{\varphi}(z))|_q \geq \sup_{|\varphi(z)| > r} \frac{|u(z)|(1 - |z|^2)^q}{(1 - |\varphi(z)|^2)^{\frac{q+1}{2}}}.\tag{2.7}
\end{equation}
Denote $H(r) = \sup_{|\varphi(z)| > r} |Q(k_{\varphi}(z))|_q$. Since $H(r)$ decreases as $r$ increases, $\lim_{r \to 1^{-}} H(r)$ exists. We claim that this limit is necessarily zero. For the purpose of obtaining a contradiction, assume that $\lim_{r \to 1^{-}} H(r) = L > 0$. Then there is a sequence \( \{z_m\} \subset \mathbb{B} \) satisfying $|\varphi(z_m)| \to 1$ as $m \to \infty$, and for each $m \in \mathbb{N}$,
\begin{equation}
|Q(k_{\varphi(z_m)})|_q > L/2.\tag{2.8}
\end{equation}
By Corollary 2.6, \( \{k_{\varphi(z_m)}\} \) converges weakly to zero in $\mathcal{N}_p$. Since $Q$ is compact, we have \( \{Q(k_{\varphi(z_m)})\}_q \) converges to zero as $m \to \infty$, which contradicts (2.8). Therefore, $\lim_{r \to 1^{-}} \sup_{|\varphi(z)| > r} |Q(k_{\varphi(z)})|_q = 0$.

Letting $r \to 1^{-}$ on both sides of (2.7), we conclude that for any compact operator $Q \in \mathcal{K}$,
\begin{equation}
\|W_{u,\varphi} - Q\| \geq \lim_{r \to 1^{-}} \sup_{|\varphi(z)| > r} \frac{|u(z)|(1 - |z|^2)^q}{(1 - |\varphi(z)|^2)^{\frac{q+1}{2}}}.\tag{2.9}
\end{equation}
From this, it follows that
\begin{equation}
\|W_{u,\varphi}\|_c = \inf_{Q \in \mathcal{K}} \{\|W_{u,\varphi} - Q\|\} \geq \lim_{r \to 1^{-}} \sup_{|\varphi(z)| > r} \frac{|u(z)|(1 - |z|^2)^q}{(1 - |\varphi(z)|^2)^{\frac{q+1}{2}}}. \tag*{\Box}
\end{equation}

In conclusion, combining Theorems 2.4 and 2.7, we obtain a full description of the essential norm of $W_{u,\varphi}$.

**Theorem 2.8.** Let $p$ and $q$ be two positive numbers. Let $\varphi : \mathbb{B} \to \mathbb{B}$ be a holomorphic self-mapping and $u : \mathbb{B} \to \mathbb{C}$ be a holomorphic function. Suppose that $W_{u,\varphi}$ is a bounded operator acting from $\mathcal{N}_p$ to $A^{-q}$. Then
\begin{equation}
\|W_{u,\varphi}\|_c \simeq \lim_{r \to 1^{-}} \sup_{|\varphi(z)| > r} \frac{|u(z)|(1 - |z|^2)^q}{(1 - |\varphi(z)|^2)^{\frac{q+1}{2}}}.
\end{equation}

Theorem 2.8 provides us a characterization of compact weighted composition operators from $\mathcal{N}_p$ to $A^{-q}$ as in Theorem 1.2(2).

**Corollary 2.9.** Suppose that $W_{u,\varphi}$ is a bounded operator acting from $\mathcal{N}_p$ to $A^{-q}$ as in Theorem 2.8. Then $W_{u,\varphi}$ is compact if and only if
\begin{equation}
\lim_{r \to 1^{-}} \sup_{|\varphi(z)| > r} \frac{|u(z)|(1 - |z|^2)^q}{(1 - |\varphi(z)|^2)^{\frac{q+1}{2}}} = 0.
\end{equation}
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