Invertible Composition Operators: The product of a composition operator with the adjoint of a composition operator.

John H. Clifford, Trieu Le and Alan Wiggins

Abstract. In this paper, we study the product of a composition operator C_φ with the adjoint of a composition operator C_ψ^* on the Hardy space $H^2(\mathbb{D})$. The order of the product gives rise to two different cases. We completely characterize when the operator $C_\varphi C_\psi^*$ is invertible, isometric, and unitary and when the operator $C_\psi^* C_\varphi$ is isometric and unitary . If one of the inducing maps φ or ψ is univalent, we completely characterize when $C_\psi^* C_\varphi$ is invertible.

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1. Introduction

Let $H^2 = H^2(\mathbb{D})$ be the set of all holomorphic functions on the open unit disc $\mathbb{D}$ having square summable complex coefficients. Let φ be a holomorphic self-map of $\mathbb{D}$. The composition operator C_φ induced by φ is defined by

$$C_\varphi f = f \circ \varphi \quad \text{for all} \quad f \in H^2.$$

A classical result of Littlewood [7] shows that C_φ is bounded. There are three excellent expositions on composition operators [6, 8, 11]. Cowen and MacCluer's book [6] is comprehensive, Martinez and Rosenthal's book [8] provides an enjoyable introductory treatment for operators on H^2, and Shapiro's book [11] is a wonderful introduction to composition operators.

This paper examines the invertibility of the products $C_\varphi C_\psi^*$ and $C_\psi^* C_\varphi$ on the Hardy space H^2 in terms of the inducing maps φ and ψ. The compactness of these two products has been previously studied in [3, 4, 5]. We take as our motivation two classical results in the theory: in 1969, H. J. Schwartz

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proved in his thesis [10] that a composition operator is invertible if and only if its inducing map is a disc automorphism, and in 1968, Eric Nordgren showed that a composition operator on H^2 is an isometry if and only if the inducing map is inner and fixes the origin [9].

In Section 3, we completely generalize these results for the product $C_\varphi C_\psi^*$. In short, $C_\varphi C_\psi^*$ is invertible if and only if each factor is invertible and $C_\varphi C_\psi^*$ is an isometry if and only if C_ψ is unitary and C_φ is an isometry. In addition, we determine that $C_\varphi C_\psi^*$ is unitary if and only if each factor is unitary.

The operator $C_\psi^* C_\varphi$, on the other hand, is considerably more interesting. In the case when the inducing maps are monomials it is not hard to see that $C_\psi^* C_\varphi$ is isometric exactly when the product is an isometric composition operator. We prove this in the following example.

Example. Suppose that $\varphi(z) = z^n$ and $\psi(z) = z^m$. The operator $C_\psi^* C_\varphi$ is an isometry if and only if there exists $p \in \mathbb{N}$ such that $n = pm$.

Proof. Suppose that $C_\psi^* C_\varphi$ is an isometry and m does not divide n, that is, $n \neq km$ for any $k \in \mathbb{N}$. Fix $f(z) = \sum_{k=0}^\infty a_k z^k \in H^2$. Then

$$\langle C_\psi^* C_\varphi z, f\rangle = \langle C_\varphi z, C_\psi f\rangle = \sum_{k=0}^\infty \overline{a}_k \langle z^n, z^{km}\rangle = 0.$$

Hence $C_\psi^* C_\varphi$ is not one-to-one and so cannot be an isometry.

Now if $\varphi(z) = z^n$ and $\psi(z) = z^m$ with $n = pm$ for some $p \in \mathbb{N}$, then $C_\varphi = C_\psi C_{z^p}$. Hence

$$C_\psi^* C_\varphi = C_\psi^* C_\psi C_{z^p} = C_{z^p}$$

is an isometry as z^p is an inner function that fixes the origin. □

Section 4, we extend this example, showing that $C_\psi^* C_\varphi$ is an isometry if and only if $\varphi = \alpha \circ \psi$ where both α and ψ are inner and fix the origin. We obtain as a corollary that $C_\psi^* C_\varphi$ is unitary if and only if ψ is inner and fixes the origin and $\phi = \lambda\psi$ for some $\lambda \in \mathbb{T}$.

The most interesting phenomena occurs when attempting to characterize invertibility of $C_\psi^* C_\varphi$. We observe that when $\varphi = \psi$ is nonconstant, invertibility of the product is equivalent to C_φ having closed range. Many equivalent characterizations in terms of the inducing map exist in the literature for C_φ to have closed range [2], [12]. However, when one of the inducing maps is univalent, we can completely determine when $C_\psi^* C_\varphi$ is invertible, partially recovering the results in Section 3.

2. Preliminaries

In this section, we record definitions and results necessary for the sequel. By a *disc automorphism* we shall mean a one-to-one and onto holomorphic map of $\mathbb{D}$ that necessarily takes the form

$$\varphi(z) = \lambda \frac{a - z}{1 - \overline{a}z}, \quad a \in \mathbb{D}, \quad |\lambda| = 1.$$

In [10], Schwartz showed that C_φ is invertible if and only if φ is a disc automorphism and $C_\varphi^{-1} = C_{\varphi^{-1}}$ (see [6], pages 4 and 5, and [8] pages 173 and 174).

An operator T is an *isometry* if $\|Tx\| = \|x\|$ for all x, or equivalently $T^*T = I$. A function φ is an *inner function* if $\varphi \in H^\infty(\mathbb{D})$ and

$$\lim_{r\to 1^-} |\varphi(re^{i\theta})| = 1 \quad \text{a.e.} \quad \theta \in \mathbb{R}.$$

Eric Nordgren related these two notions in [9].

Nordgren's Theorem ([9]) *A composition operator C_φ is an isometry if and only if φ is an inner function and $\varphi(0) = 0$.*

Finally, let $K_p(z)$ denote the *reproducing kernel* at p in $\mathbb{D}$ for H^2, which is given by

$$K_p(z) = \frac{1}{1-\overline{p}z}, \quad z \in \mathbb{D}.$$

The defining property of the reproducing kernel is

$$f(p) = \langle f, K_p\rangle \quad \text{for all} \quad f \in H^2.$$

We will have occasion to employ one of the most useful properties in the study of composition operators, the *adjoint property*:

$$C_\varphi^* K_p(z) = K_{\varphi(p)}(z). \tag{2.1}$$

An operator $T : H \to H$ is *unitary* if and only if $T^* = T^{-1}$. By Schwartz's results, we recognize that if a composition operator C_φ is unitary, then φ must be a disc automorphism. Using the adjoint property, one can then recover the well-known fact that C_φ is unitary if and only if $\varphi(z) = \lambda z$ for $\lambda \in \partial\mathbb{D}$.

3. The operator $C_\varphi C_\psi^*$

Our first theorem characterizes the invertibility of $C_\varphi C_\psi^*$ in terms of the inducing maps φ and ψ, thus generalizing Schwartz's results.

Theorem 3.1. *The operator $C_\varphi C_\psi^*$ is invertible if and only if C_φ and C_ψ are invertible; that is, φ and ψ are disc automorphisms.*

Proof. Suppose that $C_\varphi C_\psi^*$ is invertible. Then the inducing map φ must be non-constant and C_φ is an onto operator. All composition operators induced by a non-constant function are one-to-one, thus C_φ is invertible. Now the product is invertible by hypothesis, so since C_φ is invertible, C_ψ^* is invertible, implying C_ψ is invertible.

Conversely if φ and ψ are disc automorphisms, then $C_\varphi C_\psi^*$ is invertible with inverse $C_{\psi^{-1}}^* C_{\varphi^{-1}}$. $\square$

In attempting to extend Nordgren's Theorem, we observe that if T is an isometry and S is unitary, then it is trivially true that TS^* is an isometry, as

$$(TS^*)^*TS^* = ST^*TS^* = SS^* = I.$$

We now prove that this is the only manner in which the operator $C_\varphi C_\psi^*$ can be an isometry.

Theorem 3.2. *The operator $C_\varphi C_\psi^*$ is an isometry if and only if φ is an inner function such that $\varphi(0) = 0$ and $\psi(z) = \lambda z$ for some $\lambda \in \partial\mathbb{D}$.*

Proof. Assume that $C_\varphi C_\psi^*$ is an isometry. By hypothesis, $C_\psi C_\varphi^* C_\varphi C_\psi^* = I$, so C_ψ is onto. As C_ψ is one-to-one, we conclude that C_ψ is invertible, from which it follows that ψ is a disc automorphism. Let $p \in \mathbb{D}$ be such that $\psi(p) = 0$. Now using the adjoint property (Equation 2.1), $K_0 = 1$, and $C_\varphi 1 = 1$ successively yields

$$\|C_\varphi C_\psi^* K_p\| = \|C_\varphi K_{\psi(p)}\| = \|C_\varphi 1\| = 1. \tag{3.1}$$

The fact that $C_\varphi C_\psi^*$ is an isometry and Equation 3.1 yields

$$1 = \|C_\varphi C_\psi^* K_p\| = \|K_p\| = \sqrt{\frac{1}{1-|p|^2}}.$$

Thus $p = 0$, $\psi(0) = 0$ and ψ is a disc automorphism that fixes the origin. Hence $\psi(z) = \lambda z$ for some $\lambda \in \partial\mathbb{D}$ and C_ψ is unitary.

Combining this fact with $C_\psi C_\varphi^* C_\varphi C_\psi^* = I$ yields

$$C_\varphi^* C_\varphi = C_\psi^* C_\psi = I.$$

Hence C_φ is an isometry and by Nordgren's Theorem φ is an inner function such that $\varphi(0) = 0$.

The reverse direction is trivial since the assumptions yield C_φ is an isometry and C_ψ is unitary. □

We are now in a position to easily characterize when $C_\varphi C_\psi^*$ is unitary.

Corollary 3.3. *The operator $C_\varphi C_\psi^*$ is unitary if and only if both φ and ψ are of the form λz for some $\lambda \in \partial\mathbb{D}$.*

Proof. If $C_\varphi C_\psi^*$ is unitary then $C_\varphi C_\psi^*$ and $C_\psi C_\varphi^*$ are both isometries. By Theorem 3.2 we conclude that both φ and ψ have the desired form.

The reverse implication is trivial since if both φ and ψ are of the form λz for some $\lambda \in \partial\mathbb{D}$, then both C_φ and C_ψ^* are unitaries, and the product of unitary operators is unitary. □

Note that Theorem 3.1 and Corollary 3.3 show that if $C_\varphi C_\psi^*$ is either an isometry or unitary then the product is an isometric composition operator or a unitary composition operator, respectively.

4. The operator $C_\psi^* C_\varphi$

Using the example recorded in the introduction of this paper, we see that if $\varphi(z) = z^n$ for $n \geq 2$, then $C_\varphi^* C_\varphi = I$. Hence $C_\varphi^* C_\varphi$ can be unitary even if neither C_φ^* nor C_φ is invertible. Similarly, if $m \geq 2$ and $n = pm$ for some $p \in \mathbb{N}$, then $C_{z^m}^* C_{z^n}$ is an isometry even though C_{z^m} is not unitary.

In order to examine when $C_\psi^* C_\varphi$ does admit characterizations analogous to those obtained in Section 3, we first record a useful result regarding norm-one products.

Theorem 4.1. *The norm of $C_\psi^* C_\varphi$ is 1 if and only if $\varphi(0) = \psi(0) = 0$.*

Proof. Suppose that the norm of $C_\psi^* C_\varphi$ is 1. Using $C_\varphi 1 = 1$, $K_0 = 1$ and the adjoint property (Equation (2.1)) successively, we see $C_\psi^* C_\varphi 1 = K_{\psi(0)}$. Now

$$\frac{1}{1 - |\psi(0)|^2} = \|K_{\psi(0)}\|^2 = \|C_\psi^* C_\varphi 1\|^2 \leq \|C_\psi^* C_\varphi\|^2 = 1.$$

Thus $\psi(0) = 0$. The same calculation with the adjoint $C_\varphi^* C_\psi$ shows that

$$\frac{1}{1 - |\varphi(0)|^2} \leq 1,$$

which implies that $\varphi(0) = 0$ as well.

We now prove the converse using Littlewood's subordination principle ([7], also see [6, 8, 11]),

$$\|C_\varphi\|^2 \leq \frac{1 + |\varphi(0)|}{1 - |\varphi(0)|}.$$

Thus

$$\|C_\psi^* C_\varphi\|^2 \leq \frac{(1 + |\psi(0)|)(1 + |\varphi(0)|)}{(1 - |\psi(0)|)(1 - |\varphi(0))|}$$

and the hypothesis $\varphi(0) = \psi(0) = 0$ implies $\|C_\psi^* C_\varphi\| \leq 1$. To finish, observe that

$$\|C_\psi^* C_\varphi\| \geq \|C_\psi^* C_\varphi 1\| = \|K_{\psi(0)}\| = 1.$$

Hence $\|C_\psi^* C_\varphi\| = 1$. □

We obtain an immediate corollary.

Corollary 4.2. *If $C_\psi^* C_\varphi$ is an isometry then $\varphi(0) = \psi(0) = 0$.*

Proof. Since $C_\psi^* C_\varphi$ is an isometry its norm is one. Thus by Theorem 4.1 we conclude $\varphi(0) = \psi(0) = 0$. □

We now consider the case where $C_\psi^* C_\varphi$ is an isometry and obtain a generalization of Nordgren's Theorem. We shall require some preliminary results. The first is a proposition which is valid on any Hilbert space.

Proposition 4.3. *Let S and T be contractive operators on a Hilbert space. If S^*T is an isometry then T is an isometry and we have $T = SS^*T$.*

Proof. Since S and T are contractive operators, $I - T^*T \geq 0$ and $I - SS^* \geq 0$. It follows that $T^*(I - SS^*)T \geq 0$. Now we have

$$(I - T^*T) + T^*(I - SS^*)T = I - T^*T + T^*T - T^*SS^*T = 0,$$

since S^*T is an isometry. We conclude that $T^*T = I$, i.e. T is an isometry, and $T^*(I - SS^*)T = 0$. The latter equality implies $(I - SS^*)^{1/2}T = 0$, which gives $(I - SS^*)T = 0$. We then have $T = SS^*T$. □

If H is any Hilbert space and $T : H \to H$, we say T is *almost multiplicative* if whenever $f, g \in H$ are such that $f \cdot g$ also belongs to H, we have $T(f \cdot g) = Tf \cdot Tg$. By definition, $C_\varphi(f \cdot g) = (C_\varphi f) \cdot (C_\varphi g)$ for all $f, g \in H^2$ such that $f \cdot g \in H^2$, so C_φ is almost multiplicative. In [10], Schwartz characterized the composition operators as the only bounded almost multiplicative operators on H^2. The following lemma benefits from this result.

Lemma 4.4. *Suppose φ and ψ are holomorphic self-maps of $\mathbb{D}$ such that φ is non-constant and $C_\varphi = C_\psi T$ for some $T \in B(H^2)$. Then there is a holomorphic self-map α of $\mathbb{D}$ such that $T = C_\alpha$. Consequently, $\varphi = \alpha \circ \psi$.*

Proof. It follows from $C_\varphi = C_\psi T$ that $\operatorname{ran}(C_\varphi) \subset \operatorname{ran}(C_\psi)$. Hence, there is a function u in H^2 such that $\varphi = C_\psi u = u \circ \psi$. Since φ is assumed to be non-constant, ψ is non-constant as well.

Now let $f, g \in H^2$ with $f \cdot g \in H^2$. Then we have

$$C_\psi\big(Tf \cdot Tg\big) = (C_\psi Tf) \cdot (C_\psi Tg) = (C_\varphi f) \cdot (C_\varphi g) = C_\varphi(f \cdot g) = C_\psi T(f \cdot g).$$

Note that the first equality holds even though we do not know a priori that $Tf \cdot Tg$ is in H^2. As ψ is non-constant, C_ψ is injective, and so the identity $C_\psi\big(Tf \cdot Tg\big) = C_\psi T(f \cdot g)$ implies that $Tf \cdot Tg = T(f \cdot g)$. By Schwartz's result, there exists a holomorphic self-map α of $\mathbb{D}$ so that $T = C_\alpha$ on H^2. Since $C_\varphi = C_\psi C_\alpha = C_{\alpha \circ \psi}$, we conclude that $\varphi = \alpha \circ \psi$. □

We are now in a position to analyze $C_\psi^* C_\varphi$ in the case where the product is an isometry.

Theorem 4.5. *The operator $C_\psi^* C_\varphi$ is an isometry if and only if ψ is an inner function with $\psi(0) = 0$ and $\varphi = \alpha \circ \psi$ where $\alpha : \mathbb{D} \to \mathbb{D}$ is inner with $\alpha(0) = 0$.*

Proof. Suppose $C_\psi^* C_\varphi$ is an isometry. By Corollary 4.2 we have $\psi(0) = \varphi(0) = 0$. This implies that C_ψ and C_φ are contractive on H^2. Now applying Lemma 4.3 with $S = C_\psi$ and $T = C_\varphi$, we conclude that C_φ is isometric and $C_\varphi = C_\psi C_\psi^* C_\varphi$. It follows that φ is an inner function and so in particular, φ is non-constant.

By Lemma 4.4, there is a holomorphic self-map α of $\mathbb{D}$ such that $C_\alpha = C_\psi^* C_\varphi$ and $\varphi = \alpha \circ \psi$. Since C_α is then isometric, α is inner and $\alpha(0) = 0$. The identity $\varphi = \alpha \circ \psi$ then forces $|\psi| = 1$ almost everywhere on $\partial\mathbb{D}$, and so ψ is inner as well.

The reverse implication follows from direct calculation. If ψ is an inner function with $\psi(0) = 0$ and $\varphi = \alpha \circ \psi$ where $\alpha : \mathbb{D} \to \mathbb{D}$ is inner with $\alpha(0) = 0$, then C_ψ and C_α are isometries on H^2. Using the identity $C_\varphi = C_\psi C_\alpha$, we obtain

$$C_\psi^* C_\varphi = C_\psi^* C_\psi C_\alpha = C_\alpha.$$

Therefore $C_\psi^* C_\varphi$ is an isometry. □

Much like in Section 3, we can use the isometric characterization to describe precisely when $C_\psi^* C_\varphi$ is unitary in terms of the inducing maps.

Corollary 4.6. *The operator $C_\psi^* C_\varphi$ is unitary if and only if ψ is an inner function with $\psi(0) = 0$ and there is a constant $\lambda \in \partial\mathbb{D}$ such that $\varphi = \lambda\psi$.*

Proof. Suppose $C_\psi^* C_\varphi$ is unitary. By Theorem 4.5, both ψ and φ are inner with $\psi(0) = \varphi(0) = 0$ and there is an inner function α with $\alpha(0) = 0$ such that $\varphi = \alpha \circ \psi$. As in the proof of Theorem 4.5, we have $C_\psi^* C_\varphi = C_\alpha$, and so C_α is unitary. This implies that $\alpha(z) = \lambda z$ for some constant a with $|\lambda| = 1$. Therefore, $\varphi = \lambda\psi$. The reverse implication is trivial. □

For the remainder of this work, we consider the invertibility of $C_\psi^* C_\varphi$. It is natural to start with the case $\varphi = \psi$, and here we see the first roadblocks to generalizing Schwartz's characterization of invertible composition operators. If $\varphi = \psi$, then $C_\varphi^* C_\varphi$ is positive and invertible, which implies that C_φ is bounded below and hence has closed range. Conversely, if C_φ has closed range and φ is non-constant, then the injectivity of C_φ implies C_φ is bounded below, so $C_\varphi^* C_\varphi$ is invertible. We have then shown

Observation 4.7. *The operator $C_\varphi^* C_\varphi$ is invertible if and only if φ is non-constant and C_φ has closed range.*

The closed range composition operators have been objects of protracted study. If φ is inner then the range of C_φ is certainly closed, but examples of non-inner, non-constant symbols such that C_φ has closed range were pointed out by Cima, Thomson, and Wogen in [2]. In order to generalize Schwartz's results, then, we must be more restrictive in the class of symbols we work with.

Akeroyd and Ghatage note in [1] that by combining Theorem 2.5 in their paper with Zorboska's Corollary 4.2 in [12], one obtains the following theorem. We are indebted to Paul Bourdon for pointing out this result.

Theorem 4.8 ([1],[12]). *Let φ be a univalent, holomorphic self-map of $\mathbb{D}$. Then C_φ has closed range on H^2 if and only if φ is a disc automorphism.*

Assuming one of the inducing maps φ or ψ is univalent, we are now in a position to recover the results of Theorem 3.1.

Theorem 4.9. *Let ψ be a univalent, holomorphic self-map of $\mathbb{D}$. Let φ be any other holomorphic self-map of $\mathbb{D}$. Then the product $C_\psi^* C_\varphi$ is invertible if and only if both φ and ψ are disc automorphisms.*

Proof. Suppose that $C_\psi^* C_\varphi$ is invertible. Then C_ψ^* is surjective which implies that C_ψ is bounded below and so the range of C_ψ is closed. According to Theorem 4.8, ψ is a disc automorphism. Therefore C_ψ is invertible. Consequently, C_φ is invertible, which immediately yields that φ is a disc automorphism as well. The reverse implication is trivial. □

Looking beyond univalent maps, the next natural class of examples to study seems to be $C_{z^n}^* C_\varphi$ for $n \geq 2$. We conjecture that, if $C_{z^n}^* C_\varphi$ is invertible, then $\phi(z) = \alpha(z^n)$ where α is a disk automorphism, but we are unable to show this even for the case where $n = 2$.

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John H. Clifford
Department of Mathematics and Statistics
2014 CASL Building
4901 Evergreen Road
Dearborn, Michigan 48128-2406
e-mail: jcliff@umd.umich.edu

Trieu Le
Department of Mathematics and Statistics
Mail Stop 942
2801 W. Bancroft St.
Toledo, OH 43606-3390
e-mail: trieu.le2@utoledo.edu

Alan Wiggins
Department of Mathematics and Statistics
2014 CASL Building
4901 Evergreen Road
Dearborn, Michigan 48128-2406
e-mail: adwiggin@umd.umich.edu