NARRATIVE STATEMENT ON RESEARCH

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Originated from the theory of differential equations and integral equations, Operator Theory is the study of infinite matrices, their properties and applications. Such matrices are infinite rectangular arrays of numbers that represent linear operators between spaces of functions. Since the advent of Quantum Mechanics in the early 20th century, Operator Theory has become increasingly important in mathematics, physics and other fields of science. Applications of Operator Theory have also appeared in image processing, signal processing and control theory.

I am particularly interested in the study of analytic and algebraic properties of several classes of linear operators acting on Hilbert spaces whose elements are analytic functions. I have investigated Toeplitz, Hankel, composition and weighted composition operators on the Bergman, Hardy and Fock spaces. I have also been interested in the theory of reproducing kernels and its applications to PDEs and signal processing.

Since finishing my Ph.D. in 2007, I have written 40 research papers. Please refer to the list of publications for more details.

I. RESEARCH AWARDS.

1. Summer Research Awards and Fellowships, University of Toledo, 2016
2. Proposal submitted to Simons Foundation: Travel Support for Mathematicians, Spring 2023. Proposal's title: "Finite rank Toeplitz products in several complex variables". *Funded.*

II. RECENT CONFERENCE/SEMINAR PRESENTATIONS.

1. *On range of Berezin transform in several variables and applications*, International Workshop in Operator Theory and Its Applications, Helsinki, Finland, July 31-August 4, 2023
2. *Toeplitz operators with pluriharmonic symbols on the ball*, Analysis and Data Science Seminar, University at Albany, October 26, 2021 via Zoom
3. *m-Isometries and their properties*, Trojan Mathematics Seminar, Troy University, October 14, 2021 via Teams
4. *When does $T_fT_g - T_h$ have finite rank?*, International Workshop in Operator Theory and Its Applications, Chapman University, August 9-11, 2021 via Zoom
5. *Toeplitz operators with pluriharmonic symbols on the ball*, Complex Analysis and Operator Theory Seminar, University of Toledo, March 2021 via Zoom
6. *Algebraic properties of m-isometries*, Analysis Seminar, Washington University in Saint Louis, February 2021 via Zoom
7. *Algebraic properties of m-isometries*, Analysis and Geometry Seminar, Central Michigan University, February 2019
8. *Inner Functions in Weighted Hardy Spaces*, International Workshop on Operator Theory and Applications, Shanghai, China, July 2018
9. *A construction of inner functions on weighted Hardy spaces*, Analysis Seminar, Hebei Normal University, China, July 2018
10. *Characterizations of inner functions on the unit disk*, Summer Meetings, University of Science, Ho Chi Minh city, Vietnam, July 2018
11. *Composition operators on Hilbert spaces of entire functions*, AMS Spring Southeastern Sectional Meeting, Vanderbilt University, April 2018

12. *Commutants of separately radial Toeplitz operators on the Bergman space*, Mathematical Congress of the Americas, Montreal, Canada, July 2017
13. *Commutants of separately radial Toeplitz operators*, Ohio Conference, the Ohio State University, April 2017
14. *Algebraic properties of m-isometric commuting tuples*, 33rd South Eastern Analysis Meeting, University of Tennessee, March 2017
15. *Commutants of Toeplitz operators with separately radial polynomial symbols on the Fock space*, International Workshop on Operator Theory and Applications, Washington University in St. Louis, July 2016
16. *Hilbert–Schmidt Hankel operators with conjugate holomorphic symbols*, 32nd South Eastern Analysis Meeting, University of South Florida, March 2016
17. *Hilbert–Schmidt Hankel operators with conjugate holomorphic symbols*, Complex Analysis Seminar, University of Toledo, October 2015
18. *Adjoints of linear fractional composition operators on weighted Hardy spaces*, AMS Sectional Meeting, Washington, DC, March 2015

III. RESEARCH ACCOMPLISHMENTS AND FUTURE PLANS.

1. Introduction

Everyone knows from their elementary mathematics classes that if the product of two numbers is zero, then one of the numbers must be zero as well. On the other hand, it is known to anyone who has some basic knowledge of linear algebra that the product of two non-zero $n \times n$ matrices may be a zero matrix if the size n is bigger than one. This, of course, remains true when one considers matrices of infinite sizes, or more formally, linear operators. However, if we restrict our attention to certain classes of operators, then the identity $AB = 0$ may force either A or B to be zero. In the sixties, Brown and Halmos [19] showed that this is the case for Toeplitz operators on the Hardy space. The matrix of such a Toeplitz operator has the same entries on each diagonal parallel to the main diagonal. Since the appearance of Brown–Halmos's paper, mathematicians have investigated this "*zero product problem*" for other classes of operators. I have been particularly interested in the case of Toeplitz operators on the Bergman space and on the Segal–Bargmann space (also known as the symmetric Fock space in Quantum Mechanics). Several results have been obtained but the general problem remains open. Section 2 provides more details about this problem and my contribution toward solving it. In Section 3, we discuss when a product of two (or more) Toeplitz operators is equal to a finite rank perturbation of another Toeplitz operator.

Another problem that I have worked on is the "*commuting problem*". We all know that the multiplication of numbers is commutative: $ab = ba$. On the other hand, matrix multiplication is *not* commutative. The products AB and BA may be different for matrices A and B. In certain cases, it may happen that two products are the same. We then say that A and B commute. In general, it may be broad to ask for conditions that two arbitrary matrices (or operators) need to satisfy in order for them to commute. However, reasonable conditions may be found if we restrict our attention to certain classes of operators. Brown and Halmos [19] showed that two Toeplitz operators on the Hardy space commute if and only if both are upper-triangular or both are lower-triangular or one is a linear combination of the other with the identity operator. On the Bergman space, the situation becomes more complicated. The Brown–Halmos's result remains true if certain additional conditions are imposed (as proved by Axler and Čučković [10]) but it fails in general. Čučković and Rao [30] studied the case when one of the operator is assumed to be a diagonal operator. In [38, 9, 43] my collaborators and I generalized this result to the Bergman space of the unit ball. We considered the commuting problem when one of the Toeplitz operators is given by a separately radial symbol. Our results show that several variables bring more interesting features to the problem. In [11], W. Bauer and I obtained similar results for the Segal–Bargmann space in all dimensions. More details are given in Section 4.

Inspired by the work of Cowen et al., and Bourdon and Narayan, I have also investigated *composition and weighted composition operators*. Let $\mathcal{H}$ be a Hilbert space whose elements are complex-valued functions defined on a domain Ω. A self-mapping φ of Ω gives rise to a composition operator C_φ defined on $\mathcal{H}$ by composing: $C_\varphi h = h \circ \varphi$. We are interested in how the function theoretic properties of φ interact with the operator theoretic properties of C_φ. One of the fundamental problems is to classify the mappings φ which induce bounded or compact operators C_φ. After such a classification is obtained, we then investigate the spectrum, spectral radius, numerical range, cyclicity and hypercyclicity, among other things, of the composition operator. These properties are closely related with function theoretic quantities associated with the mapping φ. The study of composition operators goes back to Littlewood's famous Subordinate Principle, which shows that any composition operator is bounded on the Hardy and Bergman spaces over the unit disk on the complex plane. On the other hand, in the multivariable context, there are polynomial self-mappings that give rise to unbounded composition operators on the Hardy and Bergman spaces over the unit ball. The books [28, 53] contain essential background and fundamental results. If f is a holomorphic function on Ω, then we can also define the weighted composition operator $W_{f,\varphi} h = f \cdot (h \circ \varphi)$ for $h \in H$. The function f is called the weight function of the operator.

In [26, 27], among other things, Cowen and his collaborators characterized self-adjoint weighted composition operators on the Hardy space and on weighted Bergman spaces of the unit disc. In [18], Bourdon and Narayan studied unitary and normal weighted composition operators on the Hardy space. I have successfully brought many of these results to higher dimensions in [41]. My approach not only overcame the complicated setting of several variables but also provided shorter and more transparent proofs of the results in one dimension. See Section 6 for these generalizations.

Around ten years ago I got interested in the theory of m-isometric operators. Recall that an operator T is called an isometry if it preserves distance. That is, for all element v, one has $\|Tv\| = \|v\|$, or equivalently, $\|Tv\|^2 - \|v\|^2 = 0$. If Q is a nonzero operator such that $Q^2 = 0$ (that is, Q is nilpotent of order 2), then a direct calculation shows that $T = I + Q$ is not isometric but for any v, we have

$$\|T^3 v\|^2 - 3\|T^2 v\|^2 + 3\|Tv\|^2 - \|v\|^2 = 0.$$

Such an operator is called a 3-isometry. (Note that the coefficients are exactly the coefficients in the expansion of $(x-1)^3$.) Generally, a bounded operator T is called m-isometric if

$$\sum_{k=0}^{m} (-1)^k \binom{m}{k} \|T^{m-k} v\|^2 = 0 \quad \text{for all elements } v.$$

It is clear that any 1-isometric operator is isometric. The theory of m-isometric operators was introduced by Agler back in the early nineties and were studied in great detail by Agler and Stankus in a series of three papers [2, 3, 4]. Recently researchers have shown interests in algebraic properties of these operators. In [15], Bermúdez, Martinón, and Noda proved that if A is an isometry and Q is a nilpotent operator of order s commuting with A, then $A+Q$ is $(2s-1)$-isometric. With a more elegant approach [42], I generalized this result to more general operators which are called hereditary roots of polynomials. In addition, in [1], undergraduate student Belal Abdullah and I investigated the structure of weighted shift operators that are m-isometric. We showed that such operators form a semigroup under the Hadamard product. Section 7 describes these results in more detail.

Section 8 discusses the notion of inner functions in weighted Hardy spaces and my contributions to current research in this direction.

In the sections below I shall describe in more technical details the problems I have recently worked on and plans for future research.

2. The zero product problem

The Hardy space H^2 is the closure of the linear span of $\{z^k : k = 0, 1, 2, \ldots\}$ in $L^2(\mathbb{T})$, the space of square integrable functions on the unit circle. For a bounded function f on the circle, the Toeplitz operator T_f is the compression on H^2 of the multiplication operator M_f, that is, $T_f = PM_f|_{H^2}$, where P is the orthogonal projection from $L^2(\mathbb{T})$ onto H^2. With respect to the standard orthonormal basis $\{1, z, z^2, \ldots\}$, each Toeplitz operator can be represented as an infinite "Toeplitz matrix" whose entries are constant along each descending diagonal. In their seminal work [19], Brown and Halmos showed that if f and g are bounded functions and $T_gT_f = 0$, then one of the functions must be zero. A more general question concerning products of several Toeplitz operators is the so-called "zero product problem".

Problem. *Suppose $f_1, \ldots, f_N$ are bounded functions such that $T_{f_1} \cdots T_{f_N} = 0$. Does it follow that one of these functions must be zero?*

This problem on the Hardy space H^2 was open for more than thirty years. The affirmative answer was settled by Aleman and Vukotic [8]. Problem 2 for Toeplitz operators on the Bergman and Segal–Bargmann spaces is described below.

2.1. **Bergman space.**

Let $\mathbb{D}$ denote the open unit disc on the complex plane. The Bergman space A^2 is the Hilbert space of all analytic functions on $\mathbb{D}$ that are square integrable with respect to Lebesgue area measure. For a bounded measurable function f, the Toeplitz operator with symbol f is defined by $T_f\varphi = PM_f(\varphi) = P(f\varphi)$ for $\varphi \in A^2$, where P is the orthogonal projection from $L^2(\mathbb{D})$ onto A^2.

While Problem 2 for Toeplitz operators on the Bergman space is still open even when $N = 2$, some special cases have been understood. In 2001, Ahern and Čučković [6] gave the affirmative answer if both functions are harmonic or one of the functions is radial. (A function f is said to be radial if $f(z) = f(|z|)$ for a.e. z.) The case of harmonic functions was generalized to the Bergman space of the unit ball by Choe and Koo [25] under certain assumptions about the continuity of the functions.

In [49], Luecking shows that if f is a bounded function on the unit disc such that T_f has finite rank, then $f = 0$. By refining Luecking's approach, I was able to establish the affirmative answer to the *finite-rank version* of Problem 2 when all, except one, of the functions are radial. The theorem below is the main result in [40].

Theorem 1. *Let $f_1, \ldots, f_N$ and $g_1, \ldots, g_M$ be non-zero, bounded radial functions on the unit disc. If f is bounded so that $T_{f_1} \cdots T_{f_N} T_f T_{g_1} \cdots T_{g_M}$ has finite rank on A^2, then f must be zero.*

Luecking's theorem was generalized to higher dimensions by Choe [24] and independently by Rozenblum and Shirokov [51]. Using these results, I obtained a several-variable version of Theorem 1 in [39].

2.2. **Segal–Bargmann space.** The Segal–Bargmann space $\mathcal{F}^2$ consists of entire functions on the plane that are square integrable with respect to the Gaussian measure $d\mu(z) = \pi^{-1}e^{-|z|^2}dA(z)$, where dA denotes Lebesgue area measure. Toeplitz operators on $\mathcal{F}^2$ are also defined as compressions of multiplication operators. In my paper with Bauer [11], we showed that if f and g are bounded functions such that one of them is radial and $T_fT_g = 0$ on $\mathcal{F}^2$, then either f or g is the zero function. This gives an affirmative answer to Problem 2 on the Segal–Bargmann space when $N = 2$, provided that one of the functions is radial. We do not know if the condition that one of the functions be radial can be removed. On the other hand, it is quite surprising that we found a counterexample when $N = 3$.

Theorem 2. *There exist bounded non-zero functions f_0, f_1, f_2 on $\mathbb{C}$ such that the operator $T_{f_0}T_{f_1}T_{f_2} = 0$ on $\mathcal{F}^2$.*

Higher dimensional versions of our results remain true and we refer the reader to [11] for more details.

3. Products of Toeplitz operators

In [19], Brown and Halmos also classified all pairs of commuting Toeplitz operators on the Hardy space over the unit disc, as well as characterized all triples of Toeplitz operators (T_f, T_g, T_h) such that $T_fT_g = T_h$. Such results are commonly referred to as the Brown–Halmos theorems. Extending these results to the Bergman space setting and to Hilbert spaces of holomorphic functions on more general domains in several complex variables has been one of the central themes of research in the theory of Toeplitz operators in the last few decades.

On the Bergman space over the unit disc, the first results in the spirit of the Brown–Halmos theorems were obtained by Axler and Čučković [10], Ahern and Čučković [6], and Ahern [5]. It was shown in these papers that Brown–Halmos theorems hold true on the Bergman space for Toeplitz operators with bounded harmonic symbols. Guo, Sun and Zheng [36] later studied finite rank semi-commutators and commutators of Toeplitz operators with harmonic symbols. It was showed that semi-commutators and commutators have finite rank if and only if they are actually zero. Čučković [29] obtained criteria for $T_fT_g - T_{h^n}$ to have finite rank, where f, g and h are bounded harmonic. More general results in this direction were investigated in [23]. In a recent paper, Ding, Qin and Zheng [31] provided a more complete answer to the possible rank of $T_fT_g - T_h$ under the assumption that f, g are bounded harmonic and h is a C^2-function and $(1-|z|^2)^2\Delta h$ is integrable.

In [47], Thilakarathna and I discovered a noncommutative binary operation $\diamond$ on the space of polynomials in z and $\bar{z}$ on the complex plane and used it to characterize polynomial functions f and g for which the Toeplitz product T_fT_g is a finite rank perturbation of another Toeplitz operator. We in fact solved the problem for finite sums of products of two Toeplitz operators.

Theorem 3. *Let F_j and G_j be polynomials in z and $\bar{z}$ for $j = 1, \ldots, N$. Then there exists an integrable function H such that $\sum_{j=1}^{N} T_{F_j}T_{G_j} - T_H$ is of finite rank if and only if $\sum_{j=1}^{N} F_j \diamond G_j$ is integrable.*

Researchers have also investigated Brown–Halmos theorems in the setting of several complex variables. A classification of pairs of commuting Toeplitz operators with pluriharmonic symbols on the unit ball was given by Zheng in [54]. Subsequently, Choe and Koo [22] studied the zero product problem for Toeplitz operators on the unit ball with harmonic symbols having continuous extensions to part of the boundary. There have been other results for the Hardy space over the unit sphere and Bergman space over the polydisk. However, there has not been much progress in the unit ball case. In a recently published paper [48], Tikaradze and I obtained several important results, which generalized the aforementioned results. Among other things, we proved

Theorem 4. *Let ϕ, ψ be bounded pluriharmonic functions on the unit ball.*

(a) If $T_\phi T_\psi = T_h$ for some smooth bounded function h, then $\bar{\phi}$ or ψ is holomorphic and $\phi\psi = h$.

(b) If $T_\phi T_\psi$ has a finite rank, then ϕ or ψ must be zero.

(c) The commutator $[T_\phi, T_\psi]$ has a finite rank if and only if both ϕ, ψ are holomorphic, or both $\bar{\phi}, \bar{\psi}$ are holomorphic, or there are constants c_1, c_2, not both zero, such that $c_1\phi + c_2\psi$ is constant.

It is interesting that the approach in [48] employed techniques ranging from Partial Differential Equations to Function and Operator Theory. We also settled an open question about $\mathcal{M}$-harmonic functions. Proposition 5.4 in [48] provided a version of Theorem 3 in the setting of several variables. More specifically, it shows that whenever f and g are polynomials in z and $\bar{z}$ in $\mathbb{C}^N$ such that the sum of the degree of f in z and the degree of g in $\bar{z}$ is at most $2N+1$, then $T_f T_g = T_h$ for some integrable function h. Unfortunately, the relation between h and f, g has not been well understood yet. My plan for future research is to obtain a full version of Theorem 3 in the setting of several complex variables. It seems that new ideas are needed. I plan to continuing my collaboration with Tikaradze, Thilakarathna and other researchers in the pursuit of this project.

4. The commuting problem

In the one dimensional setting, the work of Axler and Čučković [10] solves the commuting problem under the additional assumption that both φ and ψ are harmonic functions. In [30], among other things, Čučković and Rao resolve the case φ is a radial function, which depends only on the modulus of the variable. In [38], I obtain the multivariate version of Čučković-Rao's result.

Let $\mathbb{B}_n$ denote the unit ball in $\mathbb{C}^n$. The Bergman space $A^2(\mathbb{B}_n)$ is the space of all analytic function on $\mathbb{B}_n$ that are square integrable with respect to Lebesgue volume measure. The following theorem [38] brings Čučković–Rao result to the setting of several variables.

Theorem 5. *Let f be a non-constant, bounded radial function on $\mathbb{B}_n$. For any bounded function g, the Toeplitz operators T_g and T_f commute on the Bergman space $A^2(\mathbb{B}_n)$ if and only if $f(e^{i\theta}z) = f(z)$ for a.e. $\theta \in \mathbb{R}$, $z \in \mathbb{B}_n$.*

Applying Theorem 5 with $n = 1$, we recover Čučković–Rao result. In dimensions at least two, there are non-radial functions that satisfy the conclusion of the theorem, for example, $f(z) = z_1\bar{z}_2$.

Very recently, in [43], I investigate the case φ is a separately radial function, that is, φ depends only on the modulus of each of the variables. My result not only illustrates the rich structure of Toeplitz operators that commute with T_φ but also illuminates an interesting connection between the theory of Toeplitz operators and actions of the torus group on functions defined on the unit ball. The following theorem is one of the results in [43].

Theorem 6. *Let φ be a separately radial bounded function on the unit ball. There then exists a subgroup G_φ of the torus group $\mathbb{T}^n$ such that for any bounded function ψ on the unit ball, the operators T_ψ and T_φ commute if and only if ψ is invariant under G_φ.*

The group G_φ is given more explicitly in the main results of [43]. In particular, when φ is a separately radial polynomial, G_φ always contains the subgroup $\{\zeta\cdot(1,\ldots,1) : \zeta \in \mathbb{T}\}$.

Similar to the one-dimensional case, the Segal–Bargmann space $\mathcal{F}^2_n$ on $\mathbb{C}^n$ consists of all entire functions that are square integrable with respect to the Gaussian measure $d\mu(z) = (\pi)^{-n}e^{-|z|^2}dV(z)$, where dV denotes the Lebesgue volume measure. We proved in [11] an analogous result to Theorem 5 for Toeplitz operators on $\mathcal{F}^2_n$. Even though some of the ideas from [38] can still be used, new techniques are required for $\mathcal{F}^2_n$, due to the unboundedness of $\mathbb{C}^n$.

5. Boundedness and compactness of composition operators

My work in this direction concerns composition operators acting on Hilbert spaces whose elements are entire functions. A typical example of such spaces is the Fock space $\mathcal{F}^2(\mathbb{C}^n)$ (also known as the Segal–Bargmann or Fischer space), which consists of entire functions that are square integrable with respect to the Gaussian measure on $\mathbb{C}^n$. In [20], Carswell, MacCluer and Schuster obtained complete characterizations of the mappings φ that give rise to bounded or compact composition operators C_φ on $\mathcal{F}^2(\mathbb{C}^n)$. In [44], I bring their results to a completely new ground. I investigate composition operators on the Fock space of infinitely many variables and successfully characterize the boundedness and compactness of such operators. My approach is totally different from that of Carswell-MacCluer-Schuster. While they make use of the change-of-variables and the Singular Value Decomposition of $n \times n$ matrices (which break down in the infinite dimensional context), I use tools from the theory of positive definite kernels. My result actually allows the composition operators to act on two different Fock spaces, which is new even in the finite dimensional setting.

In order to state the results, we introduce some notation. For $\mathcal{E}$ an arbitrary complex Hilbert space, the Fock space $\mathcal{F}^2(\mathcal{E})$ is the Hilbert space of complex-valued functions over $\mathcal{E}$ with reproducing kernel $K(z,w) = \exp(\langle z, w\rangle)$. If $\varphi : \mathcal{E}_1 \to \mathcal{E}_2$ is a mapping between two Hilbert spaces, the composition operator C_φ acts from $\mathcal{F}^2(\mathcal{E}_2)$ into $\mathcal{F}^2(\mathcal{E}_1)$ by composing. The following theorem summarizes two important results that I obtain in [44].

Theorem 7. *Let $\varphi : \mathcal{E}_1 \to \mathcal{E}_2$ be a mapping. Then*

(a) C_φ is bounded if and only if $\varphi(z) = Az+b$, where $A : \mathcal{E}_1 \to \mathcal{E}_2$ is linear with $\|A\| \le 1$ and A^*b belongs to the range of $(I - A^*A)^{1/2}$. Furthermore,

$$\|C_\varphi\| = \exp\Big(\frac{1}{2}\|v\|^2 + \frac{1}{2}\|b\|^2\Big).$$

Here, v is the vector of minimum norm that solves the equation $A^*b = (I-A^*A)^{1/2}v$.

(b) C_φ is compact if and only if $\varphi(z) = Az + b$, where $A : \mathcal{E}_1 \to \mathcal{E}_2$ is a linear compact operator with $\|A\| < 1$ and $b \in \mathcal{E}_2$.

My collaborators and I [32] recently initiated the study of composition operators on a more general class of Hilbert spaces of entire functions in several variables. Let $\beta = \{\beta_m\}_{m=0}^\infty$ be a sequence of positive real numbers with $\lim\limits_{m\to\infty} \beta_m^{1/m} = \infty$. The Hilbert space $\mathcal{H}_\beta(\mathbb{C}^n)$ consists of entire functions whose homogeneous expansion $f = \sum_{m\ge 0} p_m$ satisfies

$$\|f\|^2_\beta = \sum_{m\ge 0} \beta_m^2 \cdot \|p_m\|^2_{L^2} < \infty,$$

where $\|\cdot\|_{L^2}$ denotes L^2-norm with respect to the surface measure on the unit sphere. The properties of composition operators C_φ on $\mathcal{H}_\beta(\mathbb{C}^n)$ depend heavily on the behavior of the weight sequence β. We show that the boundedness of C_φ forces φ to be an affine mapping $\varphi(z) = Az + b$, and the compactness of C_φ requires $\|A\| < 1$. However, depending on the weight β, additional interaction between the operator A and the vector b is necessary. Our results greatly generalize several one-dimensional results obtained earlier by other authors [21, 37]. In addition, the results illuminate a surprising difference between the single variable and the multivariable contexts.

6. Self-adoint and unitary weighted composition operators

For any real number $\gamma > 0$, let H_γ denote the Hilbert space of analytic functions on the unit ball $\mathbb{B}_n$ with reproducing kernel

$$K_z^\gamma(w) = K^\gamma(w, z) = \frac{1}{(1 - \langle w, z\rangle)^\gamma} \quad \text{for } z, w \in \mathbb{B}_n.$$

For special values of γ we may identify H_γ with well-known spaces. For $\gamma = n$, H_n can be identified with the Hardy space. For $\gamma > n$, H_γ is a weighted Bergman space of the unit ball.

For φ a holomorphic map from $\mathbb{B}_n$ into itself and f a holomorphic function on $\mathbb{B}_n$, we define the weighted composition operator $W_{f,\varphi}$ by $W_{f,\varphi}h = f \cdot (h \circ \varphi)$ for $h \in H_\gamma$. In a recent paper [41], I generalized the work of Cowen et al., and Bourdon and Narayan on self-adjoint and unitary $W_{f,\varphi}$ on the Hardy and weighted Bergman spaces of the unit disc to the unit ball. Descriptions of all self-adjoint and all unitary weighted composition operators on H_γ were obtained. My approach, which made essential use of the reproducing kernel functions, also provided simplified and transparent proofs of the results in one dimension. I list here the results and refer the reader to [41] for more details.

Theorem 8. *For any $\gamma > 0$, the operator $W_{f,\varphi}$ is a non-zero, self-adjoint, bounded operator on H_γ if and only if there is a vector $c \in \mathbb{B}_n$, a self-adjoint linear operator A on $\mathbb{C}^n$ and a real number α such that $f = \alpha\, K_c^\gamma$ and $\varphi(z) = \frac{c + Az}{1 - \langle z, c\rangle}$ for $z \in \mathbb{B}_n$.*

Theorem 8 in particular says that in order for $W_{f,\varphi}$ to be self-adjoint on H_γ, the inducing map φ must be a linear fractional map of the unit ball and $\varphi = \varphi^\times$, where $\varphi^\times$ is the Krein adjoint of φ.

The next result shows that the class of unitary weighted composition operators on H_γ coincides with the class of co-isometric weighted composition operators. We also have a complete description of such operators.

Theorem 9. *Suppose that $\gamma > 0$ and the operator $W_{f,\varphi}$ is bounded on H_γ. Then TFAE*

(a) $W_{f,\varphi}$ is a unitary on H_γ.
(b) $W_{f,\varphi}$ is a co-isometry (that is, $W_{f,\varphi}^*$ is an isometry) on H_γ.
(c) φ is an automorphism of $\mathbb{B}_n$ and $f = \lambda k^\gamma_{\varphi^{-1}(0)}$ for some complex number λ with $|\lambda| = 1$. Here for $a \in \mathbb{B}_n$, $k_a^\gamma = K_a^\gamma / \|K_a^\gamma\|$ is the normalized reproducing kernel function at a.

7. The structure of m-isometric commuting tuples

In the studies of linear operators, the class of isometries plays an indispensable role. On Hilbert spaces, isometries can be characterized as operators satisfying

$$-I + T^*T = 0,$$

where T^* denotes the adjoint operator of T. Generalizing this operator equation, Agler introduced the notion of m-isometric operators back in the eighties in connection with his studies of subjordan operators and Toeplitz conjugacy classes. A bounded linear operator T on a Hilbert space is m-isometric if it satisfies

$$\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}T^{*k}T^{k}=0.$$

The 1-isometric operators are exactly the isometries. In a series of papers [2, 3, 4], Agler and Stankus gave an extensive study of spectral theory and representations of m-isometric operators. It is only recently that researchers [33, 16, 15] have begun investigating algebraic properties of m-isometries and several interesting phenomena have been discovered. Of particular interest is a surprising result by Bermúdez, Martinón, and Noda [15] which says that if S is isometric and N is nilpotent of order n commuting with S, then $S+N$ is $(2n-1)$-isometric. Their proof is quite lengthy with many combinatoric manipulations. With a completely different proof using algebraic approach, I [42] generalized the result to arbitrary m-isometries as well as hereditary operator roots of polynomials.

Let p be a polynomials in z and $\bar{z}$ with complex coefficients given by $p(z)=\sum_{k,\ell}a_{k,\ell}\bar{z}^{k}z^{\ell}$. For any bounded linear operator T, we define

$$p(T)=\sum_{k,\ell}a_{k,\ell}T^{*k}T^{\ell}.$$

This functional calculus was termed the *hereditary functional calculus* and was studied by Agler in the eighties. We say that T is a (hereditary) root of p if $p(T)=0$. It is can be checked that m-isometries are hereditary roots of $(\bar{z}z-1)^{m}$. My contribution to the investigation of nilpotent perturbations of hereditary roots is the following result in [42].

Theorem 10. *Let p be a polynomial and A and Q are commuting operators on a Hilbert space. If $p^{m}(A)=0$ and $Q^{s}=0$ for some non-negative integers m and s, then $p^{m+2s-2}(A+Q)=0$.*

Setting $p(z)=\bar{z}z-1$, we recover Bermúdez-Martinón-Noda result. Choosing other polynomials gives interesting results that are discussed in greater detail in [42].

It is well known that the unilateral shift on the Hardy space is an isometry. On the other hand, the Dirichlet shift is a 2-isometry [50]. Several researchers have been interested in characterizing weighted shift operators that are m-isometric. A complete characterization was first obtained by Bermudéz et al. [14]. However, their result appears difficult to apply. In a research project in the Summer of 2014, undergraduate student Belal Abdullah and I investigated this problem further and we obtained an equivalent but more transparent criterion for a weighted shift operator to be m-isometric. We obtain in [1] the following result.

Theorem 11. *Let S be a weighted shift operator with weight sequence $\{w_n\}$. Then S is an m-isometry if and only if there is a polynomial p of degree at most $m-1$ such that $|w_n|^2=p(n+1)/p(n)$ for all n.*

As a consequence, we show that any m-isometric weighted shift operator is a Hadamard product of 2-isometries and 3-isometries. We also characterize weighted shift operators whose powers are m-isometric.

In the context of multivariable operator theory, Gleason and Richter in [35] introduced and studied spectral properties of m-isometric commuting tuples. My approach in [42] can also be used to show that the sum of an m-isometric tuple with a commuting nilpotent tuple of order n is $(2n+m-2)$-isometric.

It has been known from the work of Agler–Stankus that each m-isometry T on a finite dimensional space has a simple decomposition: $T = U + N$, where U is an isometry and T is a nilpotent operator commuting with U. It was an open question for some time whether such representation still holds in the setting of m-isometric tuples. In a recent publication [45] (Journal of Functional Analysis), I proved a very general result, which answered this question in the affirmative. To state the result, we first recall some notation. For any polynomial $p(z) = \sum_{\alpha,\beta} c_{\alpha,\beta}\bar{z}^{\alpha} z^{\beta}$ over $z \in \mathbb{C}^d$ and tuple $\mathbf{T} = [T_1, \ldots, T_d]$ of commuting operators on a Hilbert space, define $p(\mathbf{T}) = \sum_{\alpha,\beta} c_{\alpha,\beta}(T^*)^{\alpha} T^{\beta}$. Then the set

$$\mathcal{J}(\mathbf{T}) = \big\{p : \ p(\mathbf{T}) = 0\big\}.$$

turns out to be an ideal of $\mathbb{C}[\bar{z}, z]$. Let $\mathrm{Rad}(\mathcal{J}(\mathbf{T}))$ be the corresponding radical ideal. Note that $\mathbf{T}$ is an m-isometric tuple if and only if $(1 - |z|^2)^m$ belongs to $\mathcal{J}(\mathbf{T})$. In such a case, $(1 - |z|^2)$ is an element of $\mathrm{Rad}(\mathcal{J}(\mathbf{T}))$. One of the main results in [45] is the following theorem, which in particular, implies the required decomposition for m-isometric tuples.

Theorem 12. *Let $\mathbf{T}$ be a tuple of commuting algebraic operators. There then exists a decomposition $\mathbf{T} = \mathbf{U} + \mathbf{N}$ such that $\mathbf{N}$ is a nilpotent tuple commuting with $\mathbf{U}$ and $\mathrm{Rad}(\mathcal{J}(\mathbf{T})) \subseteq \mathcal{J}(\mathbf{U})$.*

8. Inner functions on weighted Hardy spaces

Inner functions are bounded holomorphic functions on the unit disk whose boundary values have absolute value one almost everywhere. Equivalently, those are holomorphic functions f with unit norm on the Hardy space H^2 such that $z^m f \perp f$ for all $m \geq 1$. The celebrated Beurling's Theorem asserts that any closed subspace of H^2 that is invariant for the operator of multiplication by z is given by φH^2 for some inner function φ. Aleman, Richter and Sundberg [7] proved an analogue of Beurling's Theorem for the Bergman space A^2: any invariant subspace of A^2 is generated by the so-called wandering subspace. Their work involves a class of functions called A^2-inner functions that satisfy the norm and orthogonality constraints as above.

Due to its importance, the notion of inner functions has been generalized to weighted Hardy spaces. These are Hilbert spaces of holomorphic functions on the unit disk on which monomials form an orthogonal basis. Recently, Bénéteau et al. [13, 12] studied inner functions and examined the connections between them and optimal polynomial approximants. They described a method to construct inner functions that are analogues of finite Blaschke products with simple zeroes. In [52], Seco discussed inner functions on Dirichlet-type spaces and characterized such functions as those whose norm and multiplier norm are both equal to one. In [46] and [34], Felder and I obtained several new operator and function theoretic characterizations of inner functions on weighted Hardy spaces, which extend several previously known results. We described a construction of analogues of finite Blaschke products, using kernel functions. Our results complement those in [12]. It is striking that the proofs in [34] do not rely on orthogonality of monomials, which was thought to be essential in the study of inner and generalized inner functions.

9. Some future projects

9.1. It follows from their matrix representations that non-zero Toeplitz operators on the Hardy space are never compact. In the contrary, there exist lots of compact Toeplitz operators on the Bergman space A^2. On the other hand, the question about the existence of non-zero, finite rank Toeplitz operators on A^2 was open for quite some time. In 2008, Luecking [49] answered the question in the negative. He in fact proved a more general result, which also implies that if f is a bounded function with compact support on $\mathbb{C}$ such that the operator T_f has finite rank on the Segal–Bargmann space, then f must be zero. Borichev and Rozenblum [17] were able to remove the restriction on the support of f. I plan to make use of this result to study the finite rank product $T_f T_g$. As discussed in Section 2, this problem is still open. My first goal is to tackle the case where one of the functions is radial.

9.2. Back in 2003, B. Carswell et al. [20] studied composition operators on the Segal–Bargman space $\mathcal{F}_n^2$ over $\mathbb{C}^n$. They found necessary and sufficient conditions on the mapping φ for which C_φ is bounded or compact on $\mathcal{F}_n^2$. They also proved a norm formula for C_φ. In [44], I generalized their results to Segal–Bargmann spaces over arbitrary Hilbert spaces. The approach in [20], which relies on many tools only available in finite dimensions, does not work in this setting. I employed a different approach that makes extensive use of kernel functions. I also investigated spectral properties of these composition operators. I would like to extend the results in [41] to weighted composition operators on $\mathcal{F}_n^2$.

9.3. Even though isometric composition operators on the Hardy space over the unit disc was completely characterized by Nordgren in 1968, much less has been known on the Hardy space $H^2(\mathbb{S}_n)$ over the unit sphere in higher dimension. Certain examples of isometric composition operators C_φ on $H^2(\mathbb{S}_n)$ have been known but a complete characterization is still lacking. I am currently investigating particular classes of the symbol φ for which the problem is more tractable. A future plan is to characterize isometric products $C_\varphi C_\psi^*$ and $C_\psi^* C_\varphi$ in several variables.

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