

Normal and isometric weighted composition operators on the Fock space

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ABSTRACT

We obtain new and simple characterizations for the boundedness and compactness of weighted composition operators on the Fock space over \mathbb{C} . We also describe all weighted composition operators that are normal or isometric.

1. Introduction

Let \mathcal{H} be a Hilbert space whose elements are holomorphic functions on a domain Ω . For f a holomorphic function on Ω and $\varphi : \Omega \rightarrow \Omega$ a holomorphic map, the weighted composition operator $W_{f,\varphi}$ is defined as $W_{f,\varphi}h = f \cdot (h \circ \varphi)$. The domain of $W_{f,\varphi}$ consists of all $h \in \mathcal{H}$ for which $f \cdot (h \circ \varphi)$ also belongs to \mathcal{H} . When the weight function f is identically one, the operator $W_{f,\varphi}$ reduces to the composition operator C_φ . Researchers are often interested in how the function theoretic properties of f and φ affect the operator theoretic properties of C_φ and $W_{f,\varphi}$, and vice versa. There is a vast literature on composition operators acting on the Hardy, Bergman and other Banach spaces of holomorphic functions. The books [9, 18] are excellent references.

Weighted composition operators arose in the work of Forelli [11] on isometries of classical Hardy spaces H^p and in Cowen's work [4, 5] on commutants of analytic Toeplitz operators on the Hardy space H^2 of the unit disk. Weighted composition operators have also been used in descriptions of adjoints of composition operators, see [6] and the references therein. Boundedness and compactness of weighted composition operators on the Hardy and Bergman spaces have been studied by several mathematicians [3, 10, 17], just to list a few. In this paper we are interested in weighted composition operators acting on the Fock space over \mathbb{C} .

The Fock space \mathcal{F}^2 , also known as the Segal–Bargmann space, consists of all entire functions on the complex plane \mathbb{C} that are square integrable with respect to the Gaussian measure $d\mu(z) = \pi^{-1}e^{-|z|^2}dA(z)$. Here dA denotes the Lebesgue measure on \mathbb{C} . The inner product on \mathcal{F}^2 is given by

$$\langle f, g \rangle = \int_{\mathbb{C}} f(z)\overline{g(z)} d\mu(z) = \frac{1}{\pi} \int_{\mathbb{C}} f(z)\overline{g(z)} e^{-|z|^2} dA(z).$$

We shall use $\|\cdot\|$ to denote the corresponding norm. It is well known that the set $\{e_m(z) = z^m/\sqrt{m!} : m \geq 0\}$ forms an orthonormal basis for \mathcal{F}^2 . It is also well known that \mathcal{F}^2 is a reproducing kernel Hilbert space with kernel $K_z(w) = K(w, z) = e^{\bar{z}w}$, that is,

$$f(z) = \langle f, K_z \rangle$$

for all $f \in \mathcal{F}^2$ and $z \in \mathbb{C}$. In particular, $\|K_z\|^2 = \langle K_z, K_z \rangle = K_z(z) = e^{|z|^2}$. For more details on \mathcal{F}^2 , see [20, Chapter 2].

About a decade ago Carswell, MacCluer and Schuster [2] studied the boundedness and compactness of C_φ on the Fock space over \mathbb{C}^n . Their result, specified to the one dimensional

case, shows that C_φ is bounded if and only if $\varphi(z) = \lambda z$ with $|\lambda| = 1$ or $\varphi(z) = \lambda z + b$ with $|\lambda| < 1$ and $b \in \mathbb{C}$. They also showed that C_φ is compact if and only if $\varphi(z) = \lambda z + b$ with $|\lambda| < 1$. These results have been generalized to the Fock space of infinitely many variables by the current author in [15]. In [19], Ueki studied the boundedness and compactness of weighted composition operators $W_{f,\varphi}$. He showed that $W_{f,\varphi}$ is bounded on \mathcal{F}^2 if and only if a certain integral transform $z \mapsto B_\varphi(|f|^2)(z)$ is bounded. Likewise, $W_{f,\varphi}$ is compact if and only if the limsup of $B_\varphi(|f|^2)(z)$ equals zero as $|z| \rightarrow \infty$. However, these criteria are quite difficult to use. In fact, it is not clear how Ueki's results would reduce to Carswell-MacCluer-Schuster's aforementioned characterizations in the special case when f is a constant function. The first goal of the current paper is to report that much easier characterizations for the boundedness and compactness of $W_{f,\varphi}$ are available. In fact, Theorem 2.2 shows that $W_{f,\varphi}$ is bounded if and only if φ is an affine function and $|f(z)|^2 e^{|\varphi(z)|^2 - |z|^2}$ is bounded on \mathbb{C} . We shall also provide a similar criterion for the compactness.

Recently researchers have started investigating other operator theoretic properties of weighted composition operators. Cowen and Ko [8] and Cowen et al. [7] characterized self-adjoint weighted composition operators and studied their spectral properties on weighted Hardy spaces on the unit disk whose kernel functions are of the form $K_z(w) = (1 - \bar{z}w)^{-\kappa}$ for $\kappa \geq 1$. In [1], Bourdon and Narayan studied normal weighted composition operators on the Hardy space. They characterized unitary weighted composition operators and applied their characterization to describe all normal operators $W_{f,\varphi}$ in the case φ fixes a point in the unit disk. However, a full characterization of such operators is still lacking. The interested reader is referred to the recent paper [16] for a generalization to Bergman spaces and to higher dimensions.

In [14] Kumar and Partington, among other things, investigated isometric weighted composition operators on the Hardy space. They showed that $W_{f,\varphi}$ is isometric if and only if φ is inner and f is φ -2-inner, that is, $\|f\|_2 = 1$ and $|f|^2$ is orthogonal to φ^m for all $m = 1, 2, \dots$. A non-trivial example is $\varphi(z) = z^2$ and $f(z) = az + b$, where $|a|^2 + |b|^2 = 1$. On the other hand, much less is known about isometric weighted composition operators on Bergman spaces. In fact, a characterization of such operators is not currently available even though certain classes of examples have been known. The second goal of the present paper is to identify all normal and all isometric weighted composition operators on \mathcal{F}^2 . Since entire functions that induce bounded weighted composition operators are quite restrictive by Theorem 2.2, complete characterizations for normality and isometry of such operators are possible.

2. Boundedness and compactness

We begin with a result about entire functions over \mathbb{C} . This is a key result which helps us obtain characterizations of boundedness and compactness of weighted composition operators on \mathcal{F}^2 .

PROPOSITION 2.1. *Let f and φ be two entire functions on \mathbb{C} such that f is not identically zero. Suppose there is a positive constant M such that*

$$|f(z)|^2 e^{|\varphi(z)|^2 - |z|^2} \leq M \quad \text{for all } z \in \mathbb{C}. \quad (2.1)$$

Then $\varphi(z) = \varphi(0) + \lambda z$ for some $|\lambda| \leq 1$. If $|\lambda| = 1$, then $f(z) = f(0)e^{-\bar{\beta}z}$, where $\beta = \bar{\lambda}\varphi(0)$.

Furthermore, if

$$\lim_{|z| \rightarrow \infty} |f(z)|^2 e^{|\varphi(z)|^2 - |z|^2} = 0, \quad (2.2)$$

then $\varphi(z) = \lambda z + b$ with $|\lambda| < 1$.

Proof. Since f is not identically zero, there is an integer $k \geq 0$ and an entire function g with $g(0) \neq 0$ such that $f(z) = z^k g(z)$. The inequality (2.1) becomes $|z^k g(z)|^2 \exp(|\varphi(z)|^2 - |z|^2) \leq M$. Taking logarithms we obtain

$$|\varphi(z)|^2 - |z|^2 + 2k \log |z| + 2 \log |g(z)| \leq \log M \quad \text{for all } z \in \mathbb{C}.$$

Here we use the convention that $\log(0) = -\infty$. For any $R > 0$, putting $z = Re^{i\theta}$ and integrating with respect to θ on $[-\pi, \pi]$ yields

$$\int_{-\pi}^{\pi} |\varphi(Re^{i\theta})|^2 \frac{d\theta}{2\pi} - R^2 + 2k \log R + 2 \int_{-\pi}^{\pi} \log |g(Re^{i\theta})| \frac{d\theta}{2\pi} \leq \log M.$$

On other hand, Jensen's inequality gives

$$\int_{-\pi}^{\pi} \log |g(Re^{i\theta})| \frac{d\theta}{2\pi} \geq \log |g(0)|.$$

It then follows that

$$\int_{-\pi}^{\pi} |\varphi(Re^{i\theta})|^2 \frac{d\theta}{2\pi} - R^2 + 2k \log R + 2 \log |g(0)| \leq \log M. \quad (2.3)$$

Now consider the power expansion $\varphi(z) = \varphi(0) + \lambda z + \sum_{j=2}^{\infty} a_j z^j$ for $z \in \mathbb{C}$. The integral on the left hand side of (2.3) can be written as

$$\int_{-\pi}^{\pi} |\varphi(Re^{i\theta})|^2 \frac{d\theta}{2\pi} = |\varphi(0)|^2 + |\lambda|^2 R^2 + \sum_{j=2}^{\infty} |a_j|^2 R^{2j}.$$

We then obtain

$$|\varphi(0)|^2 + (|\lambda|^2 - 1)R^2 + \sum_{j=2}^{\infty} |a_j|^2 R^{2j} + 2k \log R + 2 \log |g(0)| \leq M.$$

Since this inequality holds for all $R > 0$, we conclude that $|\lambda| \leq 1$ and $a_j = 0$ for all $j \geq 2$. This implies $\varphi(z) = \varphi(0) + \lambda z$ as required.

If $|\lambda| = 1$ then with $\beta = \bar{\lambda}\varphi(0)$, we have $|\varphi(z)|^2 = |z|^2 + \bar{\beta}z + \beta\bar{z} + |\varphi(0)|^2$. Inequality (2.1) can be rewritten as

$$|f(z)e^{\bar{\beta}z}|^2 \leq M e^{-|\varphi(0)|^2} \quad \text{for all } z \in \mathbb{C}.$$

Liouville's theorem implies that $f(z)e^{\bar{\beta}z}$ is a constant function, which gives $f(z) = f(0)e^{-\bar{\beta}z}$ for all $z \in \mathbb{C}$. In this case, the left hand side of (2.1) is the constant $|f(0)|^2 e^{|\varphi(0)|^2}$. Therefore, if (2.2) holds then $|\lambda| < 1$. \square

We shall also need the following well-known formula. Suppose f and φ are entire functions on \mathbb{C} such that the operator $W_{f,\varphi}$ is bounded on \mathcal{F}^2 . Then for any $z \in \mathbb{C}$,

$$W_{f,\varphi}^* K_z = \overline{f(z)} K_{\varphi(z)}. \quad (2.4)$$

We are now in a position to obtain necessary and sufficient conditions for the boundedness of $W_{f,\varphi}$.

THEOREM 2.2. *Let f and φ be entire functions on \mathbb{C} such that f is not identically zero. Then $W_{f,\varphi}$ is bounded on \mathcal{F}^2 if and only if f belongs to \mathcal{F}^2 , $\varphi(z) = \varphi(0) + \lambda z$ with $|\lambda| \leq 1$ and*

$$M(f, \varphi) := \sup \left\{ |f(z)|^2 \exp(|\varphi(z)|^2 - |z|^2) : z \in \mathbb{C} \right\} < \infty. \quad (2.5)$$

Proof. Suppose first that $W_{f,\varphi}$ is bounded on \mathcal{F}^2 . Then $f = W_{f,\varphi}1$ belongs to \mathcal{F}^2 . For $z \in \mathbb{C}$, formula 2.4 gives $W_{f,\varphi}^* K_z = \bar{f}(z) K_{\varphi(z)}$. This implies

$$\begin{aligned} |f(z)|^2 \exp(|\varphi(z)|^2 - |z|^2) &= \frac{|\bar{f}(z)|^2 \|K_{\varphi(z)}\|^2}{\|K_z\|^2} \\ &= \frac{\|W_{f,\varphi}^* K_z\|^2}{\|K_z\|^2} \leq \|W_{f,\varphi}^*\|^2 = \|W_{f,\varphi}\|^2. \end{aligned}$$

Consequently, $M(f, \varphi) \leq \|W_{f,\varphi}\|^2 < \infty$. Proposition 2.1 then shows that $\varphi(z) = \varphi(0) + \lambda z$ for some $|\lambda| \leq 1$.

Now let us prove the converse. We have two cases: $\lambda = 0$ and $\lambda \neq 0$. Consider first the case $\lambda = 0$, so $\varphi(z) = \varphi(0)$ for all z . For any $h \in \mathcal{F}^2$ we have $W_{f,\varphi}h = h(\varphi(0))f = \langle h, K_{\varphi(0)} \rangle f$. Since f belongs to \mathcal{F}^2 , the operator $W_{f,\varphi}$ is bounded with norm $\|f\| \|K_{\varphi(0)}\|$.

Now assume $\lambda \neq 0$. For $h \in \mathcal{F}^2$, we compute

$$\begin{aligned} \|W_{f,\varphi}h\|^2 &= \int_{\mathbb{C}} |f(z)|^2 |h(\varphi(z))|^2 e^{-|z|^2} \frac{dA(z)}{\pi} \\ &\leq M(f, \varphi) \int_{\mathbb{C}} |h(\varphi(z))|^2 e^{-|\varphi(z)|^2} \frac{dA(z)}{\pi} \\ &= |\lambda|^{-2} M(f, \varphi) \int_{\mathbb{C}} |h(w)|^2 e^{-|w|^2} \frac{dA(w)}{\pi} \\ &\quad (\text{by the change of variables } w = \varphi(z) = \lambda z + \varphi(0)) \\ &= |\lambda|^{-2} M(f, \varphi) \|h\|^2. \end{aligned}$$

It follows that $W_{f,\varphi}$ is bounded with norm at most $|\lambda|^{-1} (M(f, \varphi))^{1/2}$. This completes the proof of the theorem. \square

EXAMPLE 1. Suppose $m \geq 1$ is an integer and $0 < s < 1$. Put $\varphi_s(z) = sz$ and $f_m(z) = z^m$. The proof of Theorem 2.2 gives $\|W_{f_m, \varphi_s}\|^2 \geq M(f_m, \varphi_s)$. On the other hand, using Stirling's approximation we have

$$\|W_{f_m, \varphi_s}\|^2 \geq \|W_{f_m, \varphi_s}1\|^2 = \|f_m\|^2 = m! \approx \sqrt{2\pi m} \left(\frac{m}{e}\right)^m.$$

Also a direct calculation shows

$$M(f_m, \varphi_s) = \sup \left\{ |z|^{2m} \exp(s^2|z|^2 - |z|^2) : z \in \mathbb{C} \right\} = \frac{1}{(1-s^2)^m} \left(\frac{m}{e}\right)^m.$$

This implies that there does not exist a positive constant C such that

$$\|W_{f_m, \varphi_s}\|^2 \leq C \cdot M(f_m, \varphi_s)$$

for all integers $m \geq 1$ and all $0 < s < 1$. Consequently, the quantities $\|W_{f,\varphi}\|^2$ and $M(f, \varphi)$ are not equivalent.

Now we suppose that $W_{f,\varphi}$ is a non-zero compact operator on \mathcal{F}^2 . Then $W_{f,\varphi}^*$ is also compact. The well-known fact that $\|K_z\|^{-1} K_z \rightarrow 0$ weakly as $|z| \rightarrow \infty$ implies $\|K_z\|^{-2} \|W_{f,\varphi}^* K_z\|^2 \rightarrow 0$. Since $W_{f,\varphi}^* K_z = \bar{f}(z) K_{\varphi(z)}$ and $\|K_a\| = e^{|a|^2}$ for any $a \in \mathbb{C}$, we obtain

$$\lim_{|z| \rightarrow \infty} |f(z)|^2 e^{|\varphi(z)|^2 - |z|^2} = 0. \quad (2.6)$$

It turns out that the converse also holds, which is our second result.

THEOREM 2.3. *Let f, φ be entire functions such that f is not identically zero. Then $W_{f,\varphi}$ is compact on \mathcal{F}^2 if and only if $\varphi(z) = \varphi(0) + \lambda z$ for some $|\lambda| < 1$ and (2.6) holds.*

Proof. If $W_{f,\varphi}$ is compact, then the discussion preceding the theorem shows that (2.6) holds. Proposition 2.1 provides the desired form of φ .

We now prove the reverse direction. If $\lambda = 0$, then as in the proof of Theorem 2.2, the operator $W_{f,\varphi}$ has rank one, hence compact. Suppose $\lambda \neq 0$. Let $\{h_m\}$ be a sequence in \mathcal{F}^2 converging weakly to 0. It is well known that the sequence $\{h_m\}$ is bounded in norm and converges to zero uniformly on compact subsets of \mathbb{C} . We need to show that $\|W_{f,\varphi}h_m\| \rightarrow 0$ as $m \rightarrow \infty$. For $w \in \mathbb{C}$, define

$$F(w) = |\lambda|^{-2} |f(\varphi^{-1}(w))|^2 e^{|w|^2 - |\varphi^{-1}(w)|^2}.$$

Since $\lim_{|w| \rightarrow \infty} |\varphi^{-1}(w)| = \infty$, (2.6) shows that $\lim_{|w| \rightarrow \infty} F(w) = 0$.

Now for any $R > 0$, we compute

$$\begin{aligned} \|W_{f,\varphi}h_m\|^2 &= \pi^{-1} \int_{\mathbb{C}} |f(z)|^2 |h_m(\varphi(z))|^2 e^{-|z|^2} dA(z) \\ &= \pi^{-1} \int_{\mathbb{C}} |\lambda|^2 F(\varphi(z)) |h_m(\varphi(z))|^2 e^{-|\varphi(z)|^2} dA(z) \\ &= \pi^{-1} \int_{\mathbb{C}} F(w) |h_m(w)|^2 e^{-|w|^2} dA(w) \\ &\quad (\text{by the change of variables } w = \varphi(z) = \lambda z + \varphi(0)) \\ &\leq \|F\|_{\infty} \int_{|w| < R} |h_m(w)|^2 d\mu(z) + \left(\sup_{|w| > R} F(w) \right) \|h_m\|^2. \end{aligned}$$

The second quantity in the last sum can be made arbitrarily small by taking R sufficiently large. Once R has been chosen, the first quantity can be made arbitrarily small by choosing m sufficiently large. We conclude that $\|W_{f,\varphi}h_m\| \rightarrow 0$ as $m \rightarrow \infty$. Therefore, $W_{f,\varphi}$ is compact on \mathcal{F}^2 . \square

REMARK 1. Ueki [19] showed that the essential norm $\|W_{f,\varphi}\|_e$ is equivalent to $\limsup_{|z| \rightarrow \infty} B_{\varphi}(|f|^2)(z)$, where $B_{\varphi}(|f|^2)$ is the integral transform

$$B_{\varphi}(|f|^2)(z) = \int_{\mathbb{C}} |f(\zeta)|^2 |e^{\langle \varphi(\zeta), z \rangle}|^2 e^{-|z|^2} d\mu(\zeta).$$

Inspired by Theorem 2.3, we define

$$M_e(f, \varphi) := \limsup_{|z| \rightarrow \infty} |f(z)|^2 e^{|\varphi(z)|^2 - |z|^2}.$$

Theorem 2.3 shows that $\|W_{f,\varphi}\|_e = 0$ if and only if $M_e(f, \varphi) = 0$. It would be an interesting problem to explore the relationship between $\|W_{f,\varphi}\|_e$ and $M_e(f, \varphi)$.

3. Normal weighted composition operators

We first discuss a well-known class of unitary weighted composition operators on \mathcal{F}^2 . These operators arise as the change of variables with respect to the Gaussian measure. Recall that for $z \in \mathbb{C}$, the function $K_z(w) = e^{\bar{z}w}$ is the reproducing kernel function for \mathcal{F}^2 at z . We define the corresponding normalized reproducing kernel as $k_z(w) = \|K_z\|^{-1} K_z(w) = e^{\bar{z}w - |z|^2/2}$.

PROPOSITION 3.1. *Let $\varphi(z) = \lambda z - u$, where $|\lambda| = 1$ and u is an arbitrary complex number. Then $W_{k_{\bar{\lambda}u}, \varphi}$ is a unitary operator in \mathcal{F}^2 and we have*

$$W_{k_{\bar{\lambda}u}, \varphi}^{-1} = W_{k_{\bar{\lambda}u}, \varphi}^* = W_{k_{-u}, \varphi^{-1}}.$$

Proof. Let h be in \mathcal{F}^2 . The change of variables $w = \lambda z - u$ yields

$$\begin{aligned} \|h\|^2 &= \int_{\mathbb{C}} |h(w)|^2 e^{-|w|^2} \frac{dA(w)}{\pi} = \int_{\mathbb{C}} |h(\lambda z - u)|^2 e^{-|\lambda z - u|^2} \frac{dA(z)}{\pi} \\ &= \int_{\mathbb{C}} |e^{\lambda \bar{u} z - |u|^2/2} h(\lambda z - u)|^2 e^{-|z|^2} \frac{dA(z)}{\pi} \\ &= \int_{\mathbb{C}} |k_{\bar{\lambda}u}(z) h(\lambda z - u)|^2 d\mu(z) = \|W_{k_{\bar{\lambda}u}, \varphi} h\|^2. \end{aligned}$$

This implies that $W_{k_{\bar{\lambda}u}, \varphi}$ is an isometry, that is, $W_{k_{\bar{\lambda}u}, \varphi}^* W_{k_{\bar{\lambda}u}, \varphi} = I$. Now for $h \in \mathcal{F}^2$ and $z \in \mathbb{C}$, we have

$$\begin{aligned} (W_{k_{\bar{\lambda}u}, \varphi} W_{k_{-u}, \varphi^{-1}} h)(z) &= k_{\bar{\lambda}u}(z) k_{-u}(\varphi(z)) \cdot h(\varphi \circ \varphi^{-1}(z)) \\ &= k_{\bar{\lambda}u}(z) k_{-u}(\lambda z - u) h(z) \\ &= \exp(\lambda \bar{u} z - |u|^2/2) \exp(-\bar{u}(\lambda z - u) - |u|^2/2) h(z) \\ &= h(z). \end{aligned}$$

This shows that $W_{k_{\bar{\lambda}u}, \varphi} W_{k_{-u}, \varphi^{-1}} = I$. Therefore, $W_{k_{\bar{\lambda}u}, \varphi}$ is a unitary operator whose inverse is $W_{k_{-u}, \varphi^{-1}}$. \square

REMARK 2. In the case $\varphi(z) = z - u$, we write $W_u = W_{k_u, \varphi}$. These operators are known as the Weyl unitaries. They satisfy the Weyl commutation relation:

$$W_u W_v = e^{i\Im(u\bar{v})} W_{u+v}.$$

In the rest of the section, we would like to identify all normal weighted composition operators on \mathcal{F}^2 . It turns out that any such operator is either a constant multiple of a unitary in Proposition 3.1 or a normal compact operator.

Bourdon and Nayaran [1] studied normal weighted composition operators $W_{f, \varphi}$ on the Hardy space over the unit disk. They completely characterized the case where φ has an interior fixed point. On the other hand, the case where all fixed points of φ lie on the circle has not been well understood. A class of examples with φ being parabolic linear fractional maps was given in [1]. However, the general characterization remains unknown. The situation on \mathcal{F}^2 turns out to be more manageable. Due to Proposition 2.1, any mapping φ that induces a bounded weighted composition operator on \mathcal{F}^2 must be affine. It is this restriction that allows us to obtain a complete characterization of normal weighted composition operators on \mathcal{F}^2 .

THEOREM 3.2. *Let f and φ be entire functions such that f is not identically zero. Then the operator $W_{f, \varphi}$ is a normal bounded operator on \mathcal{F}^2 if and only if one of the following two cases occurs:*

(a) $\varphi(z) = \lambda z + b$ with $|\lambda| = 1$ and $f = f(0)K_{-\bar{\lambda}b}$. In this case, $W_{f, \varphi}$ is a constant multiple of a unitary operator.

(b) $\varphi(z) = \lambda z + b$ with $|\lambda| < 1$ and $f = f(0)K_c$, where $c = b(1 - \lambda)^{-1}(1 - \bar{\lambda})$. In this case, $W_{f, \varphi}$ is unitarily equivalent to $f(0)C_{\lambda z}$.

Proof. Suppose first that $W_{f,\varphi}$ is a normal bounded operator. By Theorem 2.2, the boundedness implies that $\varphi(z) = \lambda z + b$ for $|\lambda| \leq 1$, and that $|f(z)|^2 e^{|\varphi(z)|^2 - |z|^2}$ is bounded on \mathbb{C} . We consider two cases.

Case 1: $|\lambda| = 1$. Proposition 2.1 asserts that

$$f = f(0)K_{-\bar{\lambda}b} = f(0)e^{|b|^2/2}k_{-\bar{\lambda}b}.$$

Consequently, $W_{f,\varphi}$ is a constant multiple of the unitary operator $W_{k_{-\bar{\lambda}b},\varphi}$.

Case 2: $|\lambda| < 1$. In this case, the function φ has a unique fixed point $a = b(1 - \lambda)^{-1}$. By the adjoint formula (2.4),

$$W_{f,\varphi}^* K_a = \overline{f(a)} K_{\varphi(a)} = \overline{f(a)} K_a.$$

This says that K_a is an eigenvector of $W_{f,\varphi}^*$ corresponding to the eigenvalue $\overline{f(a)}$. The normality of $W_{f,\varphi}$ implies $W_{f,\varphi} K_a = f(a)K_a$, which is equivalent to $f \cdot (K_a \circ \varphi) = f(a)K_a$. Therefore, for any $z \in \mathbb{C}$,

$$f(z) = f(a) \frac{K_a(z)}{K_a(\varphi(z))} = f(a)K_a(z - \varphi(z)). \quad (3.1)$$

A direct calculation shows that $W_a^* W_{f,\varphi} W_a = W_{-a} W_{f,\varphi} W_a = W_{g,\psi}$, where

$$\psi(z) = \varphi(z + a) - a = \lambda(z + a) + b - a = \lambda z$$

and

$$\begin{aligned} g(z) &= k_{-a}(z)k_a(\varphi(z + a))f(z + a) \\ &= \exp(-|a|^2)K_a(-z + \varphi(z + a))f(a)K_a(z + a - \varphi(z + a)) = f(a). \end{aligned}$$

We conclude that $W_{f,\varphi}$ is unitarily equivalent to $f(a)C_{\lambda z}$.

To obtain the explicit formula for f , we substitute $z - \varphi(z) = (1 - \lambda)z - b$ into (3.1):

$$f(z) = f(a) \exp(-\bar{a}b)K_{a(1-\bar{\lambda})}(z) = f(a) \exp(-\bar{a}b)K_c(z).$$

Setting $z = 0$ yields $f(a) \exp(-\bar{a}b) = f(0)$. Consequently, $f = f(0)K_c$.

Conversely, if (a) holds, then by Case 1, $W_{f,\varphi}$ is a constant multiple of a unitary operator, hence normal. If (b) holds, then the calculations in Case 2 show that $W_{f,\varphi}$ is unitarily equivalent to $f(0)C_{\lambda z}$, which is diagonalizable with respect to the standard orthonormal basis $\{z^m/\sqrt{m!} : m \geq 0\}$. Consequently, $W_{f,\varphi}$ is a normal operator. \square

4. Isometric weighted composition operators

In this section we investigate isometric weighted composition operators. Recall that an operator T on \mathcal{F}^2 is isometric if $\|Th\| = \|h\|$ for all $h \in \mathcal{F}^2$, or equivalently, $\langle Th, Tg \rangle = \langle h, g \rangle$ for all $h, g \in \mathcal{F}^2$.

We shall need the following uniqueness result.

LEMMA 4.1. *Let u be a measurable function such that $|u(z)| \leq C e^{|z|^2 - \epsilon|z|}$ for some constants $\epsilon > 0$ and $C > 0$. If for all integers $m, k \geq 0$,*

$$\int_{\mathbb{C}} u(z) z^m \bar{z}^k d\mu(z) = 0, \quad (4.1)$$

then $u = 0$ a.e. on \mathbb{C} .

REMARK 3. The conclusion of Lemma 4.1 does not hold if the growth condition on u is dropped. A counterexample is $u(z) = \sin(\sqrt{|z|})e^{|z|^2 - \sqrt{|z|}}$. In fact, when $m \neq k$, the integral

(4.1) is zero by the rotation invariance of $d\mu$. When $m = k$, we have

$$\begin{aligned}
 \int_{\mathbb{C}} u(z) |z|^{2m} d\mu(z) &= \frac{1}{\pi} \int_{\mathbb{C}} \sin(\sqrt{|z|}) e^{-\sqrt{|z|}} |z|^{2m} dA(z) \\
 &= 2 \int_0^\infty \sin(\sqrt{r}) e^{-\sqrt{r}} r^{2m+1} dr \\
 &\quad \text{(by integration in polar coordinates and Fubini's theorem)} \\
 &= 4 \int_0^\infty \sin(t) e^{-t} t^{4m+3} dt \quad \text{(by the change of variables } r = t^2) \\
 &= \frac{\Gamma(4m+4)}{2^{2m}} \sin((m+1)\pi) \\
 &= 0.
 \end{aligned}$$

The second last equality follows from Formula 3.944(5) in [12, p. 498].

Proof of Lemma 4.1. For integers $m, k \geq 0$, since

$$\int_{\mathbb{C}} |u(z) z^m \bar{z}^k| d\mu(z) \leq C \int_{\mathbb{C}} e^{|z|^2 - \epsilon|z|} |z|^{m+k} d\mu(z) = \frac{C}{\pi} \int_{\mathbb{C}} |z|^{m+k} e^{-\epsilon|z|} dA(z) < \infty,$$

we may use Fubini's theorem and integration in polar coordinates to obtain

$$0 = \int_{\mathbb{C}} u(z) z^m \bar{z}^k d\mu(z) = \frac{1}{\pi} \int_0^\infty \left(\int_0^{2\pi} u(re^{i\theta}) e^{i(m-k)\theta} d\theta \right) r^{m+k+1} e^{-r^2} dr.$$

Fix $\ell \in \mathbb{Z}$. For any integer $j \geq 0$, putting $m = j + |\ell| + \ell$ and $k = j + |\ell|$ yields

$$\int_0^\infty \left(\int_0^{2\pi} u(re^{i\theta}) e^{i\ell\theta} d\theta \right) r^{2|\ell|+\ell} r^{2j+1} e^{-r^2} dr = 0.$$

Because this holds for all integers $j \geq 0$, [13, Theorem 3.6] shows that

$$\int_0^{2\pi} u(re^{i\theta}) e^{i\ell\theta} d\theta = 0$$

for a.e. $r > 0$. Since ℓ was arbitrary, we conclude that $u = 0$ a.e. on \mathbb{C} . \square

We now investigate isometric operators $W_{f,\varphi}$ when φ is a constant multiple of z . The general case will then be obtained by multiplying $W_{f,\varphi}$ by an appropriate Weyl unitary. The proof of Theorem 4.3 below will provide the details.

PROPOSITION 4.2. *Let $\varphi(z) = \lambda z$ for some $|\lambda| \leq 1$ and let f be in \mathcal{F}^2 . Then $W_{f,\varphi}$ is an isometry on \mathcal{F}^2 if and only if $|\lambda| = 1$ and f is a constant of modulus one.*

Proof. The “if” part is clear since when $|\lambda| = 1$, the operator C_φ is a unitary by Proposition 3.1. This fact also follows from a direct calculation that $C_\varphi(z^m) = \lambda^m z^m$ for all integers $m \geq 0$.

To prove the “only if” part, suppose $W_{f,\varphi}$ is an isometry. Then $\lambda \neq 0$, otherwise the kernel of $W_{f,\varphi}$ would contain all monomials of degree greater than zero. For any $h, g \in \mathcal{F}^2$, the change of variables $w = \lambda z$ gives

$$\langle h, g \rangle = \int_{\mathbb{C}} h(w) \overline{g(w)} d\mu(w) = |\lambda|^2 \int_{\mathbb{C}} h(\lambda z) \overline{g(\lambda z)} e^{(1-|\lambda|^2)|z|^2} d\mu(z),$$

since $d\mu(w) = \pi^{-1} e^{-|\lambda|^2|z|^2} dA(\lambda z) = |\lambda|^2 e^{(1-|\lambda|^2)|z|^2} d\mu(z)$. On the other hand,

$$\langle W_{f,\varphi} h, W_{f,\varphi} g \rangle = \int_{\mathbb{C}} |f(z)|^2 h(\lambda z) \overline{g(\lambda z)} d\mu(z).$$

For non-negative integers m, k , setting $h(z) = z^m$, $g(z) = z^k$ and using the identity $\langle h, g \rangle = \langle W_{f,\varphi}h, W_{f,\varphi}g \rangle$, we obtain

$$\int_{\mathbb{C}} (|f(z)|^2 - |\lambda|^2 e^{(1-|\lambda|^2)|z|^2}) z^m \bar{z}^k d\mu(z) = 0.$$

By Theorem 2.2, $|f(z)|^2 \leq C e^{(1-|\lambda|^2)|z|^2}$ for some constant $C > 0$. Consequently, we may apply Lemma 4.1 to conclude that

$$|f(z)|^2 = |\lambda|^2 e^{(1-|\lambda|^2)|z|^2} \quad \text{for all } z \in \mathbb{C}.$$

This implies

$$f(z)\overline{f(w)} = |\lambda|^2 e^{(1-|\lambda|^2)z\bar{w}} \quad \text{for all } z, w \in \mathbb{C}.$$

Setting $w = 0$ immediately yields that f is a constant function and hence $|\lambda| = 1$. This completes the proof of the proposition. \square

We are now in a position to describe all isometric weighted composition operators on \mathcal{F}^2 .

THEOREM 4.3. *Let f, φ be entire functions on \mathbb{C} . Then the operator $W_{f,\varphi}$ is isometric on \mathcal{F}^2 if and only if it is a unimodular multiple of a unitary operator in Proposition 3.1.*

Proof. It suffices to consider the “only if” part. By Theorem 2.2, we have $\varphi(z) = \lambda z + b$ with $|\lambda| \leq 1$. Put $W_{\tilde{f},\tilde{\varphi}} = W_{f,\varphi}W_b$, where W_b is the Weyl unitary corresponding to b . We have $\tilde{\varphi}(z) = \varphi(z) - b = \lambda z$. Since $W_{\tilde{f},\tilde{\varphi}}$ is isometric, Proposition 4.2 shows that $|\lambda| = 1$ and $\tilde{f} = \alpha$ for some $|\alpha| = 1$. It follows that

$$W_{f,\varphi} = W_{\tilde{f},\tilde{\varphi}}W_b^{-1} = W_{\tilde{f},\tilde{\varphi}}W_{-b} = \alpha C_{\lambda z}W_{k_{-b},z+b} = \alpha W_{k_{-\bar{\lambda}b},\varphi}.$$

Note that $W_{k_{-\bar{\lambda}b},\varphi}$ is a unitary operator described by Proposition 3.1. The proof of the theorem is now completed. \square

REMARK 4. Theorem 4.3 asserts that all isometric weighted composition operator on \mathcal{F}^2 are unitary. As we mentioned in the Introduction, [14, Theorem 2.1] shows that this is not the case on the Hardy space over the unit circle. On the Bergman space A^2 over the unit disk, there are also non-unitary isometric weighted composition operators. More specifically, let $\varphi(z) = z^m$ and $f(z) = \sqrt{m} z^{m-1}$ for $m \geq 2$. Then it can be checked that $W_{f,\varphi}$ is isometric on A^2 . On the other hand, such an operator is not surjective since every function in its range vanishes at the origin. To conclude the paper, we would like to remind the reader that a complete characterization of isometric weighted composition operators on A^2 is currently unavailable. Any progress on this problem will be of interest to many researchers.

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