A refined Luecking's theorem and finite-rank products of Toeplitz operators

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Abstract. For a bounded measurable function f on the open unit disk $\mathbb{D}$, let T_f denote the corresponding Toeplitz operator on the Bergman space $A^2(\mathbb{D})$. A recent result of D. Luecking shows that if T_f has finite rank, then f must be the zero function. Using a refined version of this result, we show that if all, except possibly one, of the functions $f_1, \ldots, f_m$ are radial and $T_{f_1} \cdots T_{f_m}$ has finite rank, then one of these functions must be zero.

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1. Introduction

As usual, let $\mathbb{D}$ denote the unit disk and $\mathbb{T}$ denote the unit circle on the complex plane $\mathbb{C}$. Let $\mathrm{d}A$ denote the Lebesgue measure on $\mathbb{C}$ which is normalized so that the unit disk has total mass 1. We have $\mathrm{d}A(z) = \frac{1}{\pi}\mathrm{d}x\mathrm{d}y$, where $z = x + \mathrm{i}y$ for x, y real numbers. We write L^2 for $L^2(\mathbb{D}, \mathrm{d}A)$. The Bergman space A^2 is the subspace of L^2 that consists of all holomorphic functions. It is well known that A^2 is a closed subspace of L^2. The standard orthonormal basis for A^2 is given by $\{e_m : m = 0, 1, \ldots\}$, where $e_m(z) = \sqrt{m+1}\ z^m$ for non-negative integers m. Let P denote the orthogonal projection from L^2 onto A^2. For any function $f \in L^2$, the Toeplitz operator with symbol f is denoted by T_f, which is densely defined on A^2 by $T_f\varphi = P(f\varphi)$ for $\varphi \in H^\infty$, the space of all bounded holomorphic functions on $\mathbb{D}$. The operator T_f is in fact an integral operator given by the formula

$$(T_f\varphi)(z) = \int_{\mathbb{D}} \frac{f(w)\varphi(w)}{(1-\bar{w}z)^2}\mathrm{d}A(w), \text{ for } z \in \mathbb{D}, \varphi \in H^\infty.$$

If f is a bounded function then T_f is a bounded operator on A^2 with $\|T_f\| \leq \|f\|_\infty$ and $(T_f)^* = T_{\bar{f}}$. However, unbounded symbols can also give rise to bounded

Toeplitz operators. In fact, since T_f is an integral operator with kernel $\dfrac{f(w)}{(1-\bar{w}z)^2}$, $z, w \in \mathbb{D}$, we see that if $f \in L^2$ is supported in a compact subset of $\mathbb{D}$, then T_f is a compact operator on A^2.

A function f on $\mathbb{D}$ is called a radial function if $f(z) = f(|z|)$ for almost all $z \in \mathbb{D}$. If $f \in L^2$ is radial, then using polar coordinates, we see that

$$\begin{aligned}\langle T_f e_m, e_k\rangle &= \sqrt{(m+1)(k+1)}\int_{\mathbb{D}} f(z) z^m \bar{z}^k \mathrm{d}A(z)\\ &= \begin{cases} 0 & \text{if } m \neq k\\ (m+1)\int_0^1 2f(t)t^{2m+1}\mathrm{d}t & \text{if } m = k\end{cases}\\ &= \begin{cases} 0 & \text{if } m \neq k\\ (m+1)\int_0^1 f(r^{1/2})r^m\mathrm{d}r & \text{if } m = k.\end{cases}\end{aligned}$$

This shows that T_f is diagonal with respect to the standard orthonormal basis. The eigenvalues of T_f are given by

$$\omega(f,m) = \langle T_f e_m, e_m\rangle = (m+1)\int_0^1 f(r^{1/2})r^m\mathrm{d}r, \quad m = 0, 1, \ldots. \tag{1.1}$$

It follows from Stone-Weierstrass's theorem that if $f \in L^2$ such that T_f is the zero operator, then f must vanish almost everywhere in $\mathbb{D}$. On the other hand, the problem of determining whether there exists a nontrivial finite-rank Toeplitz operator on A^2 was open for quite some time. Recently D. Luecking [9] has found an elegant proof that gives the negative answer to this problem.

There is an extensive literature on Toeplitz operators on the Hardy space H^2 of the unit circle. We refer the reader to [10] for definitions of H^2 and their Toeplitz operators. In the context of Toeplitz operators on H^2, it was showed by A. Brown and P.R. Halmos [3] back in the 1960's that if f and g are bounded functions on the unit circle then $T_g T_f$ is another Toeplitz operator if and only if either f or $\bar{g}$ is holomorphic. From this it is readily deduced that if $f, g \in L^\infty(\mathbb{T})$ such that $T_g T_f = 0$, then one of the symbols must be the zero function. In contrast with this, for Toeplitz operators on the Bergman space, it has not been known if it is true that for $f, g \in L^\infty(\mathbb{D})$, $T_g T_f = 0$ implies g or f is the zero function. Affirmative answers have been obtained by researchers only in special cases. In [1], P. Ahern and Ž. Čučković answered the question affirmatively under the assumption that both f and g are bounded harmonic functions on $\mathbb{D}$. Later in [4], Čučković was able to show that if f, g are bounded such that f is harmonic and $g(r\mathrm{e}^{\mathrm{i}\theta}) = \sum_{m=-\infty}^{N} g_k(r)\mathrm{e}^{\mathrm{i}m\theta}$ for $z = r\mathrm{e}^{\mathrm{i}\theta} \in \mathbb{D}$, then $T_g T_f = 0$ implies either $f = 0$ or $g = 0$. The case one of the symbols is a bounded radial function has also been understood. See [2] and [7] for more details. In fact, in [7], the author was able to show that if all, except possibly one, of the functions $f_1, \ldots, f_M$ are bounded radial functions and $T_{f_1}\cdots T_{f_M} = 0$ then one of these functions must be zero.

A more general problem than the above zero product problem is the finite-rank product problem. Recall that the above mentioned theorem of Luecking shows that if $f \in L^2$ and T_f has finite rank, then f is the zero function. What happens if $T_g T_f$ has finite rank, where f and g are bounded measurable functions on the unit disk? The answer to this question seems to be still far from completed but the following important case has been understood: If f and g are bounded harmonic functions, then one of them must be the zero function (see [6]). The purpose of this paper is to report the same answer in some other special cases.

In the first part of the paper, we use Luecking's theorem to show that if f, g are functions in L^2, where f satisfies a certain condition, and $T_g T_f$ (which is densely defined on A^2) has finite rank, then either $f = 0$ or $g = 0$. In the second part of the paper, we prove a refined version of Luecking's theorem and use it to show that if $f_1, \ldots, f_{m_1}$ and $g_1, \ldots, g_{m_2}$ are radial functions in L^∞ and f is a function in L^2 such that $T_{g_1} \cdots T_{g_{m_2}} T_f T_{f_1} \cdots T_{f_{m_1}}$ has finite rank, then one of the above functions must be zero.

2. Finite-rank products of two Toeplitz operators

We begin this section with a detailed discussion of the decomposition $L^2 = \bigoplus_{m \in \mathbb{Z}} \mathcal{R} \mathrm{e}^{\mathrm{i}m\theta}$, where

$$\mathcal{R} = \{u : [0,1) \to \mathbb{C} \text{ such that } \int_0^1 |u(r)|^2 r \mathrm{d}r < \infty\}.$$

This decomposition has been used by Čučković and Rao in their studies of Toeplitz operators (see Section 2 in [5]). Let $f \in L^2(\mathbb{D})$. Then for almost all $r \in [0,1)$, the function $\zeta \mapsto f(r\zeta)$ for $\zeta \in \mathbb{T}$ is in $L^2(\mathbb{T}, \frac{1}{2\pi}\mathrm{d}\theta)$. Since $\{\zeta^m : m \in \mathbb{Z}\}$ is an orthonormal basis for $L^2(\mathbb{T}, \frac{1}{2\pi}\mathrm{d}\theta)$, we have

$$f(r\zeta) = \sum_{m=-\infty}^{\infty} \Big(\frac{1}{2\pi} \int_0^{2\pi} f(r\mathrm{e}^{\mathrm{i}\theta}) \mathrm{e}^{-\mathrm{i}m\theta} \mathrm{d}\theta \Big) \zeta^m,$$

where the sum takes place in $L^2(\mathbb{T})$. For $m \in \mathbb{Z}$, define

$$f_m(r) = \frac{1}{2\pi} \int_0^{2\pi} f(r\mathrm{e}^{\mathrm{i}\theta}) \mathrm{e}^{-\mathrm{i}m\theta} \mathrm{d}\theta, \quad 0 \le r < 1.$$

Then the above representation becomes (with $\zeta = \mathrm{e}^{\mathrm{i}\theta}$),

$$f(r\mathrm{e}^{\mathrm{i}\theta}) = \sum_{m=-\infty}^{\infty} f_m(r) \mathrm{e}^{\mathrm{i}m\theta}. \tag{2.1}$$

This representation holds for almost all $r \in [0,1)$ and for such r, the sum on the right hand side takes place in $L^2(\mathbb{T})$. Now we have

$$\begin{aligned}\|f\|^2_{L^2(\mathbb{D})} &= \int_0^1 \Big(\frac{1}{2\pi}\int_0^{2\pi} |f(r\mathrm{e}^{\mathrm{i}\theta})|^2 \mathrm{d}\theta\Big)(2r)\mathrm{d}r \\ &= \int_0^1 \Big(\sum_{m=-\infty}^{\infty} |f_m(r)|^2\Big)(2r)\mathrm{d}r \\ &= \sum_{m=-\infty}^{\infty} \int_0^1 |f_m(r)|^2 (2r)\mathrm{d}r.\end{aligned}$$

This shows that $f_m \in \mathcal{R}$ for all $m \in \mathbb{Z}$ and the right hand side of (2.1) converges in $L^2(\mathbb{D})$. Therefore, the representation (2.1) in fact takes place in $L^2(\mathbb{D})$.

The following theorem is our first result in the paper. It has been brought to our attention recently that this result was also independently obtained by I. Louhichi, N.V. Rao and A. Yousef in [8].

Theorem 2.1. *Suppose $f \in L^2$ and $f(r\mathrm{e}^{\mathrm{i}\theta}) = \sum_{m=-\infty}^{M} f_m(r)\mathrm{e}^{\mathrm{i}m\theta}$ for $z = r\mathrm{e}^{\mathrm{i}\theta}$, where M is an integer. Assume that $\int_0^1 f_M(r)r^k \mathrm{d}r \neq 0$ for all $k \geq N$, where N is a positive integer. If $g \in L^2$ such that $T_g T_f$ has finite rank, then g is the zero function.*

Proof. Recall that A^2 has the orthonormal basis $\{e_m : m = 0, 1, \ldots\}$, where $e_m(z) = \sqrt{m+1}\ z^m$ for non-negative integers m. For any non-negative integers k, l, we have

$$\begin{aligned}\langle T_f e_k, e_l\rangle &= \sqrt{(k+1)(l+1)} \int_{\mathbb{D}} f(z) z^k \bar{z}^l \mathrm{d}A(z) \\ &= \sqrt{(k+1)(l+1)} \int_0^1 \Big(\frac{1}{2\pi}\int_0^{2\pi} f(r\mathrm{e}^{\mathrm{i}\theta})\mathrm{e}^{\mathrm{i}(k-l)\theta}\mathrm{d}\theta\Big) r^{k+l}(2r)\mathrm{d}r \\ &= \sqrt{(k+1)(l+1)} \int_0^1 2 f_{l-k}(r) r^{k+l+1}\mathrm{d}r.\end{aligned}$$

By assumption about f, $\langle T_f e_k, e_l\rangle = 0$ whenever $l - k > M$. Thus for $k \in \mathbb{N}$ such that $k + M \geq 1$, we have

$$\begin{aligned}T_f e_k &= \sum_{l=0}^{\infty} \langle T_f e_k, e_l\rangle e_l \\ &= \sum_{l=0}^{k+M} \Big(\sqrt{(k+1)(l+1)} \int_0^1 2 f_{l-k}(r) r^{k+l+1}\mathrm{d}r\Big) e_l \\ &= \sqrt{(k+1)(M+k+1)}\Big(\int_0^1 2 f_M(r) r^{2k+M+1}\mathrm{d}r\Big) e_{k+M} \\ &\quad + \sqrt{k+1} \sum_{l=0}^{k+M-1} \Big(\sqrt{l+1} \int_0^1 2 f_{l-k}(r) r^{k+l+1}\mathrm{d}r\Big) e_l\end{aligned}$$

This shows that when $k+M \geq 1$ and $2k+M+1 \geq N$, e_{k+M} can be written as a linear combination of $\{T_f e_k\} \cup \{e_0, \ldots, e_{k+M-1}\}$.

Now suppose $T_g T_f$ has finite rank and let $\{\varphi_1, \ldots, \varphi_K\}$ be a set that spans $T_g T_f(\mathcal{P})$, where $\mathcal{P}$ is the space of all holomorphic polynomials in the variable z. Then for any non-negative integer k with $k+M \geq 1$ and $2k+M+1 \geq N$, we see that $T_g e_{k+M}$ is a linear combination of $\{\varphi_1, \ldots, \varphi_K\} \cup \{T_g(e_0), \ldots, T_g(e_{k+M-1})\}$. From this, it follows that T_g is a finite-rank operator. By Luecking's theorem [9] or a refined version of it (Theorem 3.1 in Section 3), we conclude that g is the zero function. □

Remark 2.2. If $f(z) = \bar{h}(z) + p(z, \bar{z})$ where $h \in A^2$ and p a polynomial in two variables, then either f is the zero function or it satisfies the hypothesis of Theorem 2.1. Therefore, Theorem 2.1 shows that if $T_g T_f$ is of finite rank for some $g \in L^2$ then either f or g is the zero function.

3. A refined Luecking's theorem and finite-rank Toeplitz products

We begin this section with a refined version of Luecking's theorem whose proof is greatly influenced by Luecking's argument. For the rest of the paper, let $\mathcal{P}$ denote the space of all holomorphic polynomials in the variable z.

Theorem 3.1. *Let $\mathcal{S} \subset \mathbb{N}$ ($\mathbb{N}$ denotes the set of all non-negative integers) so that $\sum_{s \in \mathcal{S}} \frac{1}{s+1} < \infty$. Let $\mathcal{N}$ be the subspace of $\mathcal{P}$ spanned by the monomials $\{z^m : m \in \mathbb{N} \backslash \mathcal{S}\}$ and let $\mathcal{N}^* = \{\bar{g} : g \in \mathcal{N}\}$. Let ν be a complex regular Borel measure on $\mathbb{C}$ with compact support. Let T_ν be the operator from $\mathcal{N}$ to the space of linear functionals on $\mathcal{N}^*$ defined by $T_\nu f(\bar{g}) = \int_{\mathbb{C}} f \bar{g} \mathrm{d}\nu$ for all $f, g \in \mathcal{N}$. Then T_ν has finite rank if and only if the support of ν is finite.*

Proof. For any $z \in \mathbb{C}$, let δ_z denote the point mass measure concentrated at z. Since $T_{\nu - \nu(\{0\})\delta_0} = T_\nu - \nu(\{0\}) T_{\delta_0}$, we see that T_ν has finite rank if and only if $T_{\nu - \nu(\{0\})\delta_0}$ has finite rank. So without loss of generality, we may assume that $\nu(\{0\}) = 0$.

If the support of ν is contained in a finite set $\{z_1, \ldots, z_{N-1}\}$ for some $N \geq 2$, then $T_\nu = \sum_{j=1}^{N-1} \nu(\{z_j\}) T_{\delta_{z_j}}$. Hence T_ν has rank less than N.

Conversely, suppose T_ν has rank less than N. Following Luecking's argument in [9, p. 3], we see that for any $f_1, \ldots, f_N$ and $g_1, \ldots, g_N$ in $\mathcal{N}$,

$$\int_{\mathbb{C}^n} \big(f_1(z_1) \cdots f_N(z_N)\big) \det(\bar{g}_i(z_j)) \mathrm{d}\nu^N(Z) = 0, \tag{3.1}$$

where $Z = (z_1, \ldots, z_N) \in \mathbb{C}^N$ and ν^N is the product of N copies of ν on $\mathbb{C}^N$.

Let $m_1, \ldots, m_N$ and $k_1, \ldots, k_N$ be non-negative integers. Let

$$\begin{aligned} \mathcal{Z} &= \{s \in \mathbb{N} : s + m_j \notin \mathcal{S} \text{ and } s + k_j \notin \mathcal{S} \text{ for all } 1 \leq j \leq N\} \\ &= \mathbb{N} \backslash \Big((\cup_{j=1}^N (\mathcal{S} - m_j)) \cup (\cup_{j=1}^N (S - k_j)) \Big). \end{aligned}$$

Since $\sum_{s\in\mathcal{S}} \frac{1}{s+1} < \infty$, we have $\sum_{s\in\mathbb{N}\setminus\mathcal{Z}} \frac{1}{s+1} < \infty$. This shows that

$$\sum_{s\in\mathcal{Z}} \frac{1}{s+1} = \infty. \tag{3.2}$$

For any $s \in \mathcal{Z}$, the monomials $f_j(z) = z^{m_j+s}$ and $g_j(z) = z^{k_j+s}$ for $j = 1, \dots, N$ are in $\mathcal{N}$. So we may use (3.1) to get

$$\begin{aligned} 0 &= \int_{\mathbb{C}^n} \left(z_1^{m_1+s} \cdots z_N^{m_N+s}\right) \det(\bar{z}_j^{k_i+s}) \mathrm{d}\nu^N(Z) \\ &= \int_{\mathbb{C}^n} \left(z_1^{m_1} \cdots z_N^{m_N}\right) \det(\bar{z}_j^{k_i}) |z_1 \cdots z_N|^{2s} \mathrm{d}\nu^N(Z) \\ &= \int_{\mathbb{C}^n\setminus W} \left(z_1^{m_1} \cdots z_N^{m_N}\right) \det(\bar{z}_j^{k_i}) |z_1 \cdots z_N|^{2s} \mathrm{d}\nu^N(Z), \end{aligned} \tag{3.3}$$

where $W = \{(z_1, \dots, z_N) \in \mathbb{C}^N : z_1 \cdots z_N = 0\}$. The last identity follows from the fact that $\nu^N(W) = 0$.

Let $\mathbb{K}$ denote the open right half plane consisting of all w with $\Re(w) > 0$ and let $\overline{\mathbb{K}}$ denote the closure of $\mathbb{K}$ in $\mathbb{C}$. For any $w \in \mathbb{K}$, define

$$F(w) = \int_{\mathbb{C}^n\setminus W} \left(z_1^{m_1} \cdots z_N^{m_N}\right) \det(\bar{z}_j^{k_i}) |z_1 \cdots z_N|^{2w} \mathrm{d}\nu^N(Z).$$

Here, for a positive number t and a complex number w, $t^w = \exp(w \log t)$, where log is the principal branch of the logarithmic function.

Suppose the measure ν is supported in the disk $D(0, R)$ of radius $R > 0$ centered at the origin on the complex plane. Then ν^N is supported in the polydisk $D_N(0, R)$ of the same radius centered at the origin in $\mathbb{C}^N$. For any $w \in \mathbb{K}$ and any $Z = (z_1, \dots, z_N)$ in the above polydisk, we have

$$\left||z_1 \cdots z_N|^{2w}\right| = |z_1 \cdots z_N|^{2\Re(w)} \le R^{2N\Re(w)}.$$

Therefore,

$$|F(w)| = \left| \int_{D_N(0,R)\setminus W} \left(z_1^{m_1} \cdots z_N^{m_N}\right) \det(\bar{z}_j^{k_i}) |z_1 \cdots z_N|^{2w} \mathrm{d}\nu^N(Z) \right| \le C R^{2N\Re(w)},$$

where C is a constant independent of w. It follows that F is not only defined but also continuous on $\overline{\mathbb{K}}$. An application of Morera's theorem shows that F is holomorphic on $\mathbb{K}$. Let $G(w) = F(w) R^{-2Nw}$ for $w \in \mathbb{K}$, then G is continuous, bounded on $\overline{\mathbb{K}}$ and holomorphic on $\mathbb{K}$. Now define

$$H(\zeta) = G\Big(\frac{1+\zeta}{1-\zeta}\Big), \text{ for } |\zeta| < 1.$$

Then H is a bounded holomorphic function on the unit disk. For any $s \in \mathcal{Z}$, (3.3) and the definitions of F and G show that $G(s) = F(s) = 0$, which implies

$H(\frac{s-1}{s+1}) = 0$. Now

$$\sum_{\substack{s\in\mathcal{Z}\\ s\geq 1}}(1-|\frac{s-1}{s+1}|) = \sum_{\substack{s\in\mathcal{Z}\\ s\geq 1}}\frac{2}{s+1} = \infty \quad \text{by (3.2).}$$

Corollary to Theorem 15.23 in [11] then shows that H is identically zero on the unit disk. Hence G and F are identically zero in $\overline{\mathbb{K}}$. In particular, $F(0) = 0$, which implies that

$$\int_{\mathbb{C}^n}\left(z_1^{m_1}\cdots z_N^{m_N}\right)\det(\bar{z}_j^{k_i})\mathrm{d}\nu^N(Z) = 0.$$

Since $m_1,\ldots,m_N$ and $k_1,\ldots,k_N$ were arbitrary non-negative integers, we conclude that (3.1) holds for all $f_1,\ldots,f_N$ and $g_1,\ldots,g_N$ in $\mathcal{P}$. Following Luecking's argument again [9, Section 4 and 5], we see that the support of ν is finite. □

Let $\mathcal{S}$ and $\mathcal{N}$ be as in the hypothesis of Theorem 3.1. Let $\mathcal{M}$ denote the subspace of $\mathcal{P}$ spanned by $\{z^m : m \in \mathcal{S}\}$. Let $\bar{\mathcal{M}}$ (respectively, $\bar{\mathcal{N}}$) denote the closure of $\mathcal{M}$ (respectively, $\mathcal{N}$) in A^2.

Corollary 3.2. *Suppose $f \in L^2$ so the operator T_f is densely defined on A^2. If $T_f(\mathcal{N}) \subset \mathrm{Span}(\bar{\mathcal{M}} \cup \{\varphi_1,\ldots,\varphi_N\})$, where $\varphi_1,\ldots,\varphi_N \in A^2$, then f is the zero function.*

Proof. Let $P_{\bar{\mathcal{M}}}$ (respectively, $P_{\bar{\mathcal{N}}}$) denote the orthogonal projection from A^2 onto $\bar{\mathcal{M}}$ (respectively, $\bar{\mathcal{N}}$). Then we have $P_{\bar{\mathcal{N}}} = 1 - P_{\bar{\mathcal{M}}}$ and hence, $P_{\bar{\mathcal{M}}}P_{\bar{\mathcal{N}}} = P_{\bar{\mathcal{N}}}P_{\bar{\mathcal{M}}} = 0$. By replacing φ_j by $\varphi_j - P_{\bar{\mathcal{M}}}\varphi_j$ if necessary, we may assume that $\varphi_j \perp \mathcal{M}$ for $1 \leq j \leq N$. By using the Gram-Schmidt process if necessary, we may assume that the vectors $\varphi_1,\ldots,\varphi_N$ form an orthonormal set (we may have fewer vectors after using Gram-Schmidt process but let us still denote by N the total number of these vectors).

For any p in $\mathcal{N}$, we have $T_f p = P_{\bar{\mathcal{M}}}T_f p + \sum_{j=1}^{N}\langle T_f p, \varphi_j\rangle\varphi_j$. This shows that $P_{\bar{\mathcal{N}}}(T_f p) = \sum_{j=1}^{N}\langle T_f p, \varphi_j\rangle P_{\bar{\mathcal{N}}}\varphi_j = \sum_{j=1}^{N}\langle T_f p, \varphi_j\rangle\varphi_j = \sum_{j=1}^{N}\langle fp, \varphi_j\rangle\varphi_j$. Then for any q in $\mathcal{N}$,

$$\int_{\mathbb{D}} fp\bar{q}\ \mathrm{d}A = \langle T_f p, q\rangle = \langle P_{\bar{\mathcal{N}}}(T_f p), q\rangle = \sum_{j=1}^{N}\langle fp, \varphi_j\rangle\langle\varphi_j, q\rangle.$$

Let $\mathrm{d}\nu = f\mathrm{d}A$. From the above identities, we see that the map T_ν from $\mathcal{N}$ into the space of linear functionals on $\mathcal{N}^*$ defined by $T_\nu p(\bar{q}) = \int_{\mathbb{D}} p\bar{q}\mathrm{d}\nu = \int_{\mathbb{D}} fp\bar{q}\mathrm{d}A$ for $p, q \in \mathcal{N}$ is of finite rank. Now Theorem 3.1 shows that the support of ν is finite, which implies that $f(z) = 0$ for almost all $z \in \mathbb{D}$. □

As an application of Corollary 3.2, we obtain the following result concerning finite-rank products of Toeplitz operators when all, except possibly one, of the operators are diagonal with respect to the standard orthonormal basis.

Theorem 3.3. *Suppose $f_1, \dots, f_{m_1}$ and $g_1, \dots, g_{m_2}$ are bounded measurable radial functions, none of which is the zero function. Suppose f is a function in L^2 such that $T_{g_1} \cdots T_{g_{m_2}} T_f T_{f_1} \cdots T_{f_{m_1}}$ (which is densely defined on A^2) is of finite rank, then f must be the zero function.*

Proof. For any $h \in \{f_1, \dots, f_{m_1}, g_1, \dots, g_{m_2}\}$, the operator T_h is diagonal with eigenvalues $\omega(h, m)$ given by (1.1) for $m = 0, 1, \dots$. Let $Z(h) = \{m \in \mathbb{N} : \omega(h, m) = 0\}$. Since h is not the zero function, Müntz-Szász's theorem (see [11, Theorem 15.26]) shows that $\sum_{m \in Z(h)} \frac{1}{m+1} < \infty$.

Put $\mathcal{S} = Z(f_1) \cup \cdots \cup Z(f_{m_1}) \cup Z(g_1) \cup \cdots \cup Z(g_{m_2})$. Then $\sum_{m \in \mathcal{S}} \frac{1}{s+1} < \infty$. Let $\mathcal{N}$ (respectively, $\mathcal{M}$) be the subspace of $\mathcal{P}$ spanned by $\{e_m : m \in \mathbb{N} \backslash \mathcal{S}\}$ (respectively, $\{e_m : m \in \mathcal{S}\}$). Recall that $\mathcal{P}$ denotes the space of all holomorphic polynomials in the variable z.

Put $S_1 = T_{f_1} \cdots T_{f_{m_1}}$ and $S_2 = T_{g_1} \cdots T_{g_{m_2}}$. For $\varphi \in A^2$,

$$S_2 \varphi = T_{g_1} \cdots T_{g_{m_2}} \big(\sum_{j=1}^{\infty} \langle \varphi, e_j \rangle e_j \big) = \sum_{j=1}^{\infty} \omega(g_1, j) \cdots \omega(g_{m_2}, j) \langle \varphi, e_j \rangle e_j.$$

Hence, if $S_2 \varphi = 0$, then $\omega(g_1, j) \cdots \omega(g_{m_2}, j) \langle \varphi, e_j \rangle = 0$ for all $j \in \mathbb{N}$. It then implies that $\langle \varphi, e_j \rangle = 0$ whenever $j \in \mathbb{N} \backslash \mathcal{S}$. Thus, $\ker(S_2) \subset \bar{\mathcal{M}}$. On the other hand, if $j \in \mathbb{N} \backslash \mathcal{S}$ then $\omega(f_1, j) \cdots \omega(f_{m_1}, j) \neq 0$. Therefore,

$$e_j = \frac{1}{\omega(f_1, j) \cdots \omega(f_{m_1}, j)} T_{f_1} \cdots T_{f_{m_1}} e_j = \frac{1}{\omega(f_1, j) \cdots \omega(f_{m_1}, j)} S_1 e_j.$$

This shows that $\mathcal{N} \subset S_1(\mathcal{N}) \subset S_1(\mathcal{P})$. So the domain of the operator $S_2 T_f S_1$ contains $\mathcal{P}$, which is dense in A^2.

Now suppose that $S_2 T_f S_1(\mathcal{P})$ is of finite dimensions, spanned by the set $\{u_1, \dots, u_N\}$. Let $v_j \in A^2$ such that $S_2 v_j = u_j$ for $j = 1, \dots, N$. It then follows that $T_f S_1(\mathcal{P})$ is contained in $\mathrm{Span}(\ker(S_2) \cup \{v_1, \dots, v_N\})$, which is a subspace of $\mathrm{Span}(\bar{\mathcal{M}} \cup \{v_1, \dots, v_N\})$. But as we have seen above, $\mathcal{N}$ is a subspace of $S_1(\mathcal{P})$. So we have $T_f(\mathcal{N}) \subset \mathrm{Span}(\bar{\mathcal{M}} \cup \{v_1, \dots, v_N\})$. Corollary 3.2 then implies that f is the zero function. □

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References

[1] Patrick Ahern and Željko Čučković, *A theorem of Brown-Halmos type for Bergman space Toeplitz operators*, J. Funct. Anal. **187** (2001), no. 1, 200–210. MR **1867348** **(2002h:**47040)

[2] ———, *Some examples related to the Brown-Halmos theorem for the Bergman space*, Acta Sci. Math. (Szeged) **70** (2004), no. 1-2, 373–378. MR **2072710 (2005d:**47046)

[3] Arlen Brown and Paul R. Halmos, *Algebraic properties of Toeplitz operators*, J. Reine Angew. Math. **213** (1963/1964), 89–102. MR 0160136 (28 #3350)

[4] Željko Čučković, *Berezin versus Mellin*, J. Math. Anal. Appl. **287** (2003), no. 1, 234–243. MR **2010267 (2004g:**47037)

[5] Željko Čučković and N. V. Rao, *Mellin transform, monomial symbols, and commuting Toeplitz operators*, J. Funct. Anal. **154** (1998), no. 1, 195–214. MR **1616532 (99f:**47033)

[6] Kunyu Guo, Shunhua Sun, and Dechao Zheng, *Finite rank commutators and semi-commutators of Toeplitz operators with harmonic symbols*, Illinois J. Math. **51** (2007), no. 2, 583–596 (electronic). MR 2342676

[7] Trieu Le, *Diagonal Toeplitz operators on weighted Bergman spaces*, preprint, available at http://www.math.utoledo.edu/∼tle/Papers.html.

[8] Issam Louhichi, N.V. Rao, and Abdel Yousef, *Two questions on products of Toeplitz operators on the Bergman space*, preprint.

[9] Daniel H. Luecking, *Finite rank Toeplitz operators on the Bergman space*, Proc. Amer. Math. Soc. **136** (2008), no. 5, 1717–1723.

[10] Rubén A. Martínez-Avendaño and Peter Rosenthal, *An introduction to operators on the Hardy-Hilbert space*, Graduate Texts in Mathematics, vol. 237, Springer, New York, 2007. MR **2270722 (2007k:**47051)

[11] Walter Rudin, *Real and complex analysis*, third ed., McGraw-Hill Book Co., New York, 1987. MR **924157 (88k:**00002)

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