SELF-ADJOINT, UNITARY, AND NORMAL WEIGHTED COMPOSITION OPERATORS IN SEVERAL VARIABLES

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ABSTRACT. We study weighted composition operators on Hilbert spaces of analytic functions on the unit ball with kernels of the form $(1 - \langle z, w \rangle)^{-\gamma}$ for $\gamma > 0$. We find necessary and sufficient conditions for the adjoint of a weighted composition operator to be a weighted composition operator or the inverse of a weighted composition operator. We then obtain characterizations of self-adjoint and unitary weighted composition operators. Normality of these operators is also investigated.

1. INTRODUCTION

Let $\mathbb{B}_n$ denote the open unit ball in $\mathbb{C}^n$. For $\mathcal{H}$ a Banach space of analytic functions on $\mathbb{B}_n$ and φ an analytic self-map of $\mathbb{B}_n$, the composition operator C_φ is defined by $C_\varphi h = h \circ \varphi$ for h in $\mathcal{H}$ for which the function $h \circ \varphi$ also belongs to $\mathcal{H}$. Researchers have been interested in studying how the function theoretic behavior of φ affects the properties of C_φ on $\mathcal{H}$ and vice versa. When $\mathcal{H}$ is the classical Hardy space or a weighted Bergman space of the unit disk, it follows from Littlewood Subordination Theorem that C_φ is bounded on $\mathcal{H}$ (see, for example, [10, Section 3.1]). On the other hand, the situation becomes more complicated in higher dimensions. For $n \geq 2$, there exist unbounded composition operators on the Hardy space and Bergman spaces of $\mathbb{B}_n$, even with polynomial mappings. The interested reader is referred to [10, Chapter 3] for these examples and certain necessary and sufficient conditions for the boundedness and compactness of C_φ.

Let $f : \mathbb{B}_n \to \mathbb{C}$ be an analytic function and let φ be as above. The weighted composition operator $W_{f,\varphi}$ is defined by $W_{f,\varphi}h = f \cdot (h \circ \varphi)$ for all $h \in \mathcal{H}$ for which the function $f \cdot (h \circ \varphi)$ also belongs to $\mathcal{H}$. Weighted composition operators have arisen in the work of Forelli [13] on isometries of classical Hardy spaces H^p and in Cowen's work [5, 6] on commutants of analytic Toeplitz operators on the Hardy space H^2 of the unit disk. Weighted composition operators have also been used in descriptions of adjoints of composition operators (see [7] and the references therein). Boundedness and compactness of weighted composition operators on various Hilbert spaces of analytic functions have been studied by many mathematicians (see, for example, [4, 12, 15, 17] and references therein). Recently researchers have started investigating the relations between weighted composition operators

and their adjoints. Cowen and Ko [9] and Cowen, Gunatillake and Ko [8] characterize self-adjoint weighted composition operators and study their spectral properties on weighted Hardy spaces on the unit disk whose kernel functions are of the form $K_w(z) = (1 - \overline{w}z)^{-\kappa}$ for $\kappa \geq 1$. In [3], Bourdon and Narayan study normal weighted composition operators on the Hardy space H^2. They characterize unitary weighted composition operators and apply their characterization to describe all normal operators $W_{f,\varphi}$ in the case φ fixes a point in the unit disk.

The purpose of the current paper is to study self-adjoint, unitary and normal weighted composition operators on a class of Hilbert spaces $\mathcal{H}$ of analytic functions on the unit ball. We characterize $W_{f,\varphi}$ whose adjoint is a weighted composition operator or the inverse of a weighted composition operator. As a consequence, we generalize certain results in [3, 9, 8] to higher dimensions and also obtain results that have not been previously known in one dimension.

For any real number $\gamma > 0$, let H_γ denote the Hilbert space of analytic functions on $\mathbb{B}_n$ with reproducing kernel functions

$$K_z^\gamma(w) = K^\gamma(w, z) = \frac{1}{(1 - \langle w, z \rangle)^\gamma} \quad \text{for } z, w \in \mathbb{B}_n.$$

By definition, H_γ is the completion of the linear span of $\{K_z^\gamma : z \in \mathbb{B}_n\}$ with the inner product $\langle K_z^\gamma, K_w^\gamma \rangle = K^\gamma(w, z)$ (this is indeed an inner product due to the positive definiteness of $K^\gamma(w, z)$). It is well known that any function $f \in H_\gamma$ is analytic on $\mathbb{B}_n$ and for $z \in \mathbb{B}_n$, we have $f(z) = \langle f, K_z^\gamma \rangle$.

For any multi-index $m = (m_1, \ldots, m_n) \in \mathbb{N}_0^n$ (here $\mathbb{N}_0$ denotes the set of non-negative integers) and $z = (z_1, \ldots, z_n) \in \mathbb{B}_n$, we write $z^m = z_1^{m_1} \cdots z_n^{m_n}$. It turns out that H_γ has an orthonormal basis consisting of constant multi-plies of the monomials z^m, for $m \in \mathbb{N}_0^n$. The spaces H_γ belong to the class of weighted Hardy spaces introduced by Cowen and MacCluer in [10, Section 2.1]. They are called (generalized) weighted Bergman spaces by Zhao and Zhu in [18] because of their similarities with other standard weighted Bergman spaces on the unit ball. In fact, for $\gamma > n$, H_γ is the weighted Bergman space $A_{\gamma-n-1}^2(\mathbb{B}_n)$, which consists of all analytic functions that are square integrable with respect to the weighted Lebesgue measure $(1 - |z|^2)^{\gamma-n-1} dV(z)$, where dV is the Lebesgue volume measure on $\mathbb{B}_n$. If $\gamma = n$, H_n is the usual Hardy space on $\mathbb{B}_n$. When $n \geq 2$ and $\gamma = 1$, H_1 is the so-called Drury-Arveson space, which has been given a lot of attention lately in the study of multi-variable operator theory and interpolation (see [1, 2] and the references therein). For arbitrary $\gamma > 0$, H_γ coincides with the space $A_{\gamma-n-1}^2(\mathbb{B}_n)$ in [18] (we warn the reader that when $\gamma < n$, the space $A_{\gamma-n-1}^2(\mathbb{B}_n)$ is not defined as the space of analytic functions that are square integrable with respect to $(1 - |z|^2)^{\gamma-n-1} dV(z)$, since the latter contains only the zero function).

2. Bounded weighted composition operators

As we mentioned in the Introduction, the composition operator C_φ is not always bounded on H_γ of the unit ball $\mathbb{B}_n$ when $n \geq 2$. On the other hand, if φ is a linear fractional self-map of the unit ball, then it was shown by Cowen and MacCluer [11] that C_φ is bounded on the Hardy space and all weighted Bergman spaces of $\mathbb{B}_n$. It turns out, as we will show below, that for such φ, C_φ is always bounded on H_γ for any $\gamma > 0$. We will need the following characterization of H_γ, which follows from [18, Theorem 13].

For any multi-index $m = (m_1, \ldots, m_n)$ of non-negative integers and any analytic function h on $\mathbb{B}_n$, we write $\partial^m h = \frac{\partial^{|m|} h}{\partial z_1^{m_1} \cdots \partial z_n^{m_n}}$, where $|m| = m_1 + \cdots + m_n$. For any real number α, put $d\mu_\alpha(z) = (1 - |z|^2)^{-n-1+\alpha} dV(z)$, where dV is the usual Lebesgue measure on the unit ball $\mathbb{B}_n$.

Theorem 2.1. *Let $\gamma > 0$. The following conditions are equivalent for an analytic function h on $\mathbb{B}_n$.*
(a) h belongs to H_γ.
(b) For some non-negative integer k with $2k + \gamma > n$, all the functions $\partial^m h$, where $|m| = k$, belong to $L^2(\mathbb{B}_n, d\mu_{\gamma+2k})$.
(c) For every non-negative integer k with $2k + \gamma > n$, all the functions $\partial^m h$, where $|m| = k$, belong to $L^2(\mathbb{B}_n, d\mu_{\gamma+2k})$.

Remark 2.2. Theorem 2.1 in particular shows that for any given positive number s, the function h belongs to H_γ if and only if for any multi-index l with $|l| = s$, $\partial^l h$ belongs to $H_{\gamma+2s}$. As a consequence, $H_{\gamma_1} \subset H_{\gamma_2}$ whenever $\gamma_1 \leq \gamma_2$.

Recall that the multiplier space $\text{Mult}(H_\gamma)$ of H_γ is the space of all analytic functions f on $\mathbb{B}_n$ for which fh belongs to H_γ whenever h belongs to H_γ. Since norm convergence in H_γ implies point-wise convergence on $\mathbb{B}_n$, it follows from the closed graph theorem that f is a multiplier if and only if the multiplication operator M_f is bounded on H_γ. It is well known that $\text{Mult}(H_\gamma)$ is contained in H^∞, the space of bounded analytic functions on $\mathbb{B}_n$. For $\gamma \geq n$, it holds that $\text{Mult}(H_\gamma) = H^\infty$. This follows from the fact that for such γ the norm on H_γ comes from an integral. On the other hand, when $n \geq 2$ and $\gamma = 1$ (hence H_γ is the Drury-Arveson space), $\text{Mult}(H_\gamma)$ is strictly smaller than H^∞ (see [1, Remark 8.9] or [2, Theorem 3.3]). However we will show that if f and all of its partial derivatives are bounded on $\mathbb{B}_n$, then f is a multiplier of H_γ for all $\gamma > 0$.

Lemma 2.3. *Let f be a bounded analytic function such that for each multi-index m, the function $\partial^m f$ is bounded on $\mathbb{B}_n$. Then f belongs to $\text{Mult}(H_\gamma)$, and hence the operator M_f is bounded on H_γ for any $\gamma > 0$.*

Proof. Let $\gamma > 0$ be given. Choose a positive integer k such that $\gamma + 2k > n$. Let h belong to H_γ. For any multi-index m with $|m| = k$, the derivative $\partial^m(fh)$ is a linear combination of products of the form $(\partial^t f)(\partial^s h)$ for multi-indexes s, t with $s + t = m$. For such s and t, $\partial^s h$ belongs to $H_{\gamma+2|s|} \subset H_{\gamma+2k}$

(by Remark 2.2) and $\partial^t f$, which is bounded by the hypothesis, is a multiplier of $H_{\gamma+2k}$ (since $\mathrm{Mult}(H_{\gamma+2k}) = H^\infty$). Thus, $(\partial^t f)(\partial^s h)$ belongs to $H_{\gamma+2k}$. Therefore, $\partial^m(fh)$ belongs to $H_{\gamma+2k}$. By Theorem 2.1, fh is in H_γ. Since h was arbitrary in H_γ, we conclude that f is a multiplier of H_γ. $\square$

An analytic map from $\mathbb{B}_n$ into itself is a linear fractional map [11] if there is a linear operator A on $\mathbb{C}^n$, two vectors B, C in $\mathbb{B}_n$ and a complex number d such that

$$\varphi(z) = \frac{Az + B}{\langle z, C\rangle + d} \quad \text{for } z \in \mathbb{B}_n.$$

Using Lemma 2.3 together with the aforementioned Cowen-MacCluer's result, we show that for φ a linear fractional self-map of the unit ball, the composition operator C_φ is bounded on H_γ for all $\gamma > 0$. In [14], Jury proves that C_φ is bounded on H_γ for all $\gamma \geq 1$ by an approach using kernel functions. He also obtains an estimate for the norm of C_φ but we do not need it here.

Proposition 2.4. *Let $\gamma > 0$ be given. Suppose φ is a linear fractional map of $\mathbb{B}_n$ into itself, then C_φ is bounded on H_γ.*

Proof. Since C_φ is a closed linear operator, to show that C_φ is bounded on H_γ, it suffices to show that $h \circ \varphi$ belongs to H_γ whenever h belongs to H_γ. For $\gamma > n$, this follows from [11, Theorem 15].

Now consider $\gamma > \max\{0, n - 2\}$. Write $\varphi = (\varphi_1, \ldots, \varphi_n)$. For each j, we have $\partial_{z_j}(h \circ \varphi) = (\partial_{z_1} h \circ \varphi)(\partial_{z_j}\varphi_1) + \cdots + (\partial_{z_n} h \circ \varphi)(\partial_{z_j}\varphi_n)$. For $1 \leq k \leq n$, since $\partial_{z_k} h$ belongs to $H_{\gamma+2}$ (by Remark 2.2) and $\gamma + 2 > n$, we see that $\partial_{z_k} h \circ \varphi$ also belongs to $H_{\gamma+2}$. On the other hand, since $\partial_{z_j}\varphi_k$ is analytic in a neighborhood of the closed unit ball, it satisfies the hypothesis of Lemma 2.3. Therefore by Lemma 2.3, the product $(\partial_{z_k} h \circ \varphi)(\partial_{z_j}\varphi_k)$ belongs to $H_{\gamma+2}$. Thus, $\partial_{z_j}(h \circ \varphi)$ is in $H_{\gamma+2}$ for all $1 \leq j \leq n$. Now Remark 2.2 shows that $h \circ \varphi$ belongs to H_γ.

Repeating the above argument, we obtain the conclusion of the proposition for $\gamma > \max\{0, n - 4\}$, then $\gamma > \max\{0, n - 6\}$, and so on. Therefore the conclusion holds for all $\gamma > 0$. $\square$

Remark 2.5. Proposition 2.4 together with Lemma 2.3 shows that if φ is a linear fractional self-map of $\mathbb{B}_n$ and f is analytic on an open neighborhood of $\overline{\mathbb{B}}_n$, then the weighted composition operator $W_{f,\varphi}$ is bounded on H_γ for all $\gamma > 0$.

We close this section with some elementary properties of bounded weighted composition operators. Suppose $W_{f,\varphi}$ is bounded on H_γ for some $\gamma > 0$. Then the action of the adjoint $W_{f,\varphi}^*$ on the kernel functions can be computed easily. Indeed, for any z, w in $\mathbb{B}_n$, by the properties of the reproducing kernel functions,

$$(W_{f,\varphi}^* K_z^\gamma)(w) = \langle W_{f,\varphi}^* K_z^\gamma, K_w^\gamma\rangle = \langle K_z^\gamma, f \cdot (K_w^\gamma \circ \varphi)\rangle$$
$$= \overline{f(z)}\,\overline{K_w^\gamma(\varphi(z))} = \overline{f(z)} K_{\varphi(z)}^\gamma(w).$$

This gives the well known formula

$$W_{f,\varphi}^* K_z^\gamma = \overline{f}(z) K_{\varphi(z)}^\gamma. \tag{2.1}$$

It is straight forward that the set of bounded weighted composition operators on any H_γ is closed under operator multiplication. In fact for analytic functions f, g and analytic self-maps φ, ψ of $\mathbb{B}_n$ for which both $W_{f,\varphi}$ and $W_{g,\psi}$ are bounded on some H_γ, we have

$$W_{f,\varphi} W_{g,\psi} = W_{f \cdot g \circ \varphi, \psi \circ \varphi}. \tag{2.2}$$

Another elementary fact we would like to mention is that each non-zero weighted composition operator $W_{f,\varphi}$ is determined uniquely by the pair f and φ. In fact, suppose $W_{f,\varphi} = W_{g,\psi}$ on H_γ and f is not identically zero. Then since $f = W_{f,\varphi} K_0^\gamma$ and $g = W_{g,\psi} K_0^\gamma$, we obtain $f = g$. Now for any $h \in H_\gamma$, since $f \cdot (h \circ \varphi - h \circ \psi) = 0$ and f is not identically zero, we have $h \circ \varphi = h \circ \psi$. Write $\varphi = (\varphi_1, \ldots, \varphi_n)$ and $\psi = (\psi_1, \ldots, \psi_n)$. Choosing $h(z) = z_j$, we conclude that $\varphi_j = \psi_j$ for $j = 1, \ldots, n$. Thus, $\varphi = \psi$.

3. Unitary weighted composition operators

Unitary weighted composition operators have been used in the study of Toeplitz operators on Hardy or Bergman spaces, see for example [20, p. 189]. In this section we will characterize all unitary weighted composition operators. In fact, we will show that $W_{f,\varphi}$ is unitary on H_γ if and only if φ is an automorphism and f is a constant multiple of a reproducing kernel function associated with φ.

For $a \in \mathbb{B}_n$, we define the normalized reproducing kernel k_a^γ by

$$k_a^\gamma(w) = K_a^\gamma(w)/\|K_a^\gamma\| = \frac{(1 - |a|^2)^{\gamma/2}}{(1 - \langle w, a \rangle)^\gamma} \quad \text{for } w \in \mathbb{B}_n.$$

Let φ_a be the Moebius automorphism of the ball that interchanges 0 and a. The formulas in [16, Section 2.2.1] show that φ_a is a linear fractional map of $\mathbb{B}_n$. Put $U_a = W_{k_a^\gamma, \varphi_a}$, the weighted composition operator on H_γ given by φ_a and k_a^γ. By Remark 2.5, U_a is a bounded operator. It turns out that U_a is in fact a self-adjoint unitary operator, that is, $U_a^* = U_a$ and $U_a^2 = 1$. This fact is well known and it is a consequence of a change of variables when H_γ is a weighted Bergman space ($\gamma > n$) or the Hardy space ($\gamma = n$). See [19, Proposition 1.13] for weighted Bergman spaces and [19, Proposition 4.2] for the Hardy space. On these spaces, one has the relation [20, p. 189] $U_a T_\eta U_a = T_{\eta \circ \varphi_a}$, where T_η denotes the Toeplitz operator with symbol η.

For other values of γ, for example, the Drury-Arveson space, the inner product on H_γ does not come from a measure on $\mathbb{B}_n$ so the approach using integral formulas does not seem to work. Our approach here makes use of the kernel functions and it works for all $\gamma > 0$. We in fact show that for each given $\gamma > 0$, for each automorphism ψ of $\mathbb{B}_n$, there corresponds a weight function f for which $W_{f,\psi}$ is a unitary operator on H_γ. The function f depends on ψ and the value of γ.

Proposition 3.1. *Let ψ be an automorphism of $\mathbb{B}_n$. Put $a = \psi^{-1}(0)$ and $b = \psi(0)$. Then the weighted composition operator $W_{k_a^\gamma, \psi}$ is a unitary operator on H_γ and $W_{k_a^\gamma, \psi}^* = W_{k_a^\gamma, \psi}^{-1} = W_{k_b^\gamma, \psi^{-1}}$.*

Proof. We will make use of the the identity

$$1 - \langle \psi(z), \psi(w) \rangle = \frac{(1 - \langle a, a \rangle)(1 - \langle z, w \rangle)}{(1 - \langle z, a \rangle)(1 - \langle a, w \rangle)}, \qquad (3.1)$$

which holds for all $z, w \in \overline{\mathbb{B}}_n$ (see [16, Theorem 2.2.5]). With $z = w = 0$, (3.1) gives $|b| = |\psi(0)| = |a|$. For any $z \in \mathbb{B}_n$, we have

$$
\begin{aligned}
k_b^\gamma(\psi(z)) &= \frac{(1 - |b|^2)^{\gamma/2}}{(1 - \langle \psi(z), b \rangle)^\gamma} = \frac{(1 - |b|^2)^{\gamma/2}}{(1 - \langle \psi(z), \psi(0) \rangle)^\gamma} \\
&= \frac{(1 - |b|^2)^{\gamma/2} \cdot (1 - \langle z, a \rangle)^\gamma}{(1 - |a|^2)^\gamma} \qquad \text{(by (3.1) with } w = 0) \\
&= \Big(\frac{1 - |b|^2}{1 - |a|^2}\Big)^{\gamma/2} \frac{1}{k_a^\gamma(z)} = \frac{1}{k_a^\gamma(z)} \qquad \text{(since } |b| = |a|).
\end{aligned}
$$

We obtain

$$k_a^\gamma(z) \cdot k_b^\gamma(\psi(z)) = 1 \quad \text{ for all } z \in \mathbb{B}_n. \qquad (3.2)$$

By Remark 2.5, the operators $W_{k_a^\gamma, \psi}$ and $W_{k_b^\gamma, \psi^{-1}}$ are bounded on H_γ. For $h \in H_\gamma$, (3.2) gives $W_{k_a^\gamma, \psi} W_{k_b^\gamma, \psi^{-1}} h = k_a^\gamma \cdot (k_b^\gamma \circ \psi) \cdot h = h$. Therefore $W_{k_a^\gamma, \psi} W_{k_b^\gamma, \psi^{-1}} = I$ on H_γ. Similarly, $W_{k_b^\gamma, \psi^{-1}} W_{k_a^\gamma, \psi} = 1$ on H_γ. Hence $W_{k_a^\gamma, \psi}$ is an invertible operator with inverse $W_{k_b^\gamma, \psi^{-1}}$.

Now let z and w be in $\mathbb{B}_n$. Using (3.1), we compute

$$
\begin{aligned}
\big(W_{k_a^\gamma, \psi} K_{\psi(z)}^\gamma\big)(w) &= k_a^\gamma(w) K_{\psi(z)}^\gamma(\psi(w)) \\
&= \frac{(1 - |a|^2)^{\gamma/2}}{(1 - \langle w, a \rangle)^\gamma} \frac{1}{(1 - \langle \psi(w), \psi(z) \rangle)^\gamma} \\
&= \frac{(1 - |a|^2)^{\gamma/2}}{(1 - \langle w, a \rangle)^\gamma} \frac{(1 - \langle w, a \rangle)^\gamma}{(1 - |a|^2)^\gamma} \frac{(1 - \langle a, z \rangle)^\gamma}{(1 - \langle w, z \rangle)^\gamma} = \frac{K_z^\gamma(w)}{\overline{k_a^\gamma(z)}}.
\end{aligned}
$$

Thus $W_{k_a^\gamma, \psi} K_{\psi(z)}^\gamma = K_z^\gamma / \overline{k_a^\gamma}(z)$. Using this and formula (2.1), we obtain

$$W_{k_a^\gamma, \psi}^* W_{k_a^\gamma, \psi}(K_{\psi(z)}^\gamma) = \frac{1}{\overline{k_a^\gamma(z)}} W_{k_a^\gamma, \psi}^*(K_z^\gamma) = K_{\psi(z)}^\gamma.$$

Since z was arbitrary and ψ is surjective, this implies, by linearity, that $W_{k_a^\gamma, \psi}^* W_{k_a^\gamma, \psi} h = h$ for all h in the span $\mathcal{M}$ of $\{K_z^\gamma : z \in \mathbb{B}_n\}$. Since $W_{k_a^\gamma, \psi}$ is bounded on H_γ and $\mathcal{M}$ is dense in H_γ, we conclude that $W_{k_a^\gamma, \psi}^* W_{k_a^\gamma, \psi} = I$ on H_γ. Therefore $W_{k_a^\gamma, \psi}$ is an invertible isometry on H_γ, and hence a unitary operator. $\square$

Corollary 3.2. *For any a in $\mathbb{B}_n$, the operator $U_a = W_{k_a^\gamma, \varphi_a}$ is a self-adjoint unitary operator on H_γ.*

Proof. Since φ_a is an automorphism of $\mathbb{B}_n$ with $\varphi_a^{-1} = \varphi_a$ and $a = \varphi_a^{-1}(0)$, the corollary follows immediately from Proposition 3.1. $\qquad\square$

For any linear operator V on $\mathbb{C}^n$ with $\|V\| \leq 1$, put $\psi_V(z) = Vz$ for $z \in \mathbb{B}_n$. Then ψ_V is an analytic self-map of the unit ball. We denote by C_V is composition operator C_{ψ_V} on H_γ. Lemma 8.1 in [10] shows that C_V is bounded on any H_γ and $C_V^* = C_{V^*}$ (the boundedness of C_V also follows from Proposition 2.4). When V is unitary, we obtain

Corollary 3.3. *For any unitary operator V of $\mathbb{C}^n$, the composition operator C_V is a unitary operator on H_γ with adjoint $C_V^* = C_{V^*} = C_{V^{-1}}$.*

Proof. The corollary can be proved by using Proposition 3.1 together with the fact that ψ_V is an automorphism of $\mathbb{B}_n$ with $\psi_V^{-1} = \psi_{V^{-1}}$ and $\psi(0) = 0$. It also follows (more easily) from the identities

$$C_{V^*}C_V = C_{VV^*} = I = C_{V^*V} = C_V C_{V^*}. \qquad\qquad\square$$

Now assume that φ, ψ are analytic self-maps of the unit ball and f, g are analytic functions such that the weighted composition operators $W_{f,\varphi}$ and $W_{g,\psi}$ are bounded on H_γ. We seek necessary and sufficient conditions for which $W_{f,\varphi}W_{g,\psi}^* = I$ on H_γ.

Consider first the case $\varphi(0) = 0$. For any z in $\mathbb{B}_n$, by (2.1), we have $W_{g,\psi}^* K_z^\gamma = \overline{g}(z) K_{\psi(z)}^\gamma$, so $W_{f,\varphi}W_{g,\psi}^* K_z^\gamma = \overline{g}(z) f K_{\psi(z)}^\gamma \circ \varphi$. Therefore,

$$\overline{g}(z) f(w) K_{\psi(z)}^\gamma(\varphi(w)) = K_z^\gamma(w) \quad \text{for } z, w \in \mathbb{B}_n. \qquad (3.3)$$

Letting $w = 0$ and using the fact that $K_{\psi(z)}^\gamma(\varphi(0)) = K_{\psi(z)}^\gamma(0) = 1$ and $K_z^\gamma(0) = 1$ for all $z \in \mathbb{B}_n$, we obtain $\overline{g}(z) f(0) = 1$, which gives $g(z) = 1/\overline{f(0)}$. Thus, g is a constant function.

Letting $z = 0$ in (3.3) gives $(f(0))^{-1} f(w) K_{\psi(0)}^\gamma(\varphi(w)) = K_0^\gamma(w) = 1$, which implies $f(w) = f(0)/K_{\psi(0)}^\gamma(\varphi(w))$ for $w \in \mathbb{B}_n$. Substituting this into (3.3), we obtain $K_{\psi(z)}^\gamma(\varphi(w))/K_{\psi(0)}^\gamma(\varphi(w)) = K_z^\gamma(w)$. Thus

$$\frac{(1 - \langle\varphi(w), \psi(z)\rangle)^{-\gamma}}{(1 - \langle\varphi(w), \psi(0)\rangle)^{-\gamma}} = (1 - \langle w, z\rangle)^{-\gamma}.$$

This gives (here we need to use the continuity of φ and ψ on $\mathbb{B}_n$)

$$\frac{1 - \langle\varphi(w), \psi(z)\rangle}{1 - \langle\varphi(w), \psi(0)\rangle} = 1 - \langle w, z\rangle \quad \text{for all } z, w \in \mathbb{B}_n,$$

which implies

$$\left\langle \frac{\varphi(w)}{1 - \langle\varphi(w), \psi(0)\rangle}, \psi(z) - \psi(0) \right\rangle = \langle w, z\rangle. \qquad (3.4)$$

By Lemma 3.10 below, there is an invertible linear operator A on $\mathbb{C}^n$ such that $\psi(z) = \psi(0) + Az$ and $\varphi(w) = (1 - \langle\varphi(w), \psi(0)\rangle)(A^*)^{-1}w$ for $z, w \in \mathbb{B}_n$. The latter implies

$$\langle\varphi(w), \psi(0)\rangle = (1 - \langle\varphi(w), \psi(0)\rangle) \cdot \langle(A^*)^{-1}w, \psi(0)\rangle,$$

which gives

$$1 - \langle \varphi(w), \psi(0) \rangle = \frac{1}{1 + \langle (A^*)^{-1}w, \psi(0) \rangle} = \frac{1}{1 + \langle w, A^{-1}\psi(0) \rangle}.$$

Therefore φ is a linear fractional map given by the formula

$$\varphi(w) = \frac{(A^*)^{-1}w}{1 + \langle w, A^{-1}\psi(0) \rangle} \qquad \text{for all } w \in \mathbb{B}_n.$$

It turns out that in order for φ and ψ to be self-maps of the unit ball, $\psi(0)$ must be zero. To show this, we will make use of Cowen-MacCluer's results [11] on linear fractional maps. By the definition on [11, p. 369], the adjoint map of φ has the formula $\sigma(w) = A^{-1}w - A^{-1}\psi(0)$. Since φ is a self-map of the unit ball, [11, Proposition 11] implies that σ is also a self-map of the unit ball. On the other hand, it is clear that $\psi \circ \sigma = \sigma \circ \psi = \mathrm{id}_{\mathbb{B}_n}$, the identity map of $\mathbb{B}_n$. This shows that both ψ and σ are automorphisms of $\mathbb{B}_n$.

To finish the proof, we use the description of the automorphism group of the unit ball [16, Theorem 2.2.5], which in particular says that any automorphism that does not fix the origin must be a linear fractional map with a non-constant denominator. Since the denominator of ψ is a constant, ψ must fix the origin: $\psi(0) = 0$. Therefore we obtain $\varphi(w) = (A^*)^{-1}w$ and $\psi(w) = Aw$ for $w \in \mathbb{B}_n$. But φ and ψ map the unit ball into itself, hence A is a unitary operator. Since $(A^*)^{-1} = A$, we see that $\varphi(w) = Aw = \psi(w)$ for $w \in \mathbb{B}_n$. Furthermore, since $\psi(0) = 0$, we have

$$f(w) = f(0)/K_{\psi(0)}^{\gamma}(\varphi(w)) = f(0)/K_0^{\gamma}(\varphi(w)) = f(0),$$

which is a constant function. Since $g(w) = 1/\overline{f(0)}$, we have $f(w)\overline{g}(w) = 1$ for all $w \in \mathbb{B}_n$.

Thus we have shown the 'only if' part of the following proposition. The 'if' part is much easier and it follows from Corollary 3.3.

Proposition 3.4. *Let f, g be analytic functions on $\mathbb{B}_n$ and let φ, ψ be analytic self-maps of $\mathbb{B}_n$ with $\varphi(0) = 0$. Then $W_{f,\varphi}W_{g,\psi}^* = I$ on H_γ if and only if f, g are constant functions with $f\overline{g} \equiv 1$ and there is a unitary operator A on $\mathbb{C}^n$ so that $\varphi(w) = \psi(w) = Aw$ for $w \in \mathbb{B}_n$. In this case, $W_{f,\varphi}$ and $W_{g,\psi}$ are constant multiples of a unitary composition operator.*

The general case (without the assumption $\varphi(0) = 0$) now follows from Proposition 3.4 after multiplying both $W_{f,\varphi}$ and $W_{g,\psi}$ by a unitary operator.

Theorem 3.5. *Let f, g be analytic functions on $\mathbb{B}_n$ and let φ, ψ be analytic self-maps of $\mathbb{B}_n$. Then $W_{f,\varphi}W_{g,\psi}^* = I$ on H_γ if and only if $\varphi = \psi$, an automorphism of $\mathbb{B}_n$; and there is a constant $\lambda \neq 0$ such that $f = \lambda k_a^\gamma$ and $g = (1/\overline{\lambda})k_a^\gamma$, where $a = \varphi^{-1}(0)$. Furthermore, both $W_{f,\varphi}$ and $W_{g,\psi}$ are constant multiples of the unitary operator $W_{k_a^\gamma, \varphi}$.*

Proof. The 'if' part follows from Proposition 3.1 so we only need to prove the 'only if' part. Put $b = \varphi(0)$. Define

$$\tilde{f} = f \cdot k_b^\gamma \circ \varphi, \ \ \tilde{\varphi} = \varphi_b \circ \varphi \quad \text{and} \quad \tilde{g} = g \cdot k_b^\gamma \circ \psi, \ \tilde{\psi} = \varphi_b \circ \psi.$$

Then by (2.2), $W_{\tilde{f},\tilde{\varphi}} = W_{f,\varphi}U_b$ and $W_{\tilde{g},\tilde{\psi}} = W_{g,\psi}U_b$. Since U_b is a unitary, we have $W_{\tilde{f},\tilde{\varphi}}W^*_{\tilde{g},\tilde{\psi}} = W_{f,\varphi}W^*_{g,\psi}$. Therefore the second product is the identity operator if and only if the first product is the identity operator. Since $\tilde{\varphi}(0) = \varphi_b(\varphi(0)) = \varphi_b(b) = 0$, by Proposition 3.4, $W_{\tilde{f},\tilde{\varphi}}W^*_{\tilde{g},\tilde{\psi}} = I$ on H_γ if and only if $\tilde{f}, \tilde{g}$ are constant functions with $\tilde{f} \cdot \overline{\tilde{g}} \equiv 1$ and there exists a unitary operator A on $\mathbb{C}^n$ such that $\tilde{\varphi}(w) = \tilde{\psi}(w) = Aw$ for $w \in \mathbb{B}_n$. The identity $\varphi_b^{-1} = \varphi_b$ now implies $\varphi(z) = \psi(z) = \varphi_b(Az)$ for $z \in \mathbb{B}_n$. Thus $\varphi = \psi$ and they equal an automorphism of $\mathbb{B}_n$. Suppose $\tilde{f} \equiv \lambda \neq 0$ and $\tilde{g} \equiv 1/\overline{\lambda}$. By (3.2), we obtain

$$f = \frac{\tilde{f}}{k_b^\gamma \circ \varphi} = \frac{\lambda}{k_{\varphi(0)}^\gamma \circ \varphi} = \lambda k_{\varphi^{-1}(0)}^\gamma = \lambda k_a^\gamma.$$

Similarly, $g = (1/\overline{\lambda})k_a^\gamma$. Thus $W_{f,\varphi} = \lambda W_{k_a^\gamma,\varphi}$ and $W_{g,\psi} = (1/\overline{\lambda})W_{k_a^\gamma,\varphi}$. $\quad\square$

Corollary 3.6. *Let f be an analytic function on $\mathbb{B}_n$ and φ be an analytic self-map of $\mathbb{B}_n$ such that the operator $W_{f,\varphi}$ is bounded on H_γ for some $\gamma > 0$. Then TFAE*

(a) $W_{f,\varphi}$ is a unitary on H_γ.
(b) $W_{f,\varphi}$ is a co-isometry on H_γ.
(c) φ is an automorphism of $\mathbb{B}_n$ and $f = \lambda k_{\varphi^{-1}(0)}^\gamma$ for some complex number λ with $|\lambda| = 1$.

Proof. The implication $(a) \Rightarrow (b)$ is trivial. The implications $(b) \Rightarrow (c) \Rightarrow (a)$ follow from Theorem 3.5 in the case $g = f$ and $\psi = \varphi$. $\quad\square$

Remark 3.7. The equivalence of (a) and (b) in the above corollary is not surprising in one dimension. This follows from the fact that in one dimension most weighted composition operators are injective. In fact if f is not identically zero and φ is not a constant function, then $W_{f,\varphi}$ is injective on any H_γ on the unit disk. In dimensions greater than one, it may happen that the kernel of $W_{f,\varphi}$ is non-trivial even in the case f does not vanish and φ is a non-constant map of $\mathbb{B}_n$. Thus, it might be surprising that all co-isometric weighted composition operators are in fact unitary on H_γ. Corollary 3.6 also shows that any unitary weighted composition operator on H_γ is of the form a constant (of modulus one) multiplying a unitary operator in Proposition 3.1.

Remark 3.8. The equivalence between (a) and (c) for weighted composition operators on the Hardy space of the unit disk is shown by Bourdon and Narayan in [3] by a different route. They show that if $W_{f,\varphi}$ is unitary, then φ must be a univalent inner function, and hence, an automorphism of the unit disk.

In [3], Bourdon and Narayan go on to characterize the spectra of these unitary weighted composition operators. Their spectral characterizations are based on whether the automorphism φ is elliptic, hyperbolic or parabolic. While the case of elliptic automorphisms (which fix a point in $\mathbb{B}_n$) can be carried on to higher dimensions, we have not been able to resolve the other two cases. The following spectral description is a consequence of a result in the next section about normal weighted composition operators.

Proposition 3.9. *Let f be an analytic function and φ an automorphism of $\mathbb{B}_n$ that fixes a point $p \in \mathbb{B}_n$. Suppose $W_{f,\varphi}$ is unitary on H_γ. Then $|f(p)| = 1$; all eigenvalues of $\varphi'(p)$ belong to the unit circle; and the spectrum of $W_{f,\varphi}$ is the closure of the set*

$$\{f(p)\} \cup \{f(p) \cdot \lambda_1 \cdots \lambda_s : \lambda_j \in \sigma(\varphi'(p)) \text{ for } 1 \le j \le s \text{ and } s = 1, 2, \ldots\}.$$

Here $\sigma(\varphi'(p))$ is the set of eigenvalues of the matrix $\varphi'(p)$.

Proof. Since $W_{f,\varphi}$ is normal, the description of its spectrum follows from Proposition 4.4 in Section 4 below. Since the spectrum of $W_{f,\varphi}$ must be a subset of the unit circle, we conclude that $|f(p)| = 1$ and $|\lambda| = 1$ for any λ in $\sigma(\varphi'(p))$. $\square$

We end this section with a lemma that was used in the proof of Proposition 3.4. We only need the finite dimensional version but the infinite dimensional case is also interesting in its own right. This result might have appeared in the literature but since we are not aware of an appropriate reference, we provide here a proof.

Lemma 3.10. *Let $\mathcal{M}$ be a Hilbert space with an inner product denoted by $\langle,\rangle$. Suppose F and G are two maps from the unit ball $\mathcal{B}$ of $\mathcal{M}$ into $\mathcal{M}$ such that $\langle F(w), G(z)\rangle = \langle w, z\rangle$ for all w, z in $\mathcal{B}$. Then there is an orthogonal decomposition $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2 \oplus \mathcal{M}_3$; there are bounded linear operators A, B from $\mathcal{M}$ into $\mathcal{M}_1$ with $B^*A = 1$; and there are (possibly non-linear) maps $F_1 : \mathcal{M} \to \mathcal{M}_2$ and $G_1 : \mathcal{M} \to \mathcal{M}_3$ such that $F(w) = Aw + F_1(w)$ and $G(z) = Bz + G_1(z)$ for all w, z in $\mathcal{B}$.*

If $\mathcal{M}$ has finite dimension, then both $\mathcal{M}_2$ and $\mathcal{M}_3$ are $\{0\}$ and hence $F(w) = Aw$ and $G(z) = Bz = (A^*)^{-1}z$ for $w, z \in \mathcal{B}$. If, in addition, F and G map $\mathcal{B}$ into itself, then A is a unitary operator.

If $F = G$, then $F_1 = G_1 = 0$; $A = B$; and hence $F(z) = G(z) = Az$ for $z \in \mathcal{B}$. Furthermore, A is an isometry on $\mathcal{M}$.

Proof. Let $\mathcal{N}$ be the closure of the linear span of $\{G(z) : z \in \mathcal{B}\}$. Then we have $P_{\mathcal{N}}G = G$ (here $P_{\mathcal{N}}$ is the orthogonal projection from $\mathcal{M}$ onto $\mathcal{N}$) and for all $w, z \in \mathcal{B}$,

$$\langle P_{\mathcal{N}}F(w), G(z)\rangle = \langle F(w), G(z)\rangle = \langle w, z\rangle.$$

For any z, w_1, w_2 in $\mathcal{B}$ and complex numbers c_1, c_2 such that $c_1 w_1 + c_2 w_2$ also belongs to $\mathcal{B}$, we have

$$\Big\langle P_\mathcal{N} F(c_1 w_1 + c_2 w_2) - c_1 P_\mathcal{N} F(w_1) - c_2 P_\mathcal{N} F(w_2), G(z) \Big\rangle$$
$$= \langle c_1 w_1 + c_2 w_2, z \rangle - c_1 \langle w_1, z \rangle - c_2 \langle w_2, z \rangle = 0.$$

Since the range of $P_\mathcal{N} F$ is contained in $\mathcal{N}$ and the linear span of the set $\{G(z) : z \in \mathcal{B}\}$ is dense in $\mathcal{N}$, we conclude that $P_\mathcal{N} F(c_1 w_1 + c_2 w_2) = c_1 P_\mathcal{N} F(w_1) + c_2 P_\mathcal{N} F(w_2)$. From this, it follows that $P_\mathcal{N} F$ extends to a linear operator on $\mathcal{M}$. We call this extension A and denote the closure of its range by $\mathcal{M}_1$. So A can be regarded as an operator from $\mathcal{M}$ into $\mathcal{M}_1$. We have $\langle Aw, G(z) \rangle = \langle w, z \rangle$ for all $w, z \in \mathcal{B}$. We claim that A is a closed operator and hence by the Closed Graph Theorem, it is bounded. Suppose $\{w_m\}$ is a sequence in $\mathcal{M}$ such that $w_m \to 0$ and $Aw_m \to y$ as $m \to \infty$. For $z \in \mathcal{B}$,

$$0 = \lim_{m \to \infty} \langle w_m, z \rangle = \lim_{m \to \infty} \langle Aw_m, G(z) \rangle = \langle y, G(z) \rangle.$$

Since y belongs to $\mathcal{M}_1 \subset \mathcal{N}$ and the linear span of $\{G(z) : z \in \mathcal{B}\}$ is dense in $\mathcal{N}$, we conclude that $y = 0$. So A is a closed operator.

Now for $w, z \in \mathcal{B}$, $\langle Aw, P_{\mathcal{M}_1} G(z) \rangle = \langle Aw, G(z) \rangle = \langle w, z \rangle$. It then follows, by the same argument as before, that $P_{\mathcal{M}_1} G$ extends to a bounded linear operator on $\mathcal{M}$. Call this operator B. Then the range of B is contained in $\mathcal{M}_1$ (hence we may regard B as an operator from $\mathcal{M}$ into $\mathcal{M}_1$) and we have $\langle Aw, Bz \rangle = \langle w, z \rangle$ for $w, z \in \mathcal{B}$. As before, B can be shown to be a closed operator, hence it is bounded and we have $B^* A = 1$.

Put $\mathcal{M}_2 = \mathcal{M} \ominus \mathcal{N}$ and $\mathcal{M}_3 = \mathcal{N} \ominus \mathcal{M}_1$. Put $F_1 = P_{\mathcal{M}_2} F$ and $G_1 = P_{\mathcal{M}_3} G$. We then have, on $\mathcal{B}$,

$$F = P_\mathcal{N} F + P_{\mathcal{M}_2} F = A + F_1,$$
$$G = P_\mathcal{N} G = P_\mathcal{N} P_{\mathcal{M}_1} G + P_\mathcal{N} (I - P_{\mathcal{M}_1}) G = P_{\mathcal{M}_1} G + P_{\mathcal{M}_3} G = B + G_1.$$

If $\mathcal{M}$ is a finite dimensional space, then it follows from $B^* A = 1$ that both A and B are invertible operators from $\mathcal{M}$ onto $\mathcal{M}_1$. Therefore, $\mathcal{M}_1 = \mathcal{M}$, which forces $\mathcal{M}_2 = \mathcal{M}_3 = \{0\}$. So $F(w) = Aw$ and $G(z) = Bz = (A^*)^{-1} z$ for $w, z \in \mathcal{B}$. If both F and G maps $\mathcal{B}$ into itself, then $\|A\| \leq 1$ and $\|(A^*)^{-1}\| \leq 1$. Consequently, both A and A^{-1} are contractive operators on $\mathcal{M}$. This forces A to be unitary.

If $F = G$ then we have $F_1 = G_1 = 0$ and $A = B$. But $B^* A = 1$, so $A^* A = 1$ and hence A is an isometry on $\mathcal{M}$. $\square$

4. Normal weighted composition operators

Recall that for V a linear operator on $\mathbb{C}^n$ with $\|V\| \leq 1$, we denote by C_V the composition operator induced by the analytic self-map $\psi_V(z) = Vz$ of $\mathbb{B}_n$. If V is normal, then since $C_V C_V^* = C_{V^* V} = C_{VV^*} = C_V^* C_V$, the operator C_V is normal on H_γ. It turns out that these are all normal

composition operators on H_γ for each $\gamma > 0$. The following result is part of [10, Theorem 8.2].

Proposition 4.1. *Let $\gamma > 0$ and let φ be an analytic mapping of $\mathbb{B}_n$ into itself. Then C_φ is normal on H_γ if and only if $\varphi(z) = Az$ for some normal linear operator A on $\mathbb{C}^n$ with $\|A\| \leq 1$.*

The spectrum of a normal composition operator can be determined easily. Let A be a normal linear operator on $\mathbb{C}^n$ with $\|A\| \leq 1$, we will identify the eigenvalues and eigenvectors of C_A on H_γ. We will show that C_A is diagonalizable and hence its spectrum is the closure of the set of its eigenvalues.

Since A is normal, there is an orthonormal basis $\{u_1, \ldots, u_n\}$ of $\mathbb{C}^n$ which consists of eigenvectors of A. Write $Au_j = \lambda_j u_j$, where λ_j is the eigenvalue corresponding to u_j for $1 \leq j \leq n$ (note that some of these eigenvalues may be the same). Then the spectrum of A is given by $\sigma(A) = \{\lambda_1, \ldots, \lambda_n\}$. Let $\{e_1, \ldots, e_n\}$ be the standard orthonormal basis for $\mathbb{C}^n$ and let V be the unitary operator on $\mathbb{C}^n$ such that $Vu_j = e_j$ for $1 \leq j \leq n$. For any $z = (z_1, \ldots, z_n)$ in $\mathbb{C}^n$, we have

$$VAV^*(z) = (\lambda_1 z_1, \ldots, \lambda_n z_n). \tag{4.1}$$

Recall from the Introduction that for any $\gamma > 0$, the set of analytic monomials $\{z^m = z_1^{m_1} \cdots z_n^{m_n} : m = (m_1, \ldots, m_n) \in \mathbb{N}_0^n\}$ is a complete orthogonal set in H_γ. By (4.1), we have $C_{VAV^*}(z^m) = \lambda^m z^m$ for all $m \in \mathbb{N}_0^n$ (here we write $\lambda^m = \lambda_1^{m_1} \cdots \lambda_n^{m_n}$ and use the convention that $0^0 = 1$). Since $C_{VAV^*} = C_{V^*} C_A C_V$ and C_V is unitary with $C_V^* = C_{V^*}$ (by Corollary 3.3), we conclude that the set $\{C_V z^m : m \in \mathbb{N}_0^n\}$ is a complete orthogonal set in H_γ and for each $m \in \mathbb{N}_0^n$, the function $C_V z^m$ is an eigenfunction for C_A with eigenvalue λ^m. Thus the operator C_A is diagonalizable in H_γ and the spectrum $\sigma(C_A)$ is the closure of the set $\{\lambda^m : m \in \mathbb{N}_0^n\}$.

The eigenfunctions $C_V(z^m)$ of C_A can be described in terms of the eigenvectors of A as follows.

$$\begin{aligned}
C_V(z^m) &= C_V(z_1^{m_1} \cdots z_n^{m_n}) = C_V(\langle z, e_1 \rangle^{m_1} \cdots \langle z, e_n \rangle^{m_n}) \\
&= \langle Vz, e_1 \rangle^{m_1} \cdots \langle Vz, e_n \rangle^{m_n} = \langle z, V^* e_1 \rangle^{m_1} \cdots \langle z, V^* e_n \rangle^{m_n} \\
&= \langle z, u_1 \rangle^{m_1} \cdots \langle z, u_n \rangle^{m_n}.
\end{aligned}$$

We have thus obtained

Proposition 4.2. *Let A be a normal operator on $\mathbb{C}^n$ with $\|A\| \leq 1$. Let $\{u_1, \ldots, u_n\}$ be an orthonormal basis for $\mathbb{C}^n$ consisting of eigenvectors of A. Write $Au_j = \lambda_j u_j$ for $1 \leq j \leq n$. Then the following statements hold.*

    (a) *The set $\{f_m(z) = \langle z, u_1 \rangle^{m_1} \cdots \langle z, u_n \rangle^{m_n} : m = (m_1, \ldots, m_n) \in \mathbb{N}_0^n\}$ is a complete orthogonal set of H_γ.*

    (b) *Each f_m is an eigenfunction of C_A with eigenvalue $\lambda^m = \lambda_1^{m_1} \cdots \lambda_n^{m_n}$.*

    (c) *The spectrum of C_A is the closure of the set $\{\lambda^m : m \in \mathbb{N}_0^n\}$, where $\lambda = (\lambda_1, \ldots, \lambda_n)$. This set can also be written as $\{1\} \cup \{\alpha_1 \cdots \alpha_s : \alpha_j \in \sigma(A) \text{ for } 1 \leq j \leq s \text{ and } s = 1, 2, \ldots\}$.*

In [3], Bourdon and Narayan study normal weighted composition operators on the Hardy space of the unit disk. They provide two necessary conditions for $W_{f,\varphi}$ to be normal [3, Lemma 2 and Proposition 3]: (1) either $f \equiv 0$ or f never vanishes, and (2) if φ is not a constant function and f is not the zero function, then φ is univalent. While condition (1) is still valid in all dimensions with the same proof, condition (2) no longer holds in dimension greater than one, as Proposition 4.1 shows. On the other hand, we will see that the characterization of normal $W_{f,\varphi}$ on H_γ remains the same if the map φ fixes a point in the unit ball. Our approach here was inspired by that in [3] but the argument has been simplified. Furthermore, our proof works for all H_γ in any dimension.

Theorem 4.3. *Suppose φ is an analytic self-map of $\mathbb{B}_n$ that fixes a point p in $\mathbb{B}_n$. If $W_{f,\varphi}$ is a non-zero normal operator, then there exist a constant $\alpha \neq 0$ and a normal linear operator A on $\mathbb{C}^n$ with $\|A\| \leq 1$ such that*

$$f = \alpha \frac{k_p^\gamma}{k_p^\gamma \circ \varphi}, \quad and \quad \varphi(z) = \varphi_p(A\varphi_p(z)) \text{ for } z \in \mathbb{B}_n. \qquad (4.2)$$

Conversely, if f and φ satisfy (4.2), then $\alpha = f(p)$ and $W_{f,\varphi}$ is unitarily equivalent to the normal operator $f(p)C_A$ (in fact, $W_{f,\varphi} = U_p\big(f(p)C_A\big)U_p$) and hence it is normal.

Proof. We assume first $\varphi(0) = 0$ and $W_{f,\varphi}$ is a non-zero normal operator. By (2.1), we have

$$W_{f,\varphi}^* K_0^\gamma = \overline{f(0)} K_{\varphi(0)}^\gamma = \overline{f(0)} K_0^\gamma.$$

This shows that K_0^γ is an eigenvector of $W_{f,\varphi}^*$ with eigenvalue $\overline{f(0)}$. Since $W_{f,\varphi}$ is normal, we obtain $W_{f,\varphi} K_0^\gamma = f(0) K_0^\gamma$, which implies $f \cdot K_0^\gamma \circ \varphi = f(0) K_0^\gamma$ and hence $f = f(0)$ since $K_0^\gamma \equiv 1$. So f is a constant function (which is non-zero because $W_{f,\varphi}$ is a non-zero operator). This in turns implies that C_φ is normal on H_γ. By Proposition 4.1, there is a normal linear operator on $\mathbb{B}_n$ with $\|A\| \leq 1$ such that $\varphi(z) = Az$ for $z \in \mathbb{B}_n$.

For general p, define $\widetilde{f} = (k_p^\gamma \circ \varphi \circ \varphi_p)(f \circ \varphi_p)k_p^\gamma$ and $\widetilde{\varphi} = \varphi_p \circ \varphi \circ \varphi_p$. By (2.2), $U_p W_{f,\varphi} U_p = W_{\widetilde{f},\widetilde{\varphi}}$. Since $W_{f,\varphi}$ and $W_{\widetilde{f},\widetilde{\varphi}}$ are unitarily equivalent (recall that U_p is a self-adjoint unitary operator), one is normal if and only if the other is normal. Since $\widetilde{\varphi}(0) = 0$, the above argument shows that $W_{\widetilde{f},\widetilde{\varphi}}$ is normal if and only if $\widetilde{f}$ is a constant function, say, $\widetilde{f} \equiv \alpha$ and $\widetilde{\varphi}(z) = Az$ for some normal operator A on $\mathbb{C}^n$ with $\|A\| \leq 1$. Thus we obtain $(k_p^\gamma \circ \varphi \circ \varphi_p)(f \circ \varphi_p)k_p^\gamma \equiv \alpha$ and $\varphi_p \circ \varphi \circ \varphi_p(z) = Az$. Using the fact that $\varphi_p \circ \varphi_p$ is the identity map on $\mathbb{B}_n$, we get

$$f = \frac{\alpha}{(k_p^\gamma \circ \varphi)(k_p^\gamma \circ \varphi_p)}, \quad and \quad \varphi(z) = \varphi_p(A\varphi_p(z)) \text{ for } z \in \mathbb{B}_n. \qquad (4.3)$$

On the other hand, since $\varphi_p(0) = p = \varphi_p^{-1}(0)$, (3.2) gives $k_p^\gamma \cdot (k_p^\gamma \circ \varphi_p) = 1$. Therefore f can be written as $f = \alpha \frac{k_p^\gamma}{k_p^\gamma \circ \varphi}$. Since $k_p^\gamma(\varphi(p)) = k_p^\gamma(p)$, we see that $f(p) = \alpha$.

Conversely, if f and φ satisfy (4.2), then they satisfy (4.3) (with $\alpha = f(p)$) and hence $W_{f,\varphi}$ is unitarily equivalent to the normal operator $f(p)C_A$. $\square$

We now use Theorem 4.3 and Proposition 4.2 to discuss the spectra of normal weighted composition operators. Suppose that f and φ satisfy (4.2). Let $\{u_1, \ldots, u_n\}$ be an orthonormal basis for $\mathbb{C}^n$ consisting of eigenvectors of A. Let λ_j be the eigenvalue of A corresponding to the eigenvector u_j and put $f_j(z) = \langle z, u_j \rangle$ for $1 \leq j \leq n$. For each multi-index $m = (m_1, \ldots, m_n)$ in $\mathbb{N}_0^n$, we write $f_m = f_1^{m_1} \cdots f_n^{m_n}$. From Proposition 4.2 we know that $\{f_m : m \in \mathbb{N}_0^n\}$ is a complete orthogonal set of H_γ and $C_A(f_m) = \lambda^m f_m$ for each m, where $\lambda = (\lambda_1, \ldots, \lambda_n)$.

For $1 \leq j \leq n$, put

$$g_j(z) = (U_p f_j)(z) = k_p^\gamma(z) f_j(\varphi_p(z)) = k_p^\gamma(z) \langle \varphi_p(z), u_j \rangle.$$

Put $g_m = g_1^{m_1} \cdots g_n^{m_n} = U_p(f_m)$ for $m = (m_1, \ldots, m_n) \in \mathbb{N}_0^n$. Since U_p is unitary on H_γ, the set $\{g_m : m \in \mathbb{N}_0^n\}$ is a complete orthogonal set of H_γ. Since $W_{f,\varphi} = U_p(f(p)C_A)U_p$ by Theorem 4.3 and $U_p = U_p^{-1}$, we conclude that $W_{f,\varphi} g_m = f(p)\lambda^m g_m$ for $m \in \mathbb{N}_0^n$. Therefore the spectrum of $W_{f,\varphi}$ is the closure of the set $\{f(p)\lambda^m : m \in \mathbb{N}_0^n\}$, which is the same as

$$\{f(p)\} \cup \{f(p) \cdot \alpha_1 \cdots \alpha_s : \alpha_j \in \sigma(A) \text{ for } 1 \leq j \leq s \text{ and } s = 1, 2, \ldots\}.$$

On the other hand, by the chain rule, we have

$$\varphi'(p) = \varphi_p'(A\varphi_p(0))A\varphi_p'(p) = \varphi_p'(0)A\varphi_p'(p).$$

Since $\varphi_p \circ \varphi_p = I_{\mathbb{B}_n}, \varphi_p(0) = p$ and $\varphi_p(p) = 0$, the chain rule again gives $\varphi_p'(p)\varphi_p'(0) = \varphi_p'(0)\varphi_p'(p) = I_n$, the identity operator on $\mathbb{C}^n$. Therefore $\varphi'(p)$ and A are similar and hence they have the same set of eigenvalues, counting multiplicities. In particular, $\sigma(A) = \sigma(\varphi'(p))$. We thus obtain the description of the spectrum of $W_{f,\varphi}$ intrinsically in terms of f and φ.

Proposition 4.4. *Let f be a non-zero analytic function and φ an analytic self-map of $\mathbb{B}_n$ that fixes a point p on $\mathbb{B}_n$. Suppose $W_{f,\varphi}$ is a normal operator on H_γ. Then the spectrum of $W_{f,\varphi}$ is the closure of the set*

$$\{f(p)\} \cup \{f(p) \cdot \alpha_1 \cdots \alpha_s : \alpha_j \in \sigma(\varphi'(p)) \text{ for } 1 \leq j \leq s \text{ and } s = 1, 2, \ldots\}.$$

We have characterized normal weighted composition operators induced by analytic self-maps of $\mathbb{B}_n$ that fix a point in $\mathbb{B}_n$. Our approach (conjugating $W_{f,\varphi}$ by a unitary) does not seem to work for φ that only has fixed point on the sphere. In the rest of this section, we investigate normal weighted composition operators of a certain type.

In [3, Section 5], Bourdon and Narayan note that in one dimension, the function f in the conclusion of Theorem 4.3 is in fact a constant multiple of $K_{\sigma(0)}^\gamma$, where σ is the adjoint of the linear fractional map φ. They then

go on to find necessary and sufficient conditions for the normality of $W_{f,\varphi}$, where φ is a linear fractional map and $f = K^\gamma_{\sigma(0)}$. It turns out that in higher dimensions similar results also hold but they are less obvious because of the complicated settings of several variables.

Recall that a linear fractional map φ has the form $\varphi(z) = \frac{Az+B}{\langle z,C\rangle+d}$, where A is a linear operator on $\mathbb{C}^n$; B, C are vectors in $\mathbb{C}^n$; and d is a complex number. Given such a map φ, its adjoint is defined by

$$\sigma(z) = \sigma_\varphi(z) = \frac{A^*z - C}{-\langle z, B\rangle + \overline{d}}.$$

For more details on σ and its relation with φ, see [11].

We begin by a lemma that can be verified by a direct computation, using the formulas of φ, σ and of the reproducing kernel functions.

Lemma 4.5. *Let φ be a linear fractional self-map of $\mathbb{B}_n$ and let σ be its adjoint. Then for any point a in $\mathbb{B}_n$, we have*

$$K^\gamma_{\varphi(0)} \cdot K^\gamma_a \circ \sigma = \overline{K}^\gamma_{\sigma(0)}(a)K^\gamma_{\varphi(a)} \quad and \quad K^\gamma_{\sigma(0)} \cdot K^\gamma_a \circ \varphi = \overline{K}^\gamma_{\varphi(0)}(a)K^\gamma_{\sigma(a)}.$$

By Remark 2.5, both operators $W_{K^\gamma_{\varphi(0)},\sigma}$ and $W_{K^\gamma_{\sigma(0)},\varphi}$ are bounded on H_γ. Now the first identity in Lemma 4.5 together with (2.1) shows that

$$W_{K^\gamma_{\varphi(0)},\sigma}K^\gamma_a = W^*_{K^\gamma_{\sigma(0)},\varphi}K^\gamma_a \quad \text{for all } a \in \mathbb{B}_n,$$

which implies that

$$W_{K^\gamma_{\varphi(0)},\sigma} = W^*_{K^\gamma_{\sigma(0)},\varphi}. \tag{4.4}$$

We point out that this formula is in fact equivalent to the formula of C^*_φ given by Cowen and MacCluer in [11, Theorem 16], which can be written as

$$C^*_\varphi = M_{K^\gamma_{\varphi(0)}} C_\sigma M^*_{1/K^\gamma_{\sigma(0)}}.$$

Here for an analytic function g on the unit ball, M_g denotes the operator of multiplication by g on H_γ.

For any point p in $\mathbb{B}_n$, it follows from [16, Definition 2.2.1] that the involution φ_p of $\mathbb{B}_n$ has the form $\varphi_p(z) = \frac{Tz+p}{1-\langle z,p\rangle}$ for some self-adjoint operator T depending on p. This implies that the adjoint of φ_p is the same as φ_p. Now let f, φ satisfy (4.2) in Theorem 4.3. Then the adjoint σ of φ has the form $\sigma(z) = \varphi_p(A^*\varphi_p(z))$ for $z \in \mathbb{B}_n$. (Note that the adjoint of $\psi_1 \circ \psi_2$ is the composition of the adjoint of ψ_2 and the adjoint of ψ_1, in this order, see [11, Lemma 12].) In particular, $\sigma(p) = \varphi_p(A^*\varphi_p(p)) = p$. We thus obtain

$$f = \alpha\frac{k^\gamma_p}{k^\gamma_p \circ \varphi} = \alpha\frac{K^\gamma_p}{K^\gamma_p \circ \varphi} = \alpha\frac{K^\gamma_{\sigma(p)}}{K^\gamma_p \circ \varphi} = \frac{\alpha}{\overline{K}^\gamma_{\varphi(0)}(p)}K^\gamma_{\sigma(0)}.$$

The last equality follows from the second identity in Lemma 4.5. Therefore we see that f is a constant multiple of $K^\gamma_{\sigma(0)}$.

In the rest of this section, we assume that φ is a linear fractional map and $f = K^\gamma_{\sigma(0)}$, where as above σ is the adjoint map of φ. We look for conditions

for which the weighted composition operator $W_{f,\varphi}$ is normal. We emphasize here that in the case φ has a fixed point p in $\mathbb{B}_n$, Theorem 4.3 provides a complete answer: $W_{f,\varphi}$ is normal if and only if $\varphi(z) = \varphi_p(A\varphi_p(z))$ for some normal operator A on $\mathbb{B}_n$. The result below does not require that φ have a fixed point in $\mathbb{B}_n$.

Proposition 4.6. *Suppose φ is a linear fractional self-map of $\mathbb{B}_n$ and σ is its adjoint. Let $\gamma > 0$ and put $f = K_{\sigma(0)}^\gamma$. Then the operator $W_{f,\varphi}$ is normal on H_γ if and only if $|\varphi(0)| = |\sigma(0)|$ and $\varphi \circ \sigma = \sigma \circ \varphi$.*

Proof. Using (4.4) and (2.2), we compute

$$W_{f,\varphi}^* W_{f,\varphi} = W_{K_{\varphi(0)}^\gamma, \sigma} W_{K_{\sigma(0)}^\gamma, \varphi} = W_{K_{\varphi(0)}^\gamma \cdot K_{\sigma(0)}^\gamma \circ \sigma, \varphi \circ \sigma},$$

$$W_{f,\varphi} W_{f,\varphi}^* = W_{K_{\sigma(0)}^\gamma, \varphi} W_{K_{\varphi(0)}^\gamma, \sigma} = W_{K_{\sigma(0)}^\gamma \cdot K_{\varphi(0)}^\gamma \circ \varphi, \sigma \circ \varphi}.$$

This shows that $W_{f,\varphi}$ is normal if and only if $\varphi \circ \sigma = \sigma \circ \varphi$ and

$$K_{\varphi(0)}^\gamma \cdot K_{\sigma(0)}^\gamma \circ \sigma = K_{\sigma(0)}^\gamma \cdot K_{\varphi(0)}^\gamma \circ \varphi. \qquad (4.5)$$

By the first identity in Lemma 4.5, the left hand side of (4.5) equals

$$\overline{K}_{\sigma(0)}^\gamma(\sigma(0)) K_{\varphi(\sigma(0))}^\gamma = (1 - |\sigma(0)|^2)^{-\gamma} K_{\varphi(\sigma(0))}^\gamma.$$

Similarly, by the second identity in Lemma 4.5, the right hand side of (4.5) equals

$$\overline{K}_{\varphi(0)}^\gamma(\varphi(0)) K_{\sigma(\varphi(0))}^\gamma = (1 - |\varphi(0)|^2)^{-\gamma} K_{\sigma(\varphi(0))}^\gamma.$$

Thus (4.5) holds if and only if $|\sigma(0)| = |\varphi(0)|$ and $\varphi(\sigma(0)) = \sigma(\varphi(0))$. The latter is certainly true if $\varphi \circ \sigma = \sigma \circ \varphi$.

Therefore, the operator $W_{K_{\sigma(0)}^\gamma, \varphi}$ is normal if and only if $\varphi \circ \sigma = \sigma \circ \varphi$ and $|\varphi(0)| = |\sigma(0)|$, which completes the proof of the proposition. $\qquad \square$

Remark 4.7. Proposition 4.6 in the case of the Hardy space on the unit disk ($n = 1$) was obtained by Bourdon and Narayan in [3, Proposition 12] but their conclusion was stated in a slightly different way.

Remark 4.8. In the case $n = 1$ and $\varphi(z) = \frac{az+b}{cz+d}$ for complex numbers a, b, c, d, an easy calculation shows that the conditions obtained in Proposition 4.6 are equivalent to $|b| = |c|$ and $\bar{a}b - \bar{c}d = b\bar{d} - a\bar{c}$.

When $n \geq 2$ and $\varphi(z) = \frac{Az+B}{\langle z, C \rangle + d}$, the conditions can then be expressed in terms of A, B, C and d. We leave this to the interested reader.

We conclude the section by a result taken from [3, Proposition 13] with a slightly modified proof using Remark 4.8.

Proposition 4.9. *Suppose that φ is a linear fractional self-map of the unit disk of parabolic type (so there is an ω with $|\omega| = 1$ such that $\varphi(\omega) = \omega$ and $\varphi'(\omega) = 1$). Then the operator $W_{K_{\sigma(0)}^\gamma, \varphi}$ is normal on H_γ for any $\gamma > 0$. Here as before, σ is the adjoint map of φ.*

Proof. As it is explained in the proof of [3, Proposition 13], we only need to consider $\omega = 1$ and φ of the form

$$\varphi(z) = \frac{(2-t)z + t}{-tz + (2+t)}, \quad \text{where } \mathrm{Re}(t) \geq 0.$$

Since $a = 2 - t, b = t, c = -t$ and $d = 2 + t$, we have $|b| = |c|$ and $\bar{a}b - \bar{c}d = b\bar{d} - a\bar{c} = 4\mathrm{Re}(t)$. The conclusion now follows from Remark 4.8 and Proposition 4.6. $\qquad\square$

5. Self-adjoint weighted composition operators

In this section we characterize when the adjoint of a weighted composition operator on H_γ is another weighted composition operator. As a consequence, we determine necessary and sufficient conditions for which the operator $W_{f,\varphi}$ is a self-adjoint operator. This generalizes the characterizations obtained in [9, 8], where the one-dimensional case is considered. Furthermore, our solutions to the equation $W_{g,\psi}^* = W_{f,\varphi}$ seems to be new even in one dimension.

We will need the following elementary result regarding maps on the unit ball of a Hilbert space. The existence of the linear extensions follows from a similar argument as in the proof of Lemma 3.10. The boundedness is well known and it is a consequence of the closed graph theorem.

Lemma 5.1. *Let $\mathcal{M}$ be Hilbert space with an inner product $\langle,\rangle$. Suppose F and G are two maps from the unit ball $\mathcal{B}$ of $\mathcal{M}$ into $\mathcal{M}$ such that for all $z, w \in \mathcal{B}$, $\langle F(w), z \rangle = \langle w, G(z) \rangle$. Then there is a bounded linear operator A on $\mathcal{M}$ such that $F(w) = Aw$ and $G(w) = A^*w$ for all $w \in \mathcal{B}$.*

By (4.4) we see that the adjoint of $W_{K_{\sigma(0)}^\gamma, \varphi}$ is the weighted composition operator $W_{K_{\varphi(0)}^\gamma, \sigma}$ when φ is a linear fractional map and σ is the adjoint map of φ. Our main result in this section shows that any non-zero weighted composition operator whose adjoint is a weighted composition operator must be a constant multiple of an operator of this form.

Theorem 5.2. *Let f, g be analytic functions on $\mathbb{B}_n$ and φ, ψ be analytic self-maps of $\mathbb{B}_n$. Then $W_{f,\varphi}$ and $W_{g,\psi}$ are non-zero bounded operators on H_γ and $W_{g,\psi}^* = W_{f,\varphi}$ if and only if there are vectors c, d in $\mathbb{B}_n$, a linear operator A on $\mathbb{C}^n$ and a non-zero complex number α such that*

$$\varphi(z) = \frac{d + Az}{1 - \langle z, c \rangle} \;\; and \;\; \psi(z) = \frac{c + A^*z}{1 - \langle z, d \rangle} \;\; for \; all \; z \in \mathbb{B}_n, \qquad (5.1)$$

and $f = \alpha K_c^\gamma = \alpha K_{\psi(0)}^\gamma$, $g = \bar{\alpha} K_d^\gamma = \bar{\alpha} K_{\varphi(0)}^\gamma$. In particular, the maps φ and ψ are linear fractional maps.

Remark 5.3. Note that the map ψ in (5.1) is the adjoint of φ. Thus Theorem 5.2 says, in particular, that if $W_{g,\psi}$ is the adjoint operator of $W_{f,\varphi}$, then ψ is the adjoint of φ.

Proof. Suppose first $W_{g,\psi}^* = W_{f,\varphi}$ on H_γ and they are non-zero operators. For any z and w in $\mathbb{B}_n$, using (2.1) we have

$$f(w)K_z^\gamma(\varphi(w)) = (W_{f,\varphi}K_z^\gamma)(w) = (W_{g,\psi}^* K_z^\gamma)(w) = \overline{g}(z)K_{\psi(z)}^\gamma(w). \quad (5.2)$$

Letting $z = 0$ in (5.2) gives $f(w) = \overline{g}(0)K_{\psi(0)}^\gamma(w) = \overline{g}(0)\big(1 - \langle w, \psi(0)\rangle\big)^{-\gamma}$ for $w \in \mathbb{B}_n$. This, in particular, implies $f(0) = \overline{g}(0)$, which is non-zero by the assumption that operators are non-zero.

Letting $w = 0$ in (5.2) gives

$$\overline{g}(z) = f(0)K_z^\gamma(\varphi(0)) = f(0)\big(1 - \langle \varphi(0), z\rangle\big)^{-\gamma} \text{ for } z \in \mathbb{B}_n.$$

Substituting the formulas for f, g and $K(\cdot, \cdot)$ into (5.2) and canceling the constants, we obtain

$$(1 - \langle w, \psi(0)\rangle)^{-\gamma}(1 - \langle \varphi(w), z\rangle)^{-\gamma} = (1 - \langle \varphi(0), z\rangle)^{-\gamma}(1 - \langle w, \psi(z)\rangle)^{-\gamma}.$$

This identity implies

$$(1 - \langle w, \psi(0)\rangle)(1 - \langle \varphi(w), z\rangle) = (1 - \langle \varphi(0), z\rangle)(1 - \langle w, \psi(z)\rangle). \quad (5.3)$$

An easy calculation then gives

$$\Big\langle \big(1 - \langle w, \psi(0)\rangle\big)\varphi(w) - \varphi(0), z \Big\rangle = \Big\langle w, \big(1 - \langle z, \varphi(0)\rangle\big)\psi(z) - \psi(0)\Big\rangle.$$

Using Lemma 5.1, we conclude that there exists a linear operator A on $\mathbb{C}^n$ such that

$$\varphi(w) = \frac{\varphi(0) + Aw}{1 - \langle w, \psi(0)\rangle} \text{ and } \psi(z) = \frac{\psi(0) + A^*z}{1 - \langle z, \varphi(0)\rangle} \text{ for all } w, z \in \mathbb{B}_n.$$

Put $\alpha = f(0), c = \psi(0)$ and $d = \varphi(0)$, we see that f, g and φ, ψ satisfy (5.1).

For the converse, suppose f, g and φ, ψ are as above such that φ and ψ map the unit ball into itself. Since $W_{f,\varphi} = \alpha W_{K_{\psi(0)}^\gamma, \varphi}$ and $W_{g,\psi} = \overline{\alpha} W_{K_{\varphi(0)}^\gamma, \psi}$, (4.4) gives $W_{g,\psi} = W_{f,\varphi}^*$ on H_γ, which is equivalent to $W_{f,\varphi} = W_{g,\psi}^*$. The boundedness of these operators on H_γ follows from Remark 2.5. $\square$

As an immediate application of Theorem 5.2, we obtain a characterization of self-adjoint weighted composition operators.

Corollary 5.4. *Let f be an analytic function and φ an analytic self-map of $\mathbb{B}_n$. Then $W_{f,\varphi}$ is a non-zero self-adjoint bounded operator on H_γ if and only if there is a vector $c \in \mathbb{B}_n$, a self-adjoint linear operator A on $\mathbb{C}^n$ and a real number α such that $f = \alpha K_c^\gamma = \alpha K_{\varphi(0)}^\gamma$ and $\varphi(z) = \frac{c+Az}{1-\langle z,c\rangle}$ for $z \in \mathbb{B}_n$.*

Proof. Since $W_{f,\varphi}^* = W_{f,\varphi}$, Theorem 5.2 shows that there are vectors c, d in $\mathbb{B}_n$, a linear operator A on $\mathbb{C}^n$ and a complex number α such that for all $z \in \mathbb{B}_n$, $f(z) = \alpha K_d^\gamma = \overline{\alpha} K_c^\gamma$ and $\varphi(z) = \frac{d+Az}{1-\langle z,c\rangle} = \frac{c+A^*z}{1-\langle z,d\rangle}$. This shows that $\overline{\alpha} = \alpha$, $c = d$ and $A^* = A$ and hence f, φ have the required form. $\square$

In [9, 8], the authors go on to describe the eigenvectors, eigenvalues and other spectral properties of self-adjoint weighted composition operators on H_γ ($\gamma \geq 1$) of the unit disk. Their analysis is based on the classification of linear fractional self-maps of the unit disk.

In dimension $n \geq 2$ and in the case φ has a fixed point in $\mathbb{B}_n$ (the elliptic case), eigenvectors, eigenvalues and the spectrum of the self-adjoint operator $W_{f,\varphi}$ can be described as in Proposition 4.4 and in the discussion preceding this proposition. The cases where all the fixed points of φ lie on the unit sphere (the parabolic and hyperbolic cases) are, we believe, more complicated and seem to require more careful analysis. We leave this open for future research.

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