SELF-COMMUTATOR NORM OF HYPONORMAL TOEPLITZ OPERATORS

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Abstract. Chu and Khavinson recently obtained a lower bound for the norm of the self-commutator of a certain class of hyponormal Toeplitz operators on the Hardy space. Via a different approach, we offer a generalization of their result.

1. Introduction

We denote by $\mathbb{D}$ the open unit disk in the complex plane and $\partial\mathbb{D}$ its boundary, the unit circle. Recall that the Hardy space H^2 is the closed subspace of $L^2 = L^2(\partial\mathbb{D})$ consisting of all functions whose negative Fourier coefficients vanish. Let $P : L^2 \to H^2$ denote the orthogonal projection. For a bounded function $\varphi \in L^\infty = L^\infty(\partial\mathbb{D})$, the Toeplitz operator $T_\varphi : H^2 \to H^2$ is defined as

$$T_\varphi(u) = P(\varphi\, u) \;\text{ for all } u \in H^2.$$

The study of Toeplitz operators on the Hardy space was initiated by the seminal paper [3] of Brown and Halmos in the sixties.

Define a linear operator $J : L^2 \to L^2$ by $J(u)(z) = \bar{z}u(\bar{z})$ for $u \in L^2$ and $z \in \partial\mathbb{D}$. It is immediate that J is a unitary operator on L^2 and it is not hard to verify that J maps $(H^2)^\perp$ onto H^2. For $\varphi \in L^\infty$, the Hankel operator $H_\varphi : H^2 \to H^2$ is defined as

$$H_\varphi(u) = J(I - P)(\varphi u) \quad \text{for } u \in H^2.$$

We list below a few properties of Toeplitz and Hankel operators that shall be useful for us.

(a) For any $\varphi \in L^\infty$, we have $\|T_\varphi\| = \|\varphi\|_\infty$ and $T_\varphi^* = T_{\bar{\varphi}}$.

(b) For $f, g \in H^2$,
$$\langle T_\varphi f, g\rangle = \langle \varphi f, g\rangle.$$

(c) If h belongs to H^∞ (which is $H^2 \cap L^\infty$), then $T_h = M_h$, the operator of multiplication by h. In addition,
$$T_{\bar{h}}(1) = \overline{h(0)}.$$

(d) For any $h_1, h_2 \in H^\infty$,
$$T_{\bar{h}_1 \varphi h_2} = T_{\bar{h}_1} T_\varphi T_{h_2}.$$

(e) For $u, f \in H^\infty$, we have $H_{\bar{f}}^*(u) = T_{\bar{z}} T_{u(\bar{z})}(f)$.

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A bounded linear operator T on a Hilbert space is **hyponormal** if its self-commutator $[T^*, T] = T^*T - TT^*$ is positive. Recall that the spectrum of an operator T, denoted by $\mathrm{sp}(T)$, is the set of all complex numbers λ for which $T - \lambda I$ is not invertible, where I is the identity operator. The celebrated Putnam's Inequality [9] asserts that for any hyponormal operator T,

$$\big\|[T^*, T]\big\| \leq \frac{\mathrm{Area}(\mathrm{sp}(T))}{\pi}.$$

For a function $\varphi \in H^\infty$, it is well known that the Toeplitz operator T_φ is hyponormal and $\mathrm{sp}(T_\varphi) = \overline{\varphi(\mathbb{D})}$. A lower bound for the norm of the commutator $[T^*_\varphi, T_\varphi]$ was obtained by D. Khavinson [7]. Combining with Putnam inequality, one deduces the isoperimetric inequality (see, e.g. [2]). Recently, several papers [1, 6, 8] have investigated similar problems for analytic Toeplitz operators on the Bergman space.

On the Hardy space, hyponormal Toeplitz operators were characterized by Cowen in [5].

Theorem 1 (Cowen)**.** *If φ is in $L^\infty(\partial\mathbb{D})$, where $\varphi = f + \bar{g}$ for f, g in H^2, then T_φ is hyponormal if and only if*

$$g = c + T_{\bar{h}} f$$

for some constant c and some function $h \in H^\infty$ with $\|h\|_\infty \leq 1$.

Remark 2. In general, the function h is not unique. For more details on this, see the discussion following the proof of [5, Theorem 1].

Under the additional hypothesis $h(0) = 0$, Chu and Khavinson [4] recently obtained a lower bound for the norm of the self-commutator of T_φ.

Theorem 3 (Chu-Khavinson)**.** *If $\varphi = f + \overline{T_{\bar{h}} f}$ for f, h in H^∞, $\|h\|_\infty \leq 1$ and $h(0) = 0$, then*

$$\|[T^*_\varphi, T_\varphi]\| \geq \|f - f(0)\|_2^2 = \|P(\varphi) - \varphi(0)\|_2^2. \tag{1}$$

Due to the additional condition that $h(0) = 0$, Theorem 3 is not applicable in more general situations. As an example, take $\varphi(z) = z + \bar{z}/2$. A direct calculation shows that

$$[T^*_\varphi, T_\varphi] = \frac{3}{4} e_0 \otimes e_0,$$

where $e_0(z) = 1$ for $z \in \partial\mathbb{D}$. Recall that for elements u, v in H^2, we use $u \otimes v$ to denote the operator given by $(u \otimes v)(x) = \langle x, v \rangle u$ for $x \in H^2$. We then have $\|[T^*_\varphi, T_\varphi]\| = \frac{3}{4}$ while $\|P(\varphi) - \varphi(0)\|_2 = 1$. This shows that inequality (1) is false in this case. More generally, as we shall see later, Theorem 3 is not applicable if $\varphi = f + \lambda \bar{f}$ with $0 < |\lambda| < 1$ and f an inner function vanishing at the origin.

The purpose of this note is twofold. First, we offer an improved version of Theorem 3 which is applicable even in the case $h(0) \neq 0$. Second, we present a different and more operator-theoretic proof than that of Chu and Khavinson.

We state here our main result.

Theorem 4. *Let $\varphi = f + \overline{T_{\bar{h}} f}$ be a bounded harmonic function on the unit disk, where $f, h \in H^\infty$ with $\|h\|_\infty \leq 1$ and $|h(0)| < 1$. Put $\psi = f - h(0)\, T_{\bar{h}} f$. Then*

$$\big\|[T^*_\varphi, T_\varphi]\big\| \geq \frac{\|\psi - \psi(0)\|_2^2}{1 - |h(0)|^2}. \tag{2}$$

Remark 5. In the case $h(0) = 0$, we see that (2) reduces to Chu-Khavinson's result. In the case $|h(0)| = 1$, the function h is a unimodular constant function. It then follows that $[T_\varphi^*, T_\varphi] = 0$, which means that T_φ is normal. We assume $|h(0)| < 1$ to avoid such trivial case.

Remark 6. If we fix f and let h vary in the unit ball of H^∞, the norm $\|[T_\varphi^*, T_\varphi]\|$ is always bounded by $\|f\|_\infty^2$ (which follows from (3) in Section 2). As a consequence, the right-hand side of (2) remains bounded. We provide here a more direct argument to explain this. From the definition of ψ, we have $\psi = T_{1-h(0)\bar{h}}(f)$. We then compute

$$\begin{aligned}\|\psi - \psi(0)\|_2^2 \leq \|\psi\|_2^2 &= \|T_{1-h(0)\bar{h}} f\|_2^2 = \|T_f(1 - h(0)\bar{h})\|_2^2 \\ &\leq \|f\|_\infty^2 \, \|1 - h(0)\bar{h}\|_2^2 = \|f\|_\infty^2 \left(1 + |h(0)|^2 \, \|h\|_2^2 - 2|h(0)|^2\right) \\ &\leq \|f\|_\infty^2 \, (1 - |h(0)|^2) \quad (\text{since } \|h\|_\infty \leq 1),\end{aligned}$$

which implies that the ratio $\|\psi - \psi(0)\|_2^2/(1 - |h(0)|^2)$ is bounded by $\|f\|_\infty^2$.

2. Proof of the main result

In this section we offer a proof of our result and discuss an application. We begin with a simple and probably well-known fact from Functional Analysis. We present here a quick proof.

Lemma 7. *Let T be a positive operator on a Hilbert space $\mathcal{H}$. Then for any $v \in \mathcal{H}$, the operator $S = \langle Tv, v\rangle T - (Tv) \otimes (Tv)$ is positive as well.*

Proof. Let $T^{1/2}$ denotes the positive square root of T. For any $u \in \mathcal{H}$,

$$\langle Su, u\rangle = \|T^{1/2} v\|^2 \, \|T^{1/2} u\|^2 - |\langle T^{1/2} v, T^{1/2} u\rangle|^2 \geq 0$$

by Cauchy-Schwarz's inequality. The conclusion of the lemma now follows. □

Lemma 7 provides us with the following immediate consequence for Toeplitz operators with holomorphic symbols.

Proposition 8. *Suppose that h is a bounded holomorphic function with $\|h\|_\infty \leq 1$ and $|h(0)| < 1$. Put $\xi = (1 - \overline{h(0)}\, h)/\sqrt{1 - |h(0)|^2}$. Then*

$$I - T_h T_{\bar{h}} \geq \xi \otimes \xi$$

on the Hardy space H^2.

Proof. Because $\|h\|_\infty \leq 1$, the operator $T_{\bar{h}}$ has norm at most 1. This implies that $I - T_h T_{\bar{h}} \geq 0$. Applying Lemma 7 with $T = I - T_h T_{\bar{h}}$ and $v = 1$ gives

$$\big(1 - \|T_{\bar{h}}(1)\|^2\big)\big(I - T_h T_{\bar{h}}\big) - \big(1 - T_h T_{\bar{h}}(1)\big) \otimes \big(1 - T_h T_{\bar{h}}(1)\big) \geq 0.$$

Since $T_{\bar{h}} 1 = \overline{h(0)}$ and $T_h T_{\bar{h}} 1 = \overline{h(0)}\, h$, we obtain the the desired operator inequality. □

We are now ready for the proof of our main result.

Proof of Theorem 4. Write $\varphi = f + \bar{g}$ with $g = T_{\bar{h}} f$. We have $H_{\bar{g}} = T_{\overline{h^*}} H_{\bar{f}}$ (see [5, p. 811]), where $h^*(z) = \overline{h(\bar{z})}$. We now compute

$$\begin{aligned}[T_\varphi^*, T_\varphi] &= [T_{\bar{f}+g}, T_{f+\bar{g}}] = [T_{\bar{f}}, T_f] - [T_{\bar{g}}, T_g] \\ &= H_{\bar{f}}^* H_{\bar{f}} - H_{\bar{g}}^* H_{\bar{g}} = H_{\bar{f}}^* \big(I - T_{h^*} T_{\overline{h^*}}\big) H_{\bar{f}}. \qquad (3)\end{aligned}$$

Proposition 8 then implies

$$[T_\varphi^*, T_\varphi] \geq H_{\bar{f}}^*\big(\eta \otimes \eta\big) H_{\bar{f}} = H_{\bar{f}}^*(\eta) \otimes H_{\bar{f}}^*(\eta),$$

where

$$\eta = \frac{1 - \overline{h^*(0)}\, h^*}{\sqrt{1 - |h^*(0)|^2}} = \frac{1 - h(0)\, h^*}{\sqrt{1 - |h(0)|^2}}.$$

As a consequence,

$$\big\|[T_\varphi^*, T_\varphi]\big\| \geq \big\|H_{\bar{f}}^*(\eta) \otimes H_{\bar{f}}^*(\eta)\big\| = \big\|H_{\bar{f}}^*(\eta)\big\|_2^2. \tag{4}$$

On the other hand,

$$\begin{aligned} H_{\bar{f}}^*(\eta) = T_{\bar{z}} T_{\eta(\bar{z})}(f) &= \frac{1}{\sqrt{1 - |h(0)|^2}}\, T_{\bar{z}}\big(f - h(0) T_{\bar{h}} f\big) \\ &= \frac{1}{\sqrt{1 - |h(0)|^2}}\, T_{\bar{z}}\big(f - h(0) g\big) \quad (\text{since } g = T_{\bar{h}} f) \\ &= \frac{1}{\sqrt{1 - |h(0)|^2}}\, T_{\bar{z}} \psi, \end{aligned}$$

since $\psi = f - h(0)g$. We then have

$$\|H_{\bar{f}}^*(\eta)\| = \frac{\|T_{\bar{z}}\psi\|_2^2}{1 - |h(0)|^2} = \frac{\|\psi - \psi(0)\|_2}{\sqrt{1 - |h(0)|^2}},$$

which, together with (4), gives the required inequality(2)

$$\big\|[T_\varphi^*, T_\varphi]\big\| \geq \frac{\big\|\psi - \psi(0))\big\|_2^2}{1 - |h(0)|^2}. \qquad \square$$

Theorem 4 provides a lower bound for the norm of $\|[T_\varphi^*, T_\varphi]\|$ in terms of both the holomorphic and anti-holomorphic parts of φ. In the following corollary, we obtain a weaker estimate which only depends on the holomorphic part of φ.

Corollary 9. *Let $\varphi = f + \overline{T_{\bar{h}} f}$ be a bounded harmonic function on the unit disk, where $f, h \in H^\infty$ with $\|h\|_\infty \leq 1$ and $|h(0)| < 1$. Then*

$$\big\|[T_\varphi^*, T_\varphi]\big\| \geq \frac{\big(1 - |h(0)|\, \|h\|_\infty\big)^2}{1 - |h(0)|^2} \|f - f(0)\|_2^2. \tag{5}$$

Proof. Let us estimate the numerator of the right hand side of (2) in Theorem 4.

$$\begin{aligned} \|\psi - \psi(0)\|_2 &= \|(f - f(0)) - h(0)(g - g(0))\|_2 \\ &\geq \|f - f(0)\|_2 - |h(0)| \cdot \|g - g(0)\|_2 \\ &= \|f - f(0)\|_2 - |h(0)| \cdot \|T_{\bar{z}} g\|_2 \\ &= \|f - f(0)\|_2 - |h(0)| \cdot \|T_{\bar{z}} T_{\bar{h}}(f)\|_2 \\ &= \|f - f(0)\|_2 - |h(0)| \cdot \|T_{\bar{h}} T_{\bar{z}}(f - f(0))\|_2 \quad (\text{since } T_{\bar{z}}(f(0)) = 0) \\ &\geq \|f - f(0)\|_2 - |h(0)|\, \|h\|_\infty \cdot \|f - f(0)\|_2 \quad (\text{since } \|T_{\bar{h}} T_{\bar{z}}\| \leq \|h\|_\infty). \end{aligned}$$

Combing with (2) gives

$$\frac{\|\psi - \psi(0)\|_2^2}{1 - |h(0)|^2} \geq \frac{(1 - |h(0)|\, \|h\|_\infty)^2}{1 - |h(0)|^2} \|f - f(0)\|_2^2$$

as desired. $\square$

Example 10. Consider $\varphi(z) = z + \bar{z}/2$ so that $f(z) = z$ and $h(z) = \frac{1}{2}$. We have seen in Introduction that $\|[T_\varphi^*, T_\varphi]\| = \frac{3}{4}$. On the other hand, the right-hand side of (5) is also equal to $\frac{3}{4}$. Consequently, (5) is in fact an equality in this case.

Using Putnam's Inequality and Corollary 9, we obtain

Corollary 11. *If $\varphi = f + \overline{T_{\bar{h}} f}$ for $f, h \in H^\infty$ with $\|h\|_\infty \le 1$ and $|h(0)| < 1$, then*

$$\mathrm{Area}(\mathrm{sp}(T_\varphi)) \ge \pi \frac{\big(1 - |h(0)|\,\|h\|_\infty\big)^2}{1 - |h(0)|^2} \|f - f(0)\|_2^2.$$

Given a bounded function φ for which T_φ is hyponormal, the existence of the function h in the representation $\varphi = f + \overline{T_{\bar{h}} f}$ is not unique. Our lower estimate of the norm of $[T_\varphi^*, T_\varphi]$ in Theorem 4 depends on the value of $h(0)$. In some cases, it turns out that $h(0)$ is independent of the choice of h. We illustrate this in the following example.

Example 12. Let χ be an inner function with $\chi(0) = 0$. Suppose f is a non-constant polynomial of χ and g belongs to H^∞ such that $T_{f+\bar{g}}$ is hyponormal. Then for any function $h \in H^\infty$ satisfying $\|h\|_\infty \le 1$ and $g = c + T_{\bar{h}} f$ for some constant c, the value $h(0)$ is independent of h. In the case $f = \chi$, we have $h(0) = \langle f, g \rangle$.

Proof. Since f is a non-constant polynomial of χ, there exist $M \ge 1$ and complex numbers $c_1, \ldots, c_M$ such that $c_M \ne 0$ and

$$f = c_0 + \cdots + c_M \chi^M.$$

We then compute

$$\begin{aligned} \langle \chi^M, g \rangle &= \langle \chi^M, T_{\bar{h}} f + c \rangle = \langle \chi^M, T_{\bar{h}} f \rangle \qquad (\text{since } \chi^M(0) = 0) \\ &= \langle \chi^M h, f \rangle = \sum_{j=0}^{M} \bar{c}_j \langle \chi^M h, \chi^j \rangle = \bar{c}_M h(0) = \langle \chi^M, f \rangle h(0). \end{aligned}$$

It follows that

$$h(0) = \frac{\langle \chi^M, g \rangle}{\langle \chi^M, f \rangle},$$

which is independent of the choice of h. In the case $f = \chi$, we have $M = 1$ and hence $h(0) = \langle f, g \rangle$. ◻

Remark 13. The lower estimate in Theorem 4 makes use of the value of h at the origin. For any $a \in \mathbb{D}$, we briefly discuss here how an estimate involving $h(a)$ may be obtained. However, the formula is a little more complicated. Recall that $k_a(z) = \sqrt{1 - |a|^2}/(1 - \bar{a}z)$ is the normalized reproducing kernel of the Hardy space at a. We shall write

$$\varphi_a(z) = \frac{a - z}{1 - \bar{a}z}$$

for the Mobius automorphism of the unit disk that interchanges a and the origin. Note that $\varphi_a \circ \varphi_a(z) = z$ for all $z \in \mathbb{D}$. Define the operator W_a by

$$W_a(u) = k_a \cdot (u \circ \varphi_a), \quad u \in L^2(\partial\mathbb{D}).$$

A change-of-variables on the unit circle shows that that W_a is a unitary operator on $L^2(\partial\mathbb{D})$. It is well known that H^2 is a reducing subspace of W_a and

$$W_a^* T_\varphi W_a = T_{\varphi \circ \varphi_a}$$

for any bounded φ. As a consequently, T_φ is hyponormal if and only if $T_{\varphi\circ\varphi_a}$ is hyponormal and their self-commutators have the same norm. Note that if $g = T_{\bar{h}} f$ then it can be checked that $g\circ\varphi_a = T_{\bar{h}\circ\varphi_a}(f\circ\varphi_a)+c$ for some constant c. Applying Theorem 2 for $\varphi\circ\varphi_a$ gives

$$\begin{aligned}\|[T^*_{\varphi\circ\varphi_a}, T_{\varphi\circ\varphi_a}]\| &\geq \frac{\|f\circ\varphi_a - (h\circ\varphi_a(0))g\circ\varphi_a\|_2^2 - |f\circ\varphi_a(0) - (h\circ\varphi_a(0))g\circ\varphi_a(0)|^2}{1-|h\circ\varphi_a(0)|^2}\\ &= \frac{\|f\circ\varphi_a - h(a)g\circ\varphi_a)\|_2^2 - |f(a)-h(a)g(a)|^2}{1-|h(a)|^2}. \qquad (6)\end{aligned}$$

Since W_a is a unitary operator, the first term in the numerator of (6) is equal to

$$\|W_a(f\circ\varphi_a - h(a)g\circ\varphi_a)\|_2^2 = \big\|\big(f-h(a)g\big)k_a\big\|_2^2.$$

We then have

$$\|[T^*_\varphi, T_\varphi]\| = \|[T^*_{\varphi\circ\varphi_a}, T_{\varphi\circ\varphi_a}]\| \geq \frac{\big\|(f-h(a)g)k_a\big\|_2^2 - \big|f(a)-h(a)g(a)\big|^2}{1-|h(a)|^2}. \qquad (7)$$

It is possible to obtain estimate (7) by following the proof of Theorem 4. One needs to modify Proposition 8 by setting $v = k_{\bar{a}}$ instead of $v = 1$. However, some parts of calculation are a bit more complicated. We leave this for the interested reader.

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