SPECTRA OF SOME WEIGHTED COMPOSITION OPERATORS ON THE BALL

SCOTT KASCHNER, TRIEU LE, CHLOE MAKDAD, BENJAMIN REMPFER, DEREK THOMPSON, AND DEJUAN WINTERS

Abstract. We find sufficient conditions for a self-map of the unit ball to converge uniformly under iteration to a fixed point or idempotent on the entire ball. Using these tools, we establish spectral containments for weighted composition operators on Hardy and Bergman spaces of the ball. When the compositional symbol is in the Schur-Agler class, we establish the spectral radii of these weighted composition operators.

1. Introduction

Let $\mathbb{B}$ denote the open unit ball on $\mathbb{C}^k$ for a fixed integer k. The classical *Hardy space* $H^2(\mathbb{B})$ is the Hilbert space of analytic functions h on the open unit ball $\mathbb{B}$ for which

$$\|h\| = \sup_{0<r<1} \Big(\int_{\partial\mathbb{B}} |h(r\zeta)|^2 \, d\sigma(\zeta) \Big)^{1/2} < \infty.$$

Here $d\sigma$ is the normalized surface area measure on the unit sphere $\partial\mathbb{B}$. It is well known that $H^2(\mathbb{B})$ is a reproducing kernel Hilbert space with kernel

$$K_{H^2}(z,w) = \frac{1}{(1-\langle z,w\rangle)^k}, \quad z,w \in \mathbb{B}. \tag{1}$$

Setting $K_w(z) = K(z,w)$, we then have $h(w) = \langle h, K_w\rangle$ for all $w \in \mathbb{B}$.

For any real number $\gamma > -1$, the weighted measure dV_γ is defined by

$$dV_\gamma(z) = \frac{\Gamma(k+\gamma+1)}{k!\,\Gamma(\gamma+1)}(1-|z|^2)^\gamma dV(z),$$

where dV is the normalized Lebesgue measure on $\mathbb{B}$. Note that dV_γ is a probability measure. The Bergman space A^2_γ consists of all analytic functions on $\mathbb{B}$ that are square integrable with respect to dV_γ. It is also well known that A^2_γ is a reproducing kernel Hilbert space with kernel given by

$$K_{A^2_\gamma}(z,w) = \frac{1}{(1-\langle z,w\rangle)^{k+1+\gamma}}, \quad z,w \in \mathbb{B}. \tag{2}$$

The reader is referred to [2, Section 2.1] and [11, Chapter 2] for more backgrounds on the Hardy and Bergman spaces. Note that the kernel function (2) becomes (1) when $\gamma = -1$. Consequently, we shall identify $H^2(\mathbb{B})$ and A^2_{-1}, as usually done by other authors, and all results on A^2_γ will be for $\gamma \geq -1$.

A *composition operator* C_φ on A^2_γ is given by $C_\varphi f = f \circ \varphi$. We call φ the *symbol* of the associated composition operator. While composition operators on the classical Hardy space of $\mathbb{D}$ are bounded

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whenever φ is analytic, the situation is more difficult on $\mathbb{B}$. Composition operators on $H^2(\mathbb{D})$ ($\gamma = -1$, and $\mathbb{D}$ instead of $\mathbb{B}$) have been extensively studied for several decades; [2] and [10] are seminal books on the subject. Progress has been made for composition operators on A^2_γ for other values of γ and for $\mathbb{B}$ instead of $\mathbb{D}$, but it is much slower. In this paper, we will be adapting past work on $H^2(\mathbb{D})$ to obtain results on A^2_γ.

Composition operators in several variables are not always bounded. For example, if $\varphi(z_1, z_2) = (2z_1z_2, 0)$ or $\varphi(z_1, z_2) = (z_1^2 + z_2^2, 0)$, then C_φ is not bounded on the Hardy space over the unit ball in C^2_φ. See [2, Section 3.5] for other examples and criteria for boundedness and compactness on Hardy and Bergman spaces. Following Jury [5], we are interested in mappings φ that belong to the Schur-Agler class. These are analytic functions $\varphi : \mathbb{B} \to \mathbb{B}$ for which

$$\frac{1 - \langle \varphi(z), \varphi(w) \rangle}{1 - \langle z, w \rangle}, \quad z, w \in \mathbb{B}$$

is a positive semidefinite kernel on $\mathbb{B}$. [5, Theorem 4] shows that C_φ is bounded on A^2_γ whenever φ is in the Schur-Agler class. Spectral radii of such operators were computed in [5, 6].

The results so far on composition operators naturally motivate the study of *weighted* composition operators as well. Let ψ belong to $H^\infty(\mathbb{B})$, the multiplier algebra of A^2_γ: then for any $f \in A^2_\gamma$, $\psi f \in A^2_\gamma$ as well, so the multiplication operator T_ψ is well defined. Then $T_\psi C_\varphi = W_{\psi,\varphi}$ is what we call a *weighted composition operator*.

The *spectrum* of an operator T on a Hilbert space $\mathcal{H}$, denoted $\sigma(T)$, is given by $\{\lambda \in \mathbb{C} : T - \lambda I$ is not invertible$\}$. The *spectral radius* of T is $r(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\}$. A thorough treatment of the spectrum of composition operators on $H^2(\mathbb{D})$ is given in [2, Chapter 7]. Determining the spectra of *weighted* composition operators on $H^2(\mathbb{D})$ is still largely an open question. However, new results were obtained in [1], under assumptions about φ's behavior on the open unit disk. The Denjoy-Wolf Theorem guarantees that any analytic self-map of $\mathbb{D}$ not an elliptic automorphism, will converge under iteration to a single point, and this convergence is uniform on compact subsets of $\mathbb{D}$. In [1], the authors assumed that φ converged uniformly on the *entire open disk* (not just compact subsets of the disk). This property was then exploited to calculate the spectra for a large class of weighted composition operators. This work was then extended in [4] to several other analytic function spaces over $\mathbb{D}$. In this paper, we extend these same ideas to weighted composition operators on A^2_γ, by studying uniform convergence of self-maps of $\mathbb{B}$.

We use the phrase *UCI* (uniformly convergent iterates) to specifically indicate uniform convergence of iterates on the whole domain. Define $\varphi_n = \varphi \circ \cdots \circ \varphi$, the composition of φ with itself n times.

Theorem 1 (UCI in $\mathbb{D}$ with Interior Fixed Point [1])**.** *Suppose $\varphi \colon \mathbb{D} \to \mathbb{D}$ is analytic and continuous on $\partial\mathbb{D}$. If the Denjoy-Wolff point w is in $\mathbb{D}$, then φ has UCI with φ_n converging uniformly to w, if and only if there is $N > 0$ such that $\varphi_N\left(\overline{\mathbb{D}}\right) \subseteq \mathbb{D}$.*

If any successive image of the closed unit disk by φ is strictly contained within the open disk, then uniform convergence of iterates follows. However, the situation with a boundary fixed point is a bit more delicate.

Theorem 2 (UCI in $\mathbb{D}$ with Boundary Fixed Point [1])**.** *Suppose $\varphi \colon \mathbb{D} \to \mathbb{D}$ is analytic in $\mathbb{D}$ and continuous on $\partial\mathbb{D}$ and has Denjoy-Wolff point w with $|w| = 1$ and $\varphi'(w) < 1$. If $\varphi_N\left(\overline{\mathbb{D}}\right) \subseteq \mathbb{D} \cup \{w\}$ for some $N > 0$, then φ has UCI with φ_n converging uniformly to w.*

In this note, we address the analogous questions in a several complex variables setting. The open unit ball in $\mathbb{C}^k$ is the set of all complex-valued vectors with k components that satisfy $\|z\| < 1$

(where $\|\cdot\|$ is just the standard Euclidean norm), denoted

$$\mathbb{B} = \{z \in \mathbb{C}^k \colon \|z\| < 1\}.$$

In Section 2, we prove results analogous to Theorems 1 and 2 and address additional interesting phenomena that can only occur in complex dimension greater than one.

As in [1], we use the UCI property to answer questions about operators. In Section 3, we provide spectral bounds for weighted composition operators with composition operator symbol in the Schur-Agler class, and in Section 4, we calculate spectral radii for for weighted composition operators on A^2_γ. We then end with further questions related to our work.

2. When are the iterates of φ uniformly convergent on $\mathbb{B}$?

There are many more possibilities for fixed point sets of analytic maps of unit ball in $\mathbb{C}^k$. Thus, while there is still a Denjoy-Wolff theorem for self-maps of $\mathbb{B}$, it looks quite different. To establish it, we first need some preliminaries.

Definition (Affine subset of $\mathbb{B}$). *An affine subset of $\mathbb{B}$ (or $\overline{\mathbb{B}}$) is the intersection of $\mathbb{B}$ (or $\overline{\mathbb{B}}$) with*

$$c + L = \{c + z \colon z \in L\},$$

where $c \in \mathbb{B}$ and L is a complex linear subspace of $\mathbb{C}^k$.

Theorem 3 (Fixed point sets in $\mathbb{B}$ [9, Section 8.2]). *If $\varphi\colon \mathbb{B} \to \mathbb{B}$ is analytic, then the fixed point set of φ is an affine subset of $\mathbb{B}$.*

In the following statement of the Denjoy-Wolff theorem for $\mathbb{C}^k$, the second case in the theorem describes phenomena that do not arise in $\mathbb{D}$.

Theorem 4 (Denjoy-Wolff in $\mathbb{C}^k$ [7]). *Let $\varphi\colon \mathbb{B} \to \mathbb{B}$ be analytic. Then either*

- *There exists a point $w \in \overline{\mathbb{B}}$ such that φ_n converges uniformly to a on compact subsets of $\mathbb{B} \cup \{w\}$, or*
- *The fixed point set of φ is an affine subset $\mathcal{A} \subset \overline{\mathbb{B}}$, and a subsequence φ_{m_i} of φ_n converges uniformly on compact subsets of $\mathbb{B}$ to a nonconstant idempotent h whose fixed point set contains $\mathcal{A}$. If $\mathcal{A} = \{w\}$, a single point, then φ_n converges uniformly to a on compact subsets of $\mathbb{B}$.*

For analytic $\varphi\colon \mathbb{B} \to \mathbb{B}$ and $w \in \partial\mathbb{B}$, define the radial derivative

$$d(\varphi, w) = \liminf_{z \to w} \frac{1 - \|\varphi(z)\|}{1 - \|z\|}.$$

Theorem 5 (Julia's Lemma in $\mathbb{C}^k$). *Suppose $\varphi\colon \mathbb{B} \to \mathbb{B}$ is analytic, the sequence $a_n \to w \in \partial\mathbb{B}$ satisfies*

$$d(\varphi, w) = \lim_{n\to\infty} \frac{1 - \|\varphi(a_n)\|}{1 - \|a_n\|} < \infty,$$

and $\lim_{n\to\infty} \varphi(a_n) = \eta \in \partial\mathbb{B}$. Then for all $z \in \mathbb{B}$,

$$\frac{|1 - \langle \varphi(z), \eta\rangle|^2}{1 - \|\varphi(z)\|^2} \le d(\varphi, w) \frac{|1 - \langle z, w\rangle|^2}{1 - \|z\|^2}.$$

As with the version of Julia's lemma for $\mathbb{C}$, there is an immediate, convenient geometric consequence of this inequality. For $w \in \partial\mathbb{B}$, define the ellipsoid internally tangent to $\mathbb{B}$ at w:

$$E(w, \lambda) = \{z \in \mathbb{B} \colon |1 - \langle w, z\rangle|^2 \le \lambda(1 - \|z\|^2)\}.$$

By Julia's lemma with $d(\varphi, w) < 1$ and $w \in \partial\mathbb{B}$ a fixed point, we have the following [2]:

$$\varphi(E(w,\lambda)) \subseteq E(w, \lambda d(\varphi, w)),$$

where the containment is strict except at the point w. Therefore, each image by φ of an ellipsoid in $\mathbb{B}$ internally tangent at w is contained in another ellipsoid internally tangent at w whose size is scaled by the radial derivative $d(\varphi, w)$. However, if $\mathcal{A}$, the fixed point set of φ, has positive dimension in $\mathbb{B}$, then these ellipsoids necessarily intersect $\mathcal{A}$, so we have the following corollary.

Corollary. *Suppose $\varphi\colon \mathbb{B} \to \mathbb{B}$ has fixed point set $\mathcal{A}$ with $\dim \mathcal{A} > 0$. Then for all $w \in \mathcal{A} \cap \partial\overline{\mathbb{B}}$, $d(\varphi, w) = 1$.*

Proof. If $d(\varphi, w) < 1$, then $\varphi(E(w,\lambda)) \subset E(w, \lambda d(\varphi, w))$. This is a contradiction because $\mathcal{A}$ is fixed. ∎

In light of this corollary, we must impose a different condition on φ and h to guarantee UCI when $\dim \mathcal{A} > 0$. We collect this with our UCI results when $\dim \mathcal{A} = 0$ into the following theorem, and we now begin the work original to the authors.

Theorem 6 (Theorems 1 and 2 in $\mathbb{C}^k$)**.** *Suppose $\varphi\colon \mathbb{B} \to \mathbb{B}$ is analytic and continuous on $\partial\mathbb{B}$.*

(1) If there is a unique Denjoy-Wolff point $w \in \mathbb{B}$, then φ_n converges uniformly to w on $\mathbb{B}$ if and only if for some $N > 0$, $\varphi_N(\overline{\mathbb{B}}) \subset \mathbb{B}$.
(2) If there is a unique Denjoy-Wolff point $w \in \partial\mathbb{B}$, $d(\varphi, w) < 1$, and for some $N > 0$, $\varphi_N(\overline{\mathbb{B}}) \subset \mathbb{B} \cup \{w\}$, then φ_n converges uniformly to w on $\mathbb{B}$.
(3) If the fixed point set of φ is a nontrivial affine subset $\mathcal{A} \subset \overline{\mathbb{B}}$, there is an idempotent function $h\colon \mathbb{B} \to \mathbb{B}$ whose fixed point set contains $\mathcal{A}$, $\varphi(\overline{\mathbb{B}}) \subset \mathbb{B} \cup (\mathcal{A} \cap \partial\overline{\mathbb{B}})$, and there is an $0 \le \alpha < 1$ such that

$$\frac{\|\varphi(z) - h(z)\|^2}{1 - \|\varphi(z)\|} \le \alpha \frac{\|z - h(z)\|^2}{1 - \|z\|} \tag{3}$$

in some forward invariant neighborhood of $\mathcal{A} \cap \partial\overline{\mathbb{B}}$, then $\varphi_n \to h$ uniformly on $\mathbb{B}$.

The three distinct cases correspond to the three distinct possibilities that arise from the Denjoy-Wolff theorem. Proof of parts (1) and (2) follow immediately from [1] by replacing modulus with the $\mathbb{C}^k$ norm. Part (3) regards the novel aspects of the Denjoy-Wolff theorem in more than one variable. Note that with the additional hypotheses in part (3), we are able to prove uniform convergence of φ_n to h, rather than uniform convergence of φ_{m_i} (along some specific subsequence).

Proof of Theorem 6. Let T be the forward invariant neighborhood defined by Equation (3). Note that T contracts by a multiple of α with each iterate. As in [1], we have from Equation (3) that for all $z \in T$,

$$\|\varphi_n(z) - h(z)\|^2 \le \alpha^n (1 - \|\varphi(z)\|),$$

so we have

$$\|\varphi_n(z) - h(z)\| \le \alpha^{n/2}(1 - \|\varphi_n(z)\|)^{1/2} \le \alpha^{n/2}. \tag{4}$$

It follows that φ_n converges uniformly on T.

Now suppose $\{w_n\} \subset \mathbb{B}$ is a sequence of points such that $w_n \to w \in \mathcal{A} \cap \partial\mathbb{B}$. Since $\alpha < 1$, it follows that there is an N such that for all $n \ge N$, we have $w_n \in T$. Since $\varphi_N(\overline{\mathbb{B}}) \subset \mathbb{B} \cup (\mathcal{A} \cap \partial\mathbb{B})$, it follows that

$$K = \varphi_N(\overline{\mathbb{B}}) \backslash T^\circ \subset \mathbb{B},$$

where T° is the interior of T, and K is compact. See Figure 1 for a picture of the real slice of this situation when $k = 2$. Since K is a compact subset of $\mathbb{B}$, it follows from the Denjoy-Wolff theorem that φ_n converges uniformly on K.

Since $\varphi_N(\mathbb{B}) \subset K \cup (T \cap \mathbb{B})$, φ_n converges uniformly on K, and φ_n converges uniformly on $T \cap \mathbb{B}$, φ_n also converges uniformly on $\mathbb{B}$. ■

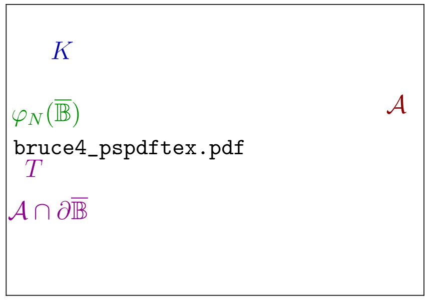

FIGURE 1. The real slice of $\mathbb{B}$ when $k = 2$ with forward invariant neighborhood T of $\mathcal{A} \cap \partial\overline{\mathbb{B}}$ and K, a forward invariant compact subset of $\mathbb{B}$

Example 7. *Let $\varphi\colon \mathbb{B} \to \mathbb{B}$ be defined by*

$$\varphi(z_1, z_2) = \left(az_1^d, az_2^d\right)$$

for some $d \geq 2$ and $|a| \leq 1$, so $0 \in \mathbb{B}$ is the unique fixed point. Note that $\varphi(\overline{\mathbb{B}}) \subset \mathbb{B}$ if and only if $|a| < 1$. As in Theorem 6 part (1), we have that φ_n converges to 0 uniformly if and only if $|a| < 1$.

Example 8. *Let $\varphi\colon \mathbb{B} \to \mathbb{B}$ be defined by*

$$\varphi(z_1, z_2) = \left(\frac{1}{2}z_1 + \frac{1}{2}, \frac{1}{2}z_2^2\right),$$

so $w := (1, 0) \in \partial\mathbb{B}$ is the unique fixed point with $d(\varphi, w) = 1/2$. Also, the ellipsoids, $E(w, \lambda)$, tangent to $\partial\mathbb{B}$ at w are forward invariant by φ and contract by a factor of $d(\varphi, w) = 1/2$:

$$\varphi(E(w, \lambda)) \subset E(w, \lambda d(\varphi, w)) = E(w, \lambda/2).$$

Then by Julia's lemma, we have

$$\|\varphi(z) - w\|^2 \leq d(\varphi, w)(1 - \|\varphi(z)\|)^2 = \frac{1}{2}(1 - \|\varphi(z)\|)^2,$$

so upon iteration,

$$\|\varphi_n(z) - w\| \leq \frac{1}{\sqrt{2}^n}(1 - \|\varphi(z)\|) < \frac{1}{\sqrt{2}^n}.$$

It follows that φ_n converges uniformly to w on $\mathbb{B}$.

Example 9. *Let $\varphi\colon \mathbb{B} \to \mathbb{B}$ be defined by*

$$\varphi(z_1, z_2) = \left(z_1, \alpha z_2^2\right)$$

for some fixed $|\alpha| < 1$. Note that $\mathcal{A} = \{(z_1, 0)\colon z_1 \in \mathbb{D}\}$, and as in Theorem 6 part (3), the idempotent map to which φ_n converges is $h(z_1, z_2) = (z_1, 0)$.

Note first that for all $w \in \mathcal{A} \cap \partial\mathbb{B}$, we have $d(\phi, w) = 1$. Note also that for all $(z_1, z_2) \in \mathbb{B}$, we have

$$\|\alpha z_2^2\|^2 \leq |\alpha| \, \|z_2\|^2,$$

which implies

$$\frac{\|\alpha z_2^2\|^2}{1 - \|\varphi(z_1, z_2)\|} \leq |\alpha| \frac{\|z_2\|^2}{1 - \|(z_1, z_2)\|}.$$

From this, we have that φ and h satisfy Equation (3)*. It follows that φ has UCI.*

Cowen and MacCluer [3] investigated linear fractional self-maps of the unit ball and the corresponding composition operators. In the case $\varphi(z) = Az+b$, where $b \in \mathbb{B}$ and A is a linear operator, we have the following necessary condition in order for φ map $\mathbb{B}$ into itself. Let $z \in \mathbb{B}$. Choose a complex number λ of modulus one such that

$$\lambda\langle Az, b\rangle = |\langle Az, b\rangle|.$$

We then have

$$1 > \|\varphi(\lambda z)\|^2 = \|\lambda Az + b\|^2 = \|Az\|^2 + 2|\langle Az, b\rangle| + \|b\|^2 \geq \|Az\|^2 + \|b\|^2.$$

Since $z \in \mathbb{B}$ was arbitrary, we conclude that

$$\|A\|^2 + \|b\|^2 \leq 1.$$

It follows that $\|A\| \leq 1$, and $b = 0$ in the case $\|A\| = 1$.

We now characterize affine maps whose iterates converge uniformly on $\mathbb{B}$.

Theorem 10. *Let $\varphi(z) = Az+b$ be a self-map of $\mathbb{B}$, where $b \in \mathbb{B}$ and A is a linear operator. Then the following statements are equivalent.*

(a) $\{\varphi_n\}$ converges uniformly on $\mathbb{B}$.
(b) $\{\varphi_n\}$ converges pointwise on $\mathbb{B}$.
(c) $\sigma(A) \subset \{1\} \cup \mathbb{D}$.

Proof. It is clear that (a) implies (b). For the implication (b)$\longrightarrow$(c), we shall prove the contrapositive. Suppose that $\sigma(A) \not\subset \{1\} \cup \mathbb{D}$. Since $\|A\| \leq 1$, we conclude that A processes an eigenvalue α with $|\alpha| = 1$ and $\alpha \neq 1$. It follows that $\|A\| = 1$ and hence $b = 0$. Let $v \in \mathbb{B}\backslash\{0\}$ be an eigenvector corresponding to α. For all integers $n \geq 1$, we then have $\varphi_n(v) = \alpha^n v$. As a consequence, $\{\varphi_n(v)\}$ does not converge, which implies that $\{\varphi_n\}$ does not converge pointwise on $\mathbb{B}$.

Now suppose that (c) holds. We consider two cases. First, assume $\sigma(A) \subset \mathbb{D}$. Then $I - A$ is invertible and since the spectral radius of A is strictly smaller than 1, we have $\lim\limits_{n\to\infty} \|A^n\| = 0$. For any integer $n \geq 2$ and $z \in \mathbb{B}$,

$$\varphi_n(z) = A^n z + (I + A + \cdots + A^{n-1})b = A^n z + (I - A^n)(I - A)^{-1}b.$$

Therefore,

$$\|\varphi_n(z) - (I - A)^{-1}b\| = \Big\|A^n\Big(z - (I - A)^{-1}b\Big)\Big\| \leq \|A^n\| \cdot \Big(1 + \|(I - A)^{-1}b\|\Big) \longrightarrow 0.$$

Consequently, $\{\varphi_n\}$ converges uniformly on $\mathbb{B}$ to the point $(I - A)^{-1}b$.

Now assume $1 \in \sigma(A)$. Then $\|A\| = 1$ and $b = 0$ so $\varphi(z) = Az$ for $z \in \mathbb{B}$. Let $\mathcal{M} = \ker(I - A)$. Then $\mathcal{M}$ is a reducing subspace for A. Decompose $\mathbb{C}^k = \mathcal{M} \oplus \mathcal{M}^\perp$ and write $A = P_{\mathcal{M}} + T$, where $P_{\mathcal{M}}$ is the orthgonal projection from $\mathbb{C}^k$ onto $\mathcal{M}$ and $T = A(I - P_{\mathcal{M}})$. Note that $TP_{\mathcal{M}} = P_{\mathcal{M}}T = 0$ and $\sigma(T) \subset \mathbb{D}$. For any integer $n \geq 1$ and any $z \in \mathbb{C}^k$, we have

$$\varphi_n(z) = P_{\mathcal{M}}z + T^n z.$$

Since $\lim\limits_{n\to\infty} \|T^n\| = 0$, we see that $\{\varphi_n\}$ converges uniformly on $\mathbb{B}$ to the projection $P_{\mathcal{M}}$. Note that $\mathcal{M} \cap \overline{\mathbb{B}}$ is the set of all fixed points of φ on $\overline{\mathbb{B}}$. ∎

3. Spectral bounds for $W_{\psi,\varphi}$

Now that we have a sense of when self-maps of $\mathbb{B}$ converge uniformly under iteration, we wish to use those results to establish facts about C_φ when bounded on A^2_γ. Our results are somewhat analogous to the results on $\mathbb{D}$ in [1], but the complexity of function theory on $\mathbb{B}$ necessitates new proofs, and in some cases, our results are stronger.

Theorem 11. *Suppose φ is an analytic self-map of $\mathbb{B}$ such that C_φ is bounded on A^2_γ, and the iterates $\{\varphi_m\}$ converge uniformly to some function h on $\mathbb{B}$. Suppose that $\psi \in H^\infty(\mathbb{B})$ is continuous on $\mathbb{B} \cup \overline{h(\mathbb{B})}$. Then $\psi \circ \varphi_m$ converges uniformly to $\psi \circ h$ on $\mathbb{B}$ as well. Consequently, $W_{\psi\circ\varphi_m,\varphi} \to W_{\psi\circ h,\varphi}$ in operator norm and we obtain the spectral inclusion:*

$$\sigma(W_{\psi,\varphi}) \subseteq \sigma(W_{\psi\circ h,\varphi}).$$

Proof. Put $K = \overline{h(\mathbb{B})}$, which is a compact subset of $\overline{\mathbb{B}}$. Let $\epsilon > 0$ be given. The continuity of ψ on the compact set K implies the existence of $\delta > 0$ such that whenever $u \in \mathbb{B}$ and $v \in K$ with $|u - v| < \delta$, we have $|\psi(u) - \psi(v)| < \epsilon$. Now since φ_m converges uniformly to h on $\mathbb{B}$, there exists $N \geq 1$ such that for all $m \geq N$ and all $z \in \mathbb{B}$, $|\varphi_m(z) - h(z)| < \delta$. Setting $u = \varphi_m(z)$ (which belongs to $\mathbb{B}$) and $v = h(z)$ (which belongs to K), we obtain

$$\Big|\psi(\varphi_m(z)) - \psi(h(z))\Big| < \epsilon$$

for all $z \in \mathbb{B}$. Therefore, $\psi \circ \varphi_m$ converges uniformly to $\psi \circ h$ on $\mathbb{B}$. Since

$$\|W_{\psi\circ\varphi_m,\varphi} - W_{\psi\circ h,\varphi}\| \leq \|T_{\psi\circ\varphi_m} - T_{\psi\circ h}\| \cdot \|C_\varphi\| \leq \|\psi\circ\varphi_m - \psi\circ h\|_\infty \cdot \|C_\varphi\| \to 0,$$

we conclude that $W_{\psi\circ\varphi_m,\varphi} \to W_{\psi\circ h,\varphi}$ in operator norm.

To obtain the spectral inclusion, we follow the arguments as in the proof of [1, Theorem 8]. We provide here some details. Consider first $\lambda \in \sigma(W_{\psi,\varphi})\backslash\{0\}$. Then since $W_{\psi,\varphi} = T_\psi C_\varphi$,

$$\lambda \in \sigma(C_\varphi T_\psi) = T_{\psi\circ\varphi} C_\varphi = W_{\psi\circ\varphi,\varphi}.$$

It follows that $\lambda \in \sigma(W_{\psi\circ\varphi_m,\varphi})$ for all $m \geq 1$. Since $(W_{\psi\circ\varphi_m,\varphi} - \lambda I)$ converges to $(W_{\psi\circ h,\varphi} - \lambda I)$ in operator norm and each operator in the sequence is not invertible, we conclude that the limit is not invertible either. That is, $\lambda \in \sigma(W_{\psi\circ h,\varphi})$. We thus have showed

$$\sigma(W_{\psi,\varphi})\backslash\{0\} \subseteq \sigma(W_{\psi\circ h,\varphi}),$$

or equivalently,

$$\sigma(W_{\psi,\varphi}) \subseteq \sigma(W_{\psi\circ h,\varphi}) \cup \{0\}.$$

Now suppose that $0 \notin \sigma(W_{\psi\circ h,\varphi})$. We claim that $0 \notin \sigma(W_{\psi,\varphi})$ either. Since $W_{\psi\circ h,\varphi}$ is invertible, $T_{\psi\circ h}$ is surjective. This implies that $\psi \circ h$ is not identically zero and so $T_{\psi\circ h}$ is injective (which is a consequence of the identity theorem for analytic functions). As a consequence, $T_{\psi\circ h}$ is invertible and hence C_φ is invertible as well. It follows that φ is an automorphism of $\mathbb{B}$. By a theorem of H. Cartan (see [8, Theorem 4 (p. 78)]), either h is an automorphism of $\mathbb{B}$ or h is a constant function. Since $\{\varphi_m\}$ converges uniformly to h on $\mathbb{B}$ and each φ_m is an automorphism, h cannot be a constant (otherwise, for m sufficiently large, the range of all φ_m must be contained in a small open ball strictly contained in $\mathbb{B}$). Therefore, h is an automorphism. It now follows that

$$W_{\psi,\varphi} = C_{h^{-1}} T_{\psi\circ h} C_h C_\varphi$$

is invertible because all factors on the right hand-side are invertible. Therefore, $0 \notin \sigma(W_{\psi,\varphi})$.

The proof of the theorem is now complete. ■

Theorem 12. *Suppose that ψ is continuous on $\mathbb{B}\cup\overline{h(\mathbb{B})}$ and $T_{\psi\circ h}$ is bounded below on A^2_γ. Furthermore, assume C_φ is bounded on A^2_γ. Let η be a complex number. Suppose there exists a sequence of nonzero functions $\{g_m\}$ such that*

$$W_{\psi\circ h,\varphi}(g_m) = \eta_m g_m,$$

where each η_m is a bounded function and $\eta_m \longrightarrow \eta$ uniformly on $\mathbb{B}$. Then $\eta \in \sigma_{ap}(W_{\psi,\varphi})$.

As a consequence,

$$\overline{\sigma_p(W_{\psi\circ h,\varphi})} \subseteq \sigma_{ap}(W_{\psi,\varphi}).$$

Proof. For each $m \geq 1$, define

$$f_m = \prod_{n=0}^{m} \psi \circ \varphi_n.$$

We have

$$W_{\psi,\varphi} f_m = \psi \cdot (f_m \circ \varphi) = (\psi \circ \varphi_{m+1}) f_m. \tag{5}$$

We then compute

$$\begin{aligned} T_{\psi\circ h} W_{\psi,\varphi}(g_m f_m) &= \big(W_{\psi\circ h,\varphi} g_m\big) \cdot \big(W_{\psi,\varphi} f_m\big) \\ &= \eta_m g_m \cdot (\psi \circ \varphi_{m+1}) f_m \\ &= \eta_m \cdot (\psi \circ \varphi_{m+1}) \cdot (g_m f_m). \end{aligned}$$

Define $G_m = \frac{g_m f_m}{\|g_m f_m\|_2}$. Then $\|G_m\|_2 = 1$ and since $T_{\psi\circ h}$ is bounded below, there exists a positive constant $\delta > 0$ such that

$$\begin{aligned} \delta \Big\| \big(W_{\psi,\varphi} - \eta I\big) G_m \Big\|_2 &\leq \Big\| T_{\psi\circ h} \big(W_{\psi,\varphi} - \eta I\big) G_m \Big\|_2 \\ &= \Big\| \Big(\eta_m \cdot (\psi \circ \varphi_{m+1}) - \eta \psi \circ h\Big) G_m \Big\|_2 \\ &\leq \Big\| \eta_m \cdot (\psi \circ \varphi_{m+1}) - \eta \psi \circ h \Big\|_\infty \cdot \|G_m\|_2 \\ &= \Big\| \eta_m \cdot (\psi \circ \varphi_{m+1}) - \eta \psi \circ h \Big\|_\infty. \end{aligned}$$

Note that

$$\begin{aligned} \eta_m \cdot (\psi \circ \varphi_{m+1}) - \eta \cdot (\psi \circ h) = (\eta_m - \eta) \cdot \big(\psi \circ \varphi_{m+1} - \psi \circ h\big) + \eta \cdot (\psi \circ \varphi_{m+1} - \psi \circ h) \\ + (\eta_m - \eta) \psi \circ h, \end{aligned}$$

which converges uniformly to zero due to Theorem 11 and the fact that $\eta_m \longrightarrow \eta$ uniformly on $\mathbb{B}$. As a consequence,

$$\Big\| \big(W_{\psi,\varphi} - \eta I\big) G_m \Big\|_2 \longrightarrow 0,$$

which means that η belongs to $\sigma_{ap}(W_{\psi,\varphi})$. ∎

Combining our previous two results, we obtain the following, our main result of this section:

Corollary 13. *Suppose that $\psi \in H^\infty(\mathbb{B})$ is continuous on $\mathbb{B} \cup \overline{h(\mathbb{B})}$ and does not vanish on $\overline{h(\mathbb{B})}$, and C_φ is bounded on A^2_γ. Then*

$$\overline{\sigma_p(W_{\psi\circ h,\varphi})} \subseteq \sigma_{ap}(W_{\psi,\varphi}) \subseteq \sigma(W_{\psi,\varphi}) \subseteq \sigma(W_{\psi\circ h,\varphi}).$$

Example 14. *There are many Schur-Agler class maps that have an affine set of fixed points. An example is* $\varphi(z) = (z_1, \frac{z_2}{2})$ *on* $\mathbb{C}^2$*. Then* $\varphi_n(z) \longrightarrow h(z) = (z_1, 0)$ *uniformly on* $\mathbb{B}$*. The fixed point set* $\mathcal{A}$ *of* φ *on* $\mathbb{B}$ *is the disk*

$$\mathcal{A} = \big\{(z_1, 0) : |z_1| < 1\big\}.$$

It is well known that $\sigma(C_\varphi) = \overline{\sigma_p(C_\varphi)}$ *and*

$$\sigma_p(C_\varphi) = \{2^{-j} : j = 0, 1, 2, \dots\}.$$

Suppose $\psi \in H^\infty(\mathbb{B})$ *is continuous on* $\mathbb{B} \cup \overline{\mathcal{A}}$*. If* ψ *is equal to a constant, say* μ *on* $\mathcal{A}$*, then Corollary 13 gives*

$$\sigma_{ap}(W_{\psi,\varphi}) = \sigma(W_{\psi,\varphi}) = \sigma(\mu C_\varphi) = \big\{\mu\, 2^{-j} : j = 0, 1, 2 \dots\big\}.$$

4. Spectral radii for $W_{\psi,\varphi}$ on A^2_γ

In the previous section, we established a series of containments between subsets of the spectrum of $W_{\psi,\varphi}$ and $W_{\psi\circ h,\varphi}$. Without knowing more information about φ or ψ, we cannot change those containments into equality or establish anything about the spectral radii of $W_{\psi,\varphi}$. Fortunately, when φ is in the Schur-Agler class, we can use prior work of Jury to establish the spectral radii for $W_{\psi,\varphi}$.

Lemma 15. *Let* φ *be in the Schur-Agler class. For any* $\psi \in H^\infty(\mathbb{B})$ *that is continuous on* $\mathbb{B} \cup \overline{h(\mathbb{B})}$*, we have*

$$r(W_{\psi\circ h,\varphi}) = \|\psi \circ h\|_\infty \cdot r(C_\varphi).$$

Combining with Theorem 11, we conclude that

$$r(W_{\psi,\varphi}) \le r(W_{\psi\circ h,\varphi}) \le \|\psi \circ h\|_\infty \cdot r(C_\varphi).$$

Proof. If h is a constant function, say $h(z) = a$ for all $z \in \mathbb{B}$. Then

$$W_{\psi\circ h,\varphi} = \psi(a) C_\varphi.$$

As a result,

$$r(W_{\psi\circ h,\varphi}) = |\psi(a)| \cdot r(C_\varphi) = \|\psi \circ h\|_\infty \cdot r(C_\varphi)$$

since $\psi \circ h$ is identically equal to $\psi(a)$.

Now assume that h is not a constant function. In this case, φ has an interior fixed point. Then $r(C_\varphi) = 1$ (see [5, Theorem 9]). Note that $h \circ \varphi = h$ so for any integer $m \ge 1$, $\big(W_{\psi\circ h,\varphi}\big)^m = W_{(\psi\circ h)^m,\varphi_m}$. For any $a \in \mathbb{B}$, we then have

$$\big(W^m_{\psi\circ h,\varphi}\big)^* K_a = \big(W_{(\psi\circ h)^m,\varphi_m}\big)^* K_a = [\overline{\psi(h(a))}]^m K_{\varphi_m(a)},$$

where K_a is the reproducing kernel at a, given by $K_a(z) = (1 - \langle z, a\rangle)^{-\beta}$ with $\beta = k + 1 + \gamma$. It follows that

$$\begin{aligned}
|\psi(h(a))|^m &= \frac{\|\big(W^m_{\psi\circ h,\varphi}\big)^* K_a\|}{\|K_{\varphi_m(a)}\|} \\
&\le \frac{\|(W^m_{\psi\circ h,\varphi})^*\| \cdot \|K_a\|}{\|K_{\varphi_m(a)}\|} \\
&= \|\big(W_{\psi\circ h,\varphi}\big)^m\| \Big(\frac{(1 - |\varphi_m(a)|^2)}{1 - |a|^2}\Big)^\beta,
\end{aligned}$$

and hence,

$$|\psi(h(a))| \le \|\big(W_{\psi\circ h,\varphi}\big)^m\|^{1/m}\Big(\frac{(1-|\varphi_m(a)|^2)}{1-|a|^2}\Big)^{\beta/m}.$$

Note that $\varphi_m(a) \to h(a)$, which satisfies $|h(a)| < 1$ since h is not a constant function. Therefore,

$$\lim_{m\to\infty}\Big(\frac{(1-|\varphi_m(a)|^2)}{1-|a|^2}\Big)^{\beta/m} = 1.$$

It follows that

$$|\psi(h(a))| \le \lim_{m\to\infty}\|\big(W_{\psi\circ h,\varphi}\big)^m\|^{1/m} = r(C_{\psi\circ h}C_\varphi).$$

Since $a \in \mathbb{B}$ was arbitrary, we conclude that

$$\|\psi\circ h\|_\infty \le r(W_{\psi\circ h,\varphi})$$

and hence

$$\|\psi\circ h\|_\infty \cdot r(C_\varphi) \le r(W_{\psi\circ h,\varphi})$$

because $r(C_\varphi) = 1$. On the other hand, since $\big(W_{\psi\circ h,\varphi}\big)^m = W_{(\psi\circ h)^m,\varphi_m}$, we have the estimate

$$\|\big(W_{\psi\circ h,\varphi}\big)^m\| \le \|\psi\circ h\|_\infty^m \, \|C_{\varphi_m}\|.$$

Taking mth root and letting $m \to \infty$, we conclude that

$$r(W_{\psi\circ h,\varphi}) \le \|\psi\circ h\|_\infty \cdot r(C_\varphi).$$

Consequently,

$$r(W_{\psi\circ h,\varphi}) = \|\psi\circ h\|_\infty \cdot r(C_\varphi).$$ ■

Theorem 16. *Suppose φ is an analytic self-map of $\mathbb{B}$ in the Schur-Agler class such that $\{\varphi_m\}$ converges uniformly to h on $\mathbb{B}$. Let $\psi \in H^\infty(\mathbb{B})$ be continuous on $\mathbb{B}\cup\overline{h(\mathbb{B})}$ and constant on $h(\mathbb{B})$, say, $\psi(h(z)) = \mu$ for all $z \in \mathbb{B}$. Consider the bounded operator $W_{\psi,\varphi}$ on A^2_γ.*

1. *(1) If φ is elliptic or parabolic, then $r(W_{\psi,\varphi}) = |\mu|$.*
2. *(2) If φ is parabolic with dilatation coefficient α, then $r(W_{\psi,\varphi}) = |\mu|\alpha^{-\beta/2}$, where $\beta = k+1+\gamma$.*

Proof. Corollary 13 shows that

$$\mu\,\overline{\sigma_p(C_\varphi)} \subseteq \sigma_{ap}(W_{\psi,\varphi}) \subseteq \sigma(W_{\psi,\varphi}) \subseteq \mu\,\sigma(C_\varphi). \tag{6}$$

Consider first the case φ is elliptic or parabolic. Since $1 \in \sigma_p(C_\varphi)$, we conclude that $r(W_{\psi,\varphi}) \ge |\mu|$. On the other hand, the last containment in (6) gives

$$r(W_{\psi,\varphi}) \le |\mu| \cdot r(C_\varphi) = |\mu|$$

because $r(C_\varphi) = 1$ by [5, Theorem 10]. It then follows that $r(W_{\psi,\varphi}) = |\mu|$.

Now assume that φ is parabolic with dilatation coefficient α. By [6, Theorem 3.2], $r(C_\varphi) = \alpha^{-\beta/2}$. By (6), we have $r(W_{\psi,\varphi}) \le r(\mu C_\varphi) = |\mu|\alpha^{-\beta/2}$. Also by [6, Theorem 3.2], we know that every value in the annulus $\alpha^{\beta/2} < |\lambda| < \alpha^{-\beta/2}$ is an eigenvalue for C_φ. Then we know by (6) again that $r(W_{\psi,\varphi}) \ge |\mu|\alpha^{-\beta/2}$, and therefore we have $r(W_{\psi,\varphi}) = |\mu|\alpha^{-\beta/2}$. ■

5. Further Questions

We end with a set of natural questions that follow from our work above.

- Can we characterize exactly when iterates of self-maps of $\mathbb{B}$ converge uniformly to a fixed point or an idempotent? To our knowledge, this has not yet been completely characterized on $\mathbb{D}$ yet; we suspect that the problem on $\mathbb{D}$ is significantly easier.
- When φ is in the Schur-Agler class, can we discover the spectrum of C_φ on A^2_γ, instead of just the spectral radius, and when doing so, will Corollary 13 immediately provide us with the spectrum of $W_{\psi,\varphi}$?
- Under what other conditions, beyond φ belonging to the Schur-Agler class, can Corollary 13 establish $\sigma(W_{\psi,\varphi})$ exactly?

Statements and Declarations

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Butler University, Indianapolis, Indiana 46208
Email address: skaschne@butler.edu, cmakdad@butler.edu, brempfer@butler.edu, dlwinter@butler.edu

The University of Toledo, Toledo, Ohio 43606
Email address: trieu.le2@utoledo.edu

Taylor University, Upland, Indiana 46989
Email address: theycallmedt@gmail.com