THE STRUCTURE OF m-ISOMETRIC WEIGHTED SHIFT OPERATORS

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Abstract. We obtain simple characterizations of unilateral and bilateral weighted shift operators that are m-isometric. We show that any such operator is a Hadamard product of 2-isometries and 3-isometries. We also study weighted shift operators whose powers are m-isometric.

1. Introduction

Throughout the paper, H denotes a separable infinite dimensional complex Hilbert space. Let $m \geq 1$ be an integer. A bounded linear operator T on H is said to be *m-isometric* if it satisfies the operator equation

$$\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}T^{*k}T^{k} = 0, \tag{1.1}$$

where T^* denotes the adjoint of T and $T^{*0} = T^0 = I$, the identity operator on H. It is immediate that T is m-isometric if and only if

$$\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}\|T^{k}x\|^2 = 0 \tag{1.2}$$

for all $x \in H$. It is well known and not difficult to check that any m-isometric operator is k-isometric for any $k \geq m$. We say that T is strictly m-isometric (or equivalently, T is a strict m-isometry) if T is m-isometric but it is not $(m-1)$-isometric. Clearly, any 1-isometric operator is isometric. This notion of m-isometries was introduced by Agler [1] back in the early nineties in connection with the study of disconjugacy of Toeplitz operators. The general theory of m-isometric operators was later investigated in great details by Agler and Stankus in a series of three papers [2–4].

In this paper, we are investigating unilateral as well as bilateral weighted shift operators that are m-isometric. Examples of such unilateral weighted shifts were given by Athavale [5] in his study of multiplication operators on certain reproducing kernel Hilbert spaces over the unit disk. In [9], Botelho and Jamison provided other examples of strictly 2-isometric and 3-isometric unilateral weighted shifts. Recently, Bermudéz et al. [8] obtained a characterization for a unilateral weighted shift to be an m-isometry. However, their

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characterization appears difficult to apply. In fact, combinatorial identities are often involved in checking whether a given unilateral weighted shift satisfies their criterion to be an m-isometry. See [8, Corollary 3.8]. Here, we offer a more simplified characterization of m-isometric weighted shifts. Our approach works not only for unilateral shifts but also for bilateral shifts. Even though our characterization is equivalent to the characterization given in [8], it is more transparent and useful. We shall see how our result quickly recovers several known examples. We further obtain an interesting structural result which says that for $m \geq 2$, any strictly m-isometric weighted shift is the Hadamard product (also known as the Schur product) of strictly 2-isometric or 3-isometric weighted shifts. We shall also study weighted shifts whose powers are m-isometric. Similar results will be proven for weighted bilateral shifts. Our characterization of m-isometric weighted bilateral shifts offers several examples which include the examples considered in a recent paper [10].

The paper is organized as follows. In Section 2, we provide a detailed study of unilateral weighted shifts which are m-isometric. The main result in this section gives a complete characterization of such operators. Several examples will be given. In Section 3, we discuss Hadamard products of m-isometric weighted shifts. We prove a factorization theorem for these operators. We then study weighted shifts whose powers are m-isometric in Section 4. Several examples are discussed. Finally, in Section 5, we investigate bilateral weighted shifts. A characterization and a factorization theorem for m-isometric bilateral weighted shifts are given.

2. m-isometric unilateral weighted shift operators

Fix an orthonormal basis $\{e_n\}_{n\geq 1}$ of H. For a sequence of complex numbers $\{w_n\}_{n\geq 1}$, the associated weighted unilateral shift operator S is a linear operator on H with

$$Se_n = w_n e_{n+1} \quad \text{for all } n \geq 1.$$

It is well known and is not difficult to see that S is a bounded operator if and only if the weight sequence $\{w_n\}_{n\geq 1}$ is bounded. We shall always assume that S is a bounded weighted shift operator. The reader is referred to [14] for an excellent source on the study of these operators. In this paper, we only focus our attention on weighted shifts that are m-isometric.

Since $Se_n = w_n e_{n+1}$ for all $n \geq 1$, we see that $S^k e_n = (\prod_{\ell=n}^{k+n-1} w_\ell) e_{n+k}$ for $k \geq 1$. Consequently,

$$S^{*k} e_n = \begin{cases} 0 & \text{if } n \leq k \\ (\prod_{\ell=n-k}^{n-1} \overline{w}_\ell) e_{n-k} & \text{if } n \geq k+1. \end{cases}$$

Therefore, $S^{*k}S^k$ is a diagonal operator with respect to the orthonormal basis $\{e_n\}_{n=1}^{\infty}$ and

$$S^{*k}S^k e_n = \Big(\prod_{\ell=n}^{k+n-1} |w_\ell|^2\Big) e_n.$$

Now assume that S is an m-isometry. That is, S satisfies equation (1.1), and equivalently, equation (1.2). We collect here two well-known facts about the weight sequence of S. See [8, Propositions 3.1 and 3.2] and also [9, Equation (4)].

(a) From (1.2), it follows that any m-isometry is bounded below, hence, injective. Consequently, $w_n \neq 0$ for all $n \geq 1$.
(b) S is m-isometric if and only if for any integer $n \geq 1$,

$$(-1)^m + \sum_{k=1}^{m} (-1)^{m-k} \binom{m}{k} \Big(\prod_{\ell=n}^{k+n-1} |w_\ell|^2\Big) = 0. \tag{2.1}$$

By studying the infinite system of equations (2.1), Bermúdez et al. [8, Theorem 3.4] gives a characterization of the weight sequence $\{w_n\}_{n\geq 1}$. Here, using a different approach, namely, the theory of Difference Equations, we obtain an equivalent but more transparent characterization. As a consequence, we derive interesting properties of m-isometric weighted shifts which have not been discovered before. The technique of Difference Equations has been used (but for a different purpose) in the study of m-isometries in [6,7].

Theorem 1. *Let S be a unilateral weighted shift with weight sequence $\{w_n\}_{n\geq 1}$. Then the following statements are equivalent.*

(a) *S is an m-isometry.*
(b) *There exists a polynomial p of degree at most $m-1$ with real coefficients such that for all integers $n \geq 1$, we have $p(n) > 0$ and*

$$|w_n|^2 = \frac{p(n+1)}{p(n)}. \tag{2.2}$$

The polynomial p may be taken to be monic.

Proof. We define a new sequence of numbers $\{u_n\}_{n\geq 1}$ as follows. Set $u_1 = 1$ and $u_n := \prod_{j=1}^{n-1} |w_j|^2$ if $n \geq 2$. Since $w_j \neq 0$ for any j as we have remarked above, all u_n are positive. We have $|w_n|^2 = u_{n+1}/u_n$ and more generally,

$$\prod_{\ell=n}^{k+n-1} |w_\ell|^2 = \frac{u_{k+n}}{u_n},$$

for all integers $n \geq 1$ and $k \geq 1$.

From (2.1), we see that S is an m-isometry if and only if the sequence $\{u_n\}_{n\geq 1}$ is a solution to the difference equation

$$(-1)^m + \sum_{k=1}^{m} (-1)^{m-k} \binom{m}{k} \frac{u_{k+n}}{u_n} = 0 \quad \text{for all } n \geq 1.$$

This equation is equivalent to

$$\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}u_{k+n}=0 \quad \text{for all } n\geq 1. \tag{2.3}$$

The characteristic polynomial of this linear difference equation is

$$f(\lambda)=\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}\lambda^{k}=(\lambda-1)^{m}.$$

Since $\lambda=1$ is the only root of f with multiplicity m, the theory of Linear Difference Equations (see, for example, [12, Section 2.3]) shows that $\{u_n\}_{n\geq 1}$ is a solution of (2.3) if and only if u_n is a polynomial in n of degree at most $m-1$.

The argument we have so far shows that S is an m-isometry if and only if there is a polynomial q of degree at most $m-1$ with real coefficients such that $u_n=q(n)$ for all $n\geq 1$.

We now prove the implication (a) $\Longrightarrow$ (b). Suppose S is an m-isometry. Consider the polynomial q given in the preceding paragraph. Since q is positive at all positive integers, the leading coefficient α of q must be positive. Put $p=q/\alpha$. Then p is a monic polynomial and for all $n\geq 1$, we have $p(n)=q(n)/\alpha>0$ and

$$w_n=\frac{u_{n+1}}{u_n}=\frac{q(n+1)}{q(n)}=\frac{p(n+1)}{p(n)}.$$

For the implication (b) $\Longrightarrow$ (a), suppose there is a polynomial p of degree at most $m-1$ with real coefficients such that $p(n)>0$ and $|w_n|^2=p(n+1)/p(n)$ for all $n\geq 1$. Set $q(n)=p(n)/p(1)$. Then we have $u_1=1=q(1)$ and for $n\geq 2$,

$$u_n=\prod_{j=1}^{n-1}|w_j|^2=\prod_{j=1}^{n-1}\frac{p(j+1)}{p(j)}=\frac{p(n)}{p(1)}=q(n).$$

Since q is of degree at most $m-1$, we conclude that $\{u_n\}_{n\geq 1}$ solves the difference equation (2.3). Consequently, S is an m-isometry. □

Remark 2. The *monic* polynomial p satisfying (b) in Theorem 1, if exists, is unique. Indeed, suppose $\tilde{p}$ is another monic polynomial such that $|w_n|^2=\tilde{p}(n+1)/\tilde{p}(n)$ and $\tilde{p}(n)>0$ for all integers $n\geq 1$. Then for any integer $k\geq 2$,

$$\frac{p(k)}{p(1)}=\prod_{\ell=1}^{k-1}|w_\ell|^2=\frac{\tilde{p}(k)}{\tilde{p}(1)}.$$

Since the polynomials $p/p(1)$ and $\tilde{p}/\tilde{p}(1)$ agree at all integer values $k\geq 2$, they must be the same polynomial. Therefore, $p/p(1)=\tilde{p}/\tilde{p}(1)$, which implies that $\tilde{p}=(\tilde{p}(1)/p(1))p$. Because both p and $\tilde{p}$ are monic, it follows that $\tilde{p}(1)/p(1)=1$ and hence, $\tilde{p}=p$.

As an immediate corollary to Theorem 1, we characterize unilateral weighted shifts that are strictly m-isometric.

Corollary 3. *A unilateral weighted shift S is strictly m-isometric if and only if there exists a polynomial p of degree $m-1$ that satisfies condition (b) in Theorem 1.*

Proof. We consider first the "only if" direction. Suppose S is a strict m-isometry. Then the polynomial p in Theorem 1 has degree at most $m-1$. If the degree of p were strictly smaller than $m-1$, then another application of Theorem 1 shows that S would be $(m-1)$-isometric, which is a contradiction. Therefore, the degree of p must be exactly $m-1$.

Now consider the "if" direction. Suppose $|w_n|^2 = p(n+1)/p(n)$ for all $n \geq 1$, where p is a polynomial of degree $m-1$. We know from Theorem 1 that S is m-isometric. By Remark 2, there does *not* exist a polynomial q with degree at most $m-2$ such that $|w_n|^2 = q(n+1)/q(n)$ for all $n \geq 1$. Theorem 1 then implies that S is not an $(m-1)$-isometry. Therefore, S is strictly m-isometric. $\square$

We now apply Corollary 3 to investigate several examples.

Example 4. A unilateral weighted shift S is a strict 2-isometry if and only if there is a monic polynomial p of degree 1 such that $p(n) > 0$ and $|w_n|^2 = p(n+1)/p(n)$ for all $n \geq 1$. Write $p(n) = n - b$ for some real number b. The positivity of p at the positive integers forces b to be smaller than 1.

We conclude that S is a strict 2-isometry if and only if there exists a real number $b < 1$ such that

$$|w_n| = \sqrt{\frac{n+1-b}{n-b}} \qquad \text{for all integers } n \geq 1.$$

Choosing $b = 0$, we recover the well-known fact [13] that the Dirichlet shift is a strict 2-isometry.

Example 5. A unilateral weighted shift S is a strict 3-isometry if and only if there is a monic polynomial p of degree 2 such that $p(n) > 0$ and $|w_n|^2 = p(n+1)/p(n)$ for all $n \geq 1$. Write $p(x) = (x-\alpha)(x-\beta)$ for some complex numbers α and β. Since p is positive at all positive integers, one of the following three cases must occur:

(1) Both α and β belong to $\mathbb{C}\backslash\mathbb{R}$. An example is $p(x) = x^2 - 5x + 7$. In this case,

$$|w_n|^2 = \frac{p(n+1)}{p(n)} = \frac{n^2-3n+3}{n^2-5n+7} \qquad \text{for all } n \geq 1.$$

This example appeared in [9, Section 2.1].

(2) There exists an integer $n_0 \geq 1$ such that both α and β belong to the open interval (n_0, n_0+1).

(3) Both α and β belong to the interval $(-\infty, 1)$.

Example 6. For each integer $m \geq 1$, consider the unilateral weighted shift S with the weight sequence given by

$$w_n = \sqrt{\frac{n+m}{n}} \quad \text{for all } n \geq 1.$$

This operator was considered in [5, Proposition 8] and [8, Corollary 3.8], where it was verified to be a strict $(m+1)$-isometry. We provide here another proof of this fact. Put $p(x) = (x+m-1)\cdots x$. Then p is a monic polynomial of degree m and for all integers $n \geq 1$, we have $p(n) > 0$ and

$$\sqrt{\frac{p(n+1)}{p(n)}} = \sqrt{\frac{(n+m)\cdots(n+1)}{(n+m-1)\cdots n}} = \sqrt{\frac{n+m}{n}} = w_n.$$

By Corollary 3, S is strictly $(m+1)$-isometric.

Theorem 1 shows that in order for S to be m-isometric, the values $|w_n|^2$ must be a rational function of n and $\lim_{n\to\infty} |w_n|^2 = 1$. This immediately raises the following question.

Question. Suppose S is a unilateral weighted shift with the weight sequence $\{w_n\}_{n\geq 1}$. Suppose there are two polynomials f and g with real coefficients such that $|w_n|^2 = f(n)/g(n)$ and that $\lim_{n\to\infty} f(n)/g(n) = 1$. What conditions must f and g satisfy to ensure that S is an m-isometry for some integer $m \geq 2$?

Example 6 shows that the relation between f and g is not at all obvious. While it is possible to obtain a criterion that involves the roots of f and g, such a criterion may not be useful or practical. On the other hand, we do not know if it is possible to find a condition that involves only the coefficients of f and g. This may have an interesting answer.

In the rest of the section, we investigate m-isometric weighted shift operators whose weight sequence starts with a given finite set of values. More specifically, let $r \geq 1$ be an integer and let $a_1, \ldots, a_r$ be nonzero complex numbers. We are interested in the question: does there exist an m-isometric unilateral weighted shift S such that $Se_k = a_k e_{k+1}$ for all $1 \leq k \leq r$? By Theorem 1, the answer to this question hinges on the existence of a polynomial p such that $p(n) > 0$ for all $n \geq 1$ and $|a_k|^2 = p(k+1)/p(k)$ for $1 \leq k \leq r$. The following result shows the existence of such a polynomial.

Proposition 7. *Let $r \geq 1$ be an integer and let $a_1, \ldots, a_r$ be nonzero complex numbers. For any $m \geq r+2$, there exists a strictly m-isometric unilateral weighted shift operator whose weight sequence starts with $a_1, \ldots, a_r$.*

Proof. By Lagrange interpolation, there exists a polynomial f of degree at most r such that $f(1) = 1$ and

$$f(k) = |a_1|^2 \cdots |a_{k-1}|^2 \quad \text{for } 2 \leq k \leq r+1.$$

Let $m \geq r+2$. We shall look for a polynomial p with degree $m-1$ in the form
$$p(x) = x^{m-r-2}(x-1)\cdots(x-r-1) + \alpha\, f(x)$$
such that $p(n) > 0$ for all integers $n \geq 1$. Here α is a positive number that we need to determine. Note that $p(k) = \alpha f(k) > 0$ for all $1 \leq k \leq r+1$ so we only need to find α such that $p(n) > 0$ for $n \geq r+2$. This is equivalent to
$$\frac{1}{\alpha} > \sup\Big\{\frac{-f(x)}{x^{m-r-2}(x-1)\cdots(x-r-1)} \;:\; x \geq r+2\Big\}.$$
Since the rational function on the right hand is continuous on $[r+2, \infty)$ and its limit at infinity is zero, the above supremum is finite. Consequently, there exists such an α. Note that p is a monic polynomial of degree $m-1$ and for $1 \leq k \leq r$,
$$|a_k|^2 = \frac{f(k+1)}{f(k)} = \frac{\alpha f(k+1)}{\alpha f(k)} = \frac{p(k+1)}{p(k)}.$$
Let S be the unilateral weighted shift operator whose weight sequence $\{w_n\}_{n\geq 1}$ is given by $w_n = a_n$ for $1 \leq n \leq r$ and
$$w_n = \sqrt{\frac{p(n+1)}{p(n)}} \quad \text{for } n \geq r+1.$$
Since p is a polynomial of degree $m-1$ and $|w_n|^2 = p(n+1)/p(n)$ for all $n \geq 1$, Corollary 3 shows that S is strictly m-isometric. $\square$

Remark 8. The condition $m \geq r+2$ in the above proposition is necessary. In fact, with an appropriate choice of $a_1, \ldots, a_r$, there does not exist an $(r+1)$-isometric unilateral weighted shift operator whose weight sequence starts with $a_1, \ldots, a_r$. For example, set $s = 1$ and take $|a_1| < 1$. Example 4 shows that there does not exist a 2-isometric weighted shift operator S with $Se_1 = a_1 e_2$ since $|a_1| < 1$.

3. The semigroup of m-isometric unilateral weighted shifts

In this section, we investigate the structure of m-isometric weighted shifts. Let us define $\mathcal{W}$ to be the set of all unilateral weighted shifts that are m-isometric for some integer $m \geq 1$. We shall see that $\mathcal{W}$ turns out to be a semigroup with an identity. The multiplication on W is the Hadamard product of operators. We shall also show that any element in $\mathcal{W}$ can be factored as a product of simpler factors.

Let us first recall the Hadamard product, which is also known as the Schur product. Suppose A and B are bounded operators on H. Let (a_{jk}) and (b_{jk}), respectively, be the matrix representations of A and B with respect to the orthonormal basis $\{e_n\}_{n=1}^{\infty}$. Then the Hadamard product of A and B, denoted by $A \odot B$, is an operator on H with matrix (c_{jk}), where $c_{jk} = a_{jk} b_{jk}$ for all integers $j, k \geq 1$. It is well known that $A \odot B$ is a bounded operator on H.

It is clear that the Hadamard product of any two unilateral weighted shifts is a unilateral weighted shift. Corollary 3 tells us more.

Proposition 9. *Let S and T be unilateral weighted shift operators such that S is strictly k-isometric and T is strictly ℓ-isometric. Then $S \odot T$ is strictly $(k+\ell-1)$-isometric. Consequently, the following statements hold.*

(i) *The pair $(\mathcal{W}, \odot)$ is a commutative semigroup with identity U, the unweighted unilateral shift.*
(ii) *If $S \odot T = U$, then both S and T are isometric operators. This shows that invertible elements in $(W, \odot)$ are exactly the isometries.*

Proof. Let $\{s_n\}_{n\geq 1}$ and $\{t_n\}_{n\geq 1}$ be the weight sequences of S and T, respectively. Then $S \odot T$ is a unilateral weighted shift with weights $w_n = s_n t_n$ for $n \geq 1$.

Since S is k-isometric, Corollary 3 shows the existence of a polynomial p of degree $k-1$ with real coefficients such that $p(n) > 0$ and $|s_n|^2 = p(n+1)/p(n)$ for all $n \geq 1$. Similarly, there is a polynomial q of degree $\ell - 1$ such that $q(n) > 0$ and $|t_n|^2 = q(n+1)/q(n)$ for all $n \geq 1$. Put $h = p \cdot q$. Then h is a polynomial with degree $k + \ell - 2$ and for any $n \geq 1$,

$$h(n) = p(n)q(n) > 0, \quad \text{and} \quad |w_n|^2 = |s_n|^2 |t_n|^2 = \frac{h(n+1)}{h(n)}.$$

By Corollary 3 again, $S \odot T$ is strictly $(k + \ell - 1)$-isometric. Therefore, $\mathcal{W}$ is closed under $\odot$ and hence, $(\mathcal{W}, \odot)$ is a semigroup. It is clear that the unweighted unilateral shift U is the identity of this semigroup.

If $S \odot T = U$, then since U is isometric, we have $k + \ell - 1 = 1$. This forces $k = \ell = 1$, which means that both S and T are isometric operators. The proof of the proposition is now completed. $\square$

In general, the operator $A \odot B$ is usually not m-isometric when A is an arbitrary k-isometry and B is an arbitrary ℓ-isometry. An obvious example is $A = I$, the identity operator, and B any ℓ-isometry whose matrix contains at least one zero on its main diagonal. Then $A \odot B$ is a diagonal operator with at least one zero on its diagonal. Since $A \odot B$ is not injective, it cannot be m-isometric for any $m \geq 1$. This shows that the property in Proposition 9 is quite special for m-isometric unilateral weighted shifts. On the other hand, we would like to explain here that a more general approach can be used to prove Proposition 9, without the need of an explicit characterization. Recall that the tensor product space $H \bar{\otimes} H$ admits the orthonormal basis $\big\{e_j \otimes e_k : j, k \geq 1\big\}$. The "diagonal subspace" $\widetilde{H}$ is a subspace of $H \bar{\otimes} H$ with the orthonormal basis $\{e_j \otimes e_j : j \geq 1\}$. It is well known that $A \odot B$ is unitarily equivalent to the *compression* of the tensor product $A \otimes B$ on $\widetilde{H}$. Duggal [11] shows that if A is k-isometric and B is ℓ-isometric, then $A \otimes B$ is m-isometric on $H \bar{\otimes} H$ with $m = k + \ell - 1$. Since the compression of an m-isometric operator on a subspace may not be m-isometric, the operator $A \odot B$ may not be m-isometric as we have seen above. However, if both A

and B are unilateral weighted shifts, then $\widetilde{H}$ turns out to be an invariant subspace of $A \otimes B$. It then follows that $A \odot B$, being unitarily equivalent to the *restriction* of $A \otimes B$ on an invariant subspace, is m-isometric as well.

As another interesting application of Theorem 1, we show that any element in the semigroup $(\mathcal{W}, \odot)$ can be written as a product of elements that are 2-isometric or 3-isometric.

Recall that $\mathbb{Z}^+$ denotes the set of all positive integers. We need the following elementary facts about polynomials with real coefficients.

Lemma 10. *Let $p \in \mathbb{R}[x]$ be a monic polynomial such that $p(n) > 0$ for all $n \in \mathbb{Z}^+$. Then the following statements hold.*

(1) Given any integer $n \in \mathbb{Z}^+$, the polynomial p has an even number of roots (counted with multiplicity) in the interval $(n, n+1)$.
(2) There are linear and quadratic monic polynomials $p_1, \ldots, p_\nu$ in $\mathbb{R}[x]$ which assumes positive values on $\mathbb{Z}^+$ such that $p = p_1 \cdots p_\nu$.

Proof. (1) Let n be a positive integer such that p has at least a root in the interval $(n, n+1)$. Let $\alpha_1, \ldots, \alpha_\ell$ be these roots, listed with multiplicity. Write $p(x) = (x - \alpha_1) \cdots (x - \alpha_\ell) r(x)$, where the polynomial $r(x)$ has no roots in $(n, n+1)$. Since $r(n+1)$ and $r(n)$ have the same sign, we see that $\operatorname{sgn}(p(n+1)) = (-1)^\ell \operatorname{sgn}(p(n))$. But $p(n+1)$ and $p(n)$ are both positive, so ℓ must be even.

(2) We know that p can be factored as a product of monic linear and irreducible quadratic (not necessarily distinct) polynomials in $\mathbb{R}[x]$. The proof of the statement is completed once we notice the following facts. Firstly, any monic irreducible quadratic factor is positive over $\mathbb{R}$, hence over $\mathbb{Z}^+$. Secondly, any linear factor of the form $q(x) = x - b$ with $b < 1$ has positive values over $[1, \infty)$, hence over $\mathbb{Z}^+$ as well. Lastly, by (1), the remaining linear factors can be grouped into pairs of the form $(x - \alpha)(x - \beta)$, where α and β lie between two consecutive positive integers. Any such quadratic polynomial also assumes positive values on $\mathbb{Z}^+$. $\square$

We are now in a position to prove a factorization theorem for non-isometric elements of $(\mathcal{W}, \odot)$.

Theorem 11. *Any non-isometric element in $(\mathcal{W}, \odot)$ is a $\odot$-product of elements that are either strictly 2-isometric or strictly 3-isometric.*

Proof. Let S be a non-isometric element in $(\mathcal{W}, \odot)$. Assume that S is strictly m isometric with $m \geq 2$. By Theorem 1, there is a monic polynomial p such that $p(n) > 0$ and $|w_n|^2 = p(n+1)/p(n)$ for all integers $n \geq 1$. Using Lemma 10, we obtain a factorization $p = p_1 \cdots p_\nu$, where each polynomial p_j is either linear or quadratic. Now for each integer $n \geq 1$, set $\gamma_n = w_n/|w_n|$ and write

$$w_n = \gamma_n |w_n| = \gamma_n \sqrt{\frac{p_1(n+1)}{p_1(n)}} \cdots \sqrt{\frac{p_\nu(n+1)}{p_\nu(n)}}.$$

Let S_1 be the unilateral weighted shift operator whose weight sequence is $\{\gamma_n\sqrt{p_1(n+1)/p_1(n)}\}_{n\geq 1}$. For $2\leq j\leq \nu$, let S_j be the unilateral weighted shift operator whose weight sequence is $\{\sqrt{p_j(n+1)/p_j(n)}\}_{n\geq 1}$. We then have $S=S_1\odot\cdots\odot S_\nu$ and each S_j is either strictly 2-isometric or strictly 3-isometric by Corollary 3. This completes the proof of the theorem. □

Remark 12. It should be noted that any strictly 2-isometric element in $(\mathcal{W},\odot)$ cannot be trivially written as a product of non-isometric elements. On the other hand, some strictly 3-isometric elements may be written as a product of strict 2-isometries. These elements arise from Case (3) in Example 5.

We close this section with a corollary to Theorem 11.

Corollary 13. *Let S be a unilateral weighted shift operator. Then S is m-isometric for some $m\geq 2$ if and only if it can be written as the Hadamard product of unilateral weighted shift operators each of which is strictly 2-isometric or 3-isometric.*

4. Unilateral shifts whose powers are m-isometric

Let $\alpha\geq 2$ be a positive integer. It is well known that if A is an m-isometry then A^α is an m-isometry as well. The converse, on the other hand, does not hold (see [7, Examples 3.3 and 3.5] and also Examples 15 and 16 that we shall discuss below).

In this section, we would like to characterize the weights of a given unilateral weighted shift S such that S^α is m-isometric. Our approach relies on the characterization of m-isometric unilateral weighted shifts obtained in Section 2. Let S be a unilateral weighted shift with weight sequence $\{w_n\}_{n\geq 1}$. Recall that $\{e_n\}_{n\geq 1}$ is an orthonormal basis of H such that $Se_n=w_ne_{n+1}$ for all $n\geq 1$. Then S^α is a shift of multiplicity α, that is, for all integers $n\geq 1$,

$$S^\alpha e_n = u_n e_{n+\alpha},$$

where $u_n = w_n\cdots w_{n+\alpha-1}$.

For each $1\leq r\leq\alpha$, let $\mathcal{X}_r$ denote the closed subspace spanned by $\{e_r, e_{r+\alpha}, e_{r+2\alpha},\ldots\}$. Then $\mathcal{X}_r$ is a reducing subspace of S^α and S^α is unitarily equivalent to the direct sum $T_1\oplus\cdots\oplus T_\alpha$, where each $T_r = S^\alpha|_{\mathcal{X}_r}$ is a unilateral weighted shift with weight sequence $\{u_{\ell\alpha+r}\}_{\ell\geq 0}$. Consequently, S^α is m-isometric on H if and only if T_r is m-isometric on $\mathcal{X}_r$ for all $1\leq r\leq\alpha$. By Theorem 1, this is equivalent to the existence of polynomials $f_1,\ldots,f_\alpha$ of degree at most $m-1$ such that $f_r(\ell)>0$ and

$$|u_{\ell\alpha+r}|^2 = \frac{f_r(\ell+1)}{f_r(\ell)} \text{ for all } \ell\geq 0 \text{ and } 1\leq r\leq\alpha. \tag{4.1}$$

Note that S^α is a strict m-isometry if and only if one of the polynomials $f_1,\ldots,f_\alpha$ has degree exactly $m-1$. With the above characterization, we would like to recover a formula for determining the weights $\{w_n\}_{n\geq 1}$ of S. The following theorem is our main result in this section.

Theorem 14. *Let S be a unilateral weighted shift with weight sequence $\{w_n\}_{n\geq 1}$. Then S^α is m-isometric if and only if there exists a function $g : \mathbb{Z}_+ \to \mathbb{R}_{>0}$ such that the following conditions hold*

(a) For each $1 \leq r \leq \alpha$, the function $\ell \mapsto g(\ell\alpha + r)$ is a polynomial of degree at most $m-1$ in ℓ.

(b) We have $|w_n|^2 = \dfrac{g(n+1)}{g(n)}$ for all integer $n \geq 1$.

Proof. Suppose first that S^α is m-isometric. Then we have (4.1). We define a function $g : \mathbb{Z}_+ \to \mathbb{R}_{>0}$ by

$$f(n) = f_r(\ell),$$

where ℓ and r are unique integer values satisfying $1 \leq r \leq \alpha$, $\ell \geq 0$ and $n = \ell\alpha + r$. Equation (4.1) can be written as

$$|u_n|^2 = |u_{\ell\alpha+r}|^2 = \frac{f_r(\ell+1)}{f_r(\ell)} = \frac{f(n+\alpha)}{f(n)}.$$

Now for $n > \alpha$, we have

$$\frac{u_{n-\alpha+1}}{u_{n-\alpha}} = \frac{w_{n-\alpha+1}\cdots w_n}{w_{n-\alpha}\cdots w_{n-1}} = \frac{w_n}{w_{n-\alpha}},$$

which implies

$$\frac{|w_n|^2}{|w_{n-\alpha}|^2} = \frac{|u_{n-\alpha+1}|^2}{|u_{n-\alpha}|^2} = \frac{f(n+1)}{f(n-\alpha+1)} \cdot \frac{f(n-\alpha)}{f(n)} = \frac{f(n+1)/f(n)}{f(n-\alpha+1)/f(n-\alpha)}.$$

Consequently, if $n = \ell\alpha + r$ with $1 \leq r \leq \alpha$, then

$$\frac{|w_n|^2}{f(n+1)/f(n)} = \frac{|w_{n-\alpha}|^2}{f(n-\alpha+1)/f(n-\alpha)} = \cdots = \frac{|w_r|^2}{f(r+1)/f(r)}.$$

Denoting this positive common ratio by c_r, we obtain the formula

$$|w_n|^2 = c_r\frac{f(n+1)}{f(n)} \qquad \text{for } n = \ell\alpha + r.$$

Since $|w_1|^2\cdots|w_\alpha|^2 = |u_1|^2 = f(\alpha+1)/f(1)$ we conclude that $c_1\cdots c_\alpha = 1$. Now set $c_0 = 1$ and define $g(\ell\alpha+r) = c_0\cdots c_{r-1}f(\ell\alpha+r)$ for $\ell \geq 0$ and $1 \leq r \leq \alpha$. It is clear that condition (a) is satisfied.

For $n = \ell\alpha + r$ with $1 \leq r \leq \alpha - 1$ and $\ell \geq 0$, we compute

$$\frac{g(n+1)}{g(n)} = \frac{g(\ell\alpha+r+1)}{g(\ell\alpha+r)} = \frac{c_0\cdots c_r f(n+1)}{c_0\cdots c_{r-1}f(n)} = c_r\frac{f(n+1)}{f(n)} = |w_n|^2.$$

On the other hand, if $n = \ell\alpha + \alpha$ for some $\ell \geq 0$, then

$$\frac{g(n+1)}{g(n)} = \frac{g((\ell+1)\alpha+1)}{g(\ell\alpha+\alpha)} = \frac{c_0 f(n+1))}{c_0\cdots c_{\alpha-1}f(n)} = c_\alpha\frac{f(n+1)}{f(n)} = |w_n|^2.$$

In the second last equality, we used the fact that $c_1\cdots c_{\alpha-1} = c_\alpha^{-1}$. Thus, we have shown that condition (b) is satisfied for any positive integer n.

Conversely, suppose there is a function $g : \mathbb{Z}_+ \to \mathbb{R}_{>0}$ such that both (a) and (b) hold. Then for any integer n, condition (b) gives

$$|u_n|^2 = \prod_{j=0}^{\alpha-1} |w_{n+j}|^2 = \prod_{j=0}^{\alpha-1} \frac{g(n+j+1)}{g(n+j)} = \frac{g(n+\alpha)}{g(n)}.$$

For integers $1 \leq r \leq \alpha$ and $\ell \geq 0$, put $f_r(\ell) = g(\ell\alpha + r)$. Then we have $|u_{\ell\alpha+r}|^2 = f_r(\ell+1)/f_r(\ell)$ and each f_r is a polynomial of degree at most $m-1$ in ℓ by (a). Consequently, condition (4.1) is satisfied and hence, S^α is m-isometric. $\square$

We now use Theorem 14 to investigate several examples.

Example 15. Define $g(2\ell+2) = g(2\ell+1) = \ell+1$ for all integers $\ell \geq 0$. Consider a unilateral weighted shift S with weights given by

$$w_n = \sqrt{\frac{g(n+1)}{g(n)}} = \begin{cases} \sqrt{\dfrac{g(2\ell+2)}{g(2\ell+1)}} & \text{if } n = 2\ell+1 \\ \sqrt{\dfrac{g(2\ell+3)}{g(2\ell+2)}} & \text{if } n = 2\ell+2 \end{cases}$$

$$= \begin{cases} 1 & \text{if } n = 2\ell+1 \\ \sqrt{\dfrac{\ell+2}{\ell+1}} & \text{if } n = 2\ell+2. \end{cases}$$

Since conditions (a) and (b) in Theorem 14 are satisfied with $\alpha = 2$ and $m = 2$, we conclude that S^2 is 2-isometric. However, S is not 2-isometric by Theorem 1.

Example 16. The above example can be generalized in the following way. Let $\alpha \geq 2$ and $m \geq 2$ be integers. Let p be a polynomial of degree $m-1$ such that $p(k) > 0$ for all integers $k \geq 0$. Consider a unilateral weighted shift S with weights defined by

$$w_n = \sqrt{\frac{p(\lfloor (n+1)/\alpha \rfloor)}{p(\lfloor n/\alpha \rfloor)}} = \begin{cases} 1 & \text{if } n = \ell\alpha + r \text{ with } 0 \leq r \leq \alpha - 2 \\ \sqrt{\dfrac{p(\ell+1)}{p(\ell)}} & \text{if } n = \ell\alpha + (\alpha - 1). \end{cases}$$

Here $\lfloor x \rfloor$ denotes the greatest integer smaller than or equal to x. It can be checked that conditions (a) and (b) in Theorem 14 are satisfied by the function $g(n) = p(\lfloor n/\alpha \rfloor)$. We conclude that S^α is m-isometric. As before, S is not m-isometric by Theorem 1.

Example 17. We now consider [6, Example 3.5]. Let S be a unilateral weighted shift with weights $w_{2\ell+1} = 4$ and $w_{2\ell+2} = \left(\frac{3\ell+4}{6\ell+2}\right)^2$ for all integers $\ell \geq 0$. Define

$$g(n) = \begin{cases} (3\ell+1)^4 & \text{if } n = 2\ell+1 \\ 16\,(3\ell+1)^4 & \text{if } n = 2\ell+2. \end{cases}$$

It can be checked that $|w_n|^2 = g(n+1)/g(n)$ for all positive integers n and that both $g(2\ell+1)$ and $g(2\ell+2)$ are polynomials in ℓ of degree 4. Theorem 14 shows that S^2 is a 5-isometry. (The statement that S^2 is a 2-isometry in [6, Example 3.5] is in fact inaccurate.)

Using Theorem 14, one can obtain other interesting examples. We leave it to the interested reader.

5. m-ISOMETRIC BILATERAL WEIGHTED SHIFT OPERATORS

In this section we discuss bilateral weighted shift operators that are m-isometric. It turns out that the characterization of m-isometric unilateral shift operators in Theorem 1 play a crucial role.

Let us fix an orthonormal basis $\{f_n\}_{n\in\mathbb{Z}}$ of H indexed by the integers $\mathbb{Z}$. A bilateral weighted shift operator T is a linear operator on H such that

$$Tf_n = w_n f_{n+1}, \qquad \text{for } n \in \mathbb{Z}.$$

As before, the sequence $\{w_n\}_{n\in\mathbb{Z}}$ of complex numbers is called the weight sequence of T. We assume that $\{w_n\}_{n\in\mathbb{Z}}$ is bounded so that T is a bounded operator. We shall obtain a description of the weight sequence of any m-isometric bilateral weighted shift operator.

Remark 18. We have already noticed that any m-isometry is injective and has a closed range. Since the range of an injective bilateral weighted shift operator is dense, it follows that any m-isometric bilateral weighted shift operator is automatically invertible.

Our first result in this section characterizes bilateral weighted shift operators that are m-isometric.

Theorem 19. *Let T be a bilateral weighted shift operator with the weight sequence $\{w_n\}_{n\in\mathbb{Z}}$. Then T is an m-isometric operator if and only if there exists a polynomial p of degree at most $m-1$ such that for any integer n, we have $p(n) > 0$ and*

$$|w_n|^2 = \frac{p(n+1)}{p(n)}.$$

Furthermore, the degree of p must be even.

Proof. For any positive integer $k \geq 0$, let H_k be the closed subspace of H that is spanned by $\{f_n\}_{n\geq -k}$. It is clear that $\{H_k\}_{k\geq 0}$ is an increasing sequence of invariant subspaces of T and $H = \overline{\cup_{k=0}^{\infty} H_k}$. Put $T_k = T|_{H_k}$. It then follows from the definition of m-isometries that T is an m-isometry on H if and only if T_k is an m-isometry on H_k for all k. Note that each T_k is a unilateral weighted shift on H_k with respect to the orthonormal basis $\{e_n\}_{n\geq -k}$. The weight sequence of T_k is $\{w_n\}_{n\geq -k}$.

We first suppose that T is m-isometric. Then each operator T_k is m-isometric on H_k. By Theorem 1, there is a *monic* polynomial p_k of degree at most $m-1$ with real coefficients such that for all $n \geq -k$, we have $p_k(n) > 0$ and

$$|w_n|^2 = \frac{p_k(n+1)}{p_k(n)}.$$

Note that we have actually applied a version of Theorem 1 with the index n starting from $-k$ instead of 1. Since $T_k|_{H_0} = T_0$, the uniqueness established in Remark 2 shows that the polynomials p_k are all the same. Let us call this polynomial p. Then p is monic and for any integer $n \in \mathbb{Z}$, we have $p(n) > 0$ and $|w_n|^2 = p(n+1)/p(n)$. The positivity of p on $\mathbb{Z}$ shows that its degree must be even.

Conversely, suppose p is a polynomial of degree at most $m-1$ with real coefficients such that $p(n) > 0$ and $|w_n|^2 = p(n+1)/p(n)$ for all $n \in \mathbb{Z}$. By Theorem 1, each unilateral weighted shift operator $T_k = T|_{H_k}$ is m-isometric on H_k. It follows that T is m-isometric on H. $\square$

With the same argument as in the proof of Corollary 3, we obtain a characterization of strict m-isometric bilateral weighted shift operator.

Corollary 20. *The bilateral weighted shift operator T is strictly m-isometric if and only if the degree of p is exactly $m-1$ and m is an odd integer.*

Remark 21. Corollary 20 shows that there only exist strict m-isometric bilateral weighted shift operators when m is odd. This fact is not surprising since it actually follows from Remark 18 and a general result [2, Proposition 1.23] (see also [10, Proposition A]) which asserts that if A is an invertible k-isometry and k is even, then A is a $(k-1)$-isometry.

Example 22. The operator T is a strict 3-isometry if and only if there is a monic polynomial p of degree 2 such that $p(n) > 0$ and $|w_n|^2 = p(n+1)/p(n)$ for all $n \in \mathbb{Z}$. Write $p(x) = (x-\alpha)(x-\beta)$ for some complex numbers α and β. Since p assumes positive values on $\mathbb{Z}$, one of the following two cases must occur:

(1) Both α and β belong to $\mathbb{C}\backslash\mathbb{R}$.
(2) There exists an integer n_0 such that both α and β belong to the open interval (n_0, n_0+1).

It should be noted that quadratic polynomials that give rise to 3-isometric *bilateral* weighted shift operators are more restrictive than quadratic polynomials that give rise to 3-isometric *unilateral* weighted shift operators (see Example 5).

Example 23. Let $\ell \geq 2$ be an even integer and b be a positive number. Define $p(x) = x(x+1)\cdots(x+\ell-1)+b$. Then p has degree ℓ and $p(n) > 0$ for all $n \in \mathbb{Z}$. Let T be a bilateral weighted shift operator with weights

$$w_n = \sqrt{\frac{p(n+1)}{p(n)}} \quad \text{for } n \in \mathbb{Z}.$$

By Corollary 20, the operator T is a strict $(\ell+1)$-isometry. This example was discussed in [10, Theorem 1].

As in the case of unilateral weighted shift operators, we also have a factorization theorem for m-isometric bilateral weighted shift operators.

Theorem 24. *Any bilateral weighted shift operator that is strictly m-isometric for some odd integer $m \geq 3$ can be written as a Hadamard product of strictly 3-isometric bilateral weighted shift operators.*

Proof. For any strictly m-isometric bilateral weighted shift operator, let p be the monic polynomial given in Theorem 19. With an argument similar to that in the proof of Lemma 10, one can factor $p = p_1 \cdots p_\nu$, where each p_j is a monic quadratic polynomial having positive values over $\mathbb{Z}$. The remaining of the proof is now the same as the proof of Theorem 11. $\square$

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