ON TOEPLITZ OPERATORS ON BERGMAN SPACES OF THE UNIT POLYDISK

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Abstract. We study Toeplitz operators on the Bergman space A^2_ϑ of the unit polydisk $\mathbb{D}^n$, where ϑ is a product of n rotation-invariant regular Borel probability measures. We show that if f is a bounded Borel function on $\mathbb{D}^n$ such that $F(w) = \lim_{\substack{z\to w\\ z\in\mathbb{D}^n}} f(z)$ exists for all $w \in \partial\mathbb{D}^n$, then T_f is compact if and only if $F = 0$ a.e. with respect to a measure γ associated with ϑ on the boundary $\partial\mathbb{D}^n$. We also discuss the commuting problem: if g is a non-constant bounded holomorphic function on $\mathbb{D}^n$, then what conditions does a bounded function f need to satisfy so that T_f commutes with T_g?

1. Introduction

For any $r > 0$, we denote by $\mathbb{D}_r$ the open disk of radius r centered at 0 in $\mathbb{C}$. As usual, we use $\mathbb{D}$ instead of $\mathbb{D}_1$ for the open unit disk. A regular Borel probability measure ν on $\mathbb{D}$ is said to be *rotation-invariant* if $\nu(\mathrm{e}^{\mathrm{i}\theta}E) = \nu(E)$ for all Borel subsets $E \subset \mathbb{D}$ and $\theta \in (0, 2\pi)$. It follows that there is a regular Borel probability measure μ on $[0, 1)$ so that for any function $f \in L^1(\mathbb{D}, \mathrm{d}\nu)$, we have

$$\int_{\mathbb{D}} f(z)\mathrm{d}\nu(z) = \int_{[0,1)} \Big\{ \int_{\mathbb{T}} f(r\zeta)\mathrm{d}\sigma(\zeta)\Big\}\mathrm{d}\mu(r), \tag{1.1}$$

where σ is the normalized arc-length measure on the unit circle $\mathbb{T}$.

We now assume that $\nu(\mathbb{D}\backslash\mathbb{D}_r) > 0$ for all $0 < r < 1$. We will use $\langle , \rangle$ and $\|\cdot\|$ to denote the inner product and the norm in $L^2(\mathbb{D}, \mathrm{d}\nu)$. The Bergman space A^2_ν consists of all functions in $L^2(\mathbb{D}, \mathrm{d}\nu)$ that are holomorphic on $\mathbb{D}$. It is a consequence of Cauchy's formula and the assumption about ν that for each compact subset $M \subset \mathbb{D}$, there is a constant $C_M > 0$ such that for $f \in A^2_\nu$,

$$\sup\{|f(z)| : z \in M\} \le C_M \Big\{ \int_{\mathbb{D}} |f(w)|^2\mathrm{d}\nu(w)\Big\}^{1/2} = C_M\|f\|. \tag{1.2}$$

This shows that A^2_ν is a closed subspace of $L^2(\mathbb{D}, \mathrm{d}\nu)$ and for each $z \in \mathbb{D}$, the map $f \mapsto f(z)$ is a bounded linear functional on A^2_ν. So there is a function $K_z \in A^2_\nu$ such that $f(z) = \langle f, K_z\rangle$. The function K_z is called the reproducing kernel for A^2_ν at z. Let $k_z = K_z/\|K_z\|$ for $z \in \mathbb{D}$. Then k_z is called the normalized reproducing kernel for A^2_ν at z. It is well known that $k_z \to 0$ weakly as $|z| \uparrow 1$.

Fix a positive integer n. Let $\mathbb{D}^n = \mathbb{D} \times \cdots \times \mathbb{D}$ denote the open unit polydisk in $\mathbb{C}^n$, which is the product of n copies of $\mathbb{D}$. Let $\nu_1, \ldots, \nu_n$ be rotation-invariant regular Borel probability measures on $\mathbb{D}$ so that $\nu_j(\mathbb{D}\backslash\mathbb{D}_r) > 0$ for all $0 < r < 1$

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and $1 \leq j \leq n$. Let us denote by ϑ the product measure $\nu_1 \times \cdots \times \nu_n$ on $\mathbb{D}^n$. Then ϑ is automatically a regular Borel probability measure. For any function f in $L^1(\mathbb{D}^n, \mathrm{d}\vartheta)$, we have

$$\begin{aligned} &\int_{\mathbb{D}^n} f(z)\mathrm{d}\vartheta(z) \\ &= \int_{[0,1)^n} \Big\{ \int_{\mathbb{T}^n} f(r_1\zeta_1, \ldots, r_n\zeta_n)\mathrm{d}\sigma(\zeta_1)\cdots\mathrm{d}\sigma(\zeta_n)\Big\}\mathrm{d}\mu_1(r_1)\cdots\mathrm{d}\mu_n(r_n), \end{aligned} \tag{1.3}$$

where for $1 \leq j \leq n$, μ_j is the measure on $[0,1)$ corresponding to ν_j as in (1.1).

Let $H(\mathbb{D}^n)$ denote the space of all holomorphic functions on $\mathbb{D}^n$. Similar to the one-dimensional case above, we define the Bergman space A^2_ϑ to be the space of all functions $f \in H(\mathbb{D}^n)$ which also belong to $L^2(\mathbb{D}^n, \mathrm{d}\vartheta)$. As before, for any compact subset $M \subset \mathbb{D}^n$, there is a positive constant C_M such that (1.2) holds true for all f in A^2_ϑ. This implies that for each $z = (z_1, \ldots, z_n) \in \mathbb{D}^n$, there is a reproducing kernel $K_z \in A^2_\vartheta$ such that $f(z) = \langle f, K_z\rangle$ for all $f \in A^2_\vartheta$. It is well known that $K_z = K^{(1)}_{z_1}\cdots K^{(1)}_{z_n}$, where each $K^{(j)}_{z_j}$ is the reproducing kernel for $A^2_{\nu_j}(\mathbb{D})$ at z_j. For any compact subset $M \subset \mathbb{D}^n$ and any $z \in M$, since $\|K_z\|^2 = |K_z(z)| \leq C_M\|K_z\|$, we conclude that $\|K_z\| \leq C_M$.

If $\{u_j : j \in \mathbb{N}\}$ is an orthonormal basis for A^2_ϑ then for $z, w \in \mathbb{D}^n$, we have

$$K_z(w) = \langle K_z, K_w\rangle = \sum_{j=0}^{\infty}\langle K_z, u_j\rangle\langle u_j, K_w\rangle = \sum_{j=0}^{\infty} u_j(w)\bar{u}_j(z).$$

For $m = (m_1, \ldots, m_n) \in \mathbb{N}^n$ (here $\mathbb{N}$ denotes the set of all non-negative integers) and $z = (z_1, \ldots, z_m) \in \mathbb{D}^n$, we write $z^m = z_1^{m_1}\cdots z_n^{m_n}$ and $\bar{z}^m = \bar{z}_1^{m_1}\cdots\bar{z}_n^{m_n}$. For m and k in $\mathbb{N}^n$, by (1.3), we have

$$\int_{\mathbb{D}^n} z^m\bar{z}^k\mathrm{d}\vartheta(z) = \begin{cases} 0 & \text{if } m \neq k, \\ \int_{[0,1)^n} r_1^{2m_1}\cdots r_n^{2m_n}\mathrm{d}\mu_1(r_1)\cdots\mathrm{d}\mu_n(r_n) & \text{if } m = k. \end{cases}$$

Let $\alpha_m = \int_{[0,1)^n} r_1^{2m_1}\cdots r_n^{2m_n}\mathrm{d}\mu_1(r_1)\cdots\mathrm{d}\mu_n(r_n)$. Then since the span of $\{z^m : m \in \mathbb{N}^n\}$ is dense in A^2_ϑ, the set $\{e_m(z) = \frac{z^m}{\sqrt{\alpha_m}} : m \in \mathbb{N}^n\}$ is an orthonormal basis for A^2_ϑ. It is usually referred to as the standard orthonormal basis.

Example 1.1. If $\nu_1 = \cdots = \nu_n$ is the normalized Lebesgue measure on $\mathbb{D}$, then ϑ is the normalized Lebesgue measure on $\mathbb{D}^n$ and A^2_ϑ in this case is the usual Bergman space on $\mathbb{D}^n$. For each $z = (z_1, \ldots, z_n) \in \mathbb{D}^n$, the reproducing kernel K_z has the form $K_z(w) = \frac{1}{(1-\bar{z}_1w_1)^2}\cdots\frac{1}{(1-\bar{z}_nw_n)^2}$, for $w = (w_1, \ldots, w_n) \in \mathbb{D}^n$. The standard orthonormal basis for A^2_ϑ is $\{\sqrt{(m_1+1)\cdots(m_n+1)}\ z^m : m \in \mathbb{N}^n\}$.

Let P denote the orthogonal projection from $L^2(\mathbb{D}^n, \mathrm{d}\vartheta)$ onto A^2_ϑ. For any bounded Borel function f on $\mathbb{D}$, the Toeplitz operator T_f is defined by

$$T_f : A^2_\vartheta \longrightarrow A^2_\vartheta, \quad (T_f)(\psi) = P(f\psi) \quad \text{for all } \psi \in A^2_\vartheta.$$

The function f is called the symbol of T_f. Since $\|P\| = 1$, it follows that T_f is a bounded operator with $\|T_f\| \leq \|f\|_\infty$. Here, $\|f\|_\infty$ is the norm of f as an element of $L^\infty(\mathbb{D}^n, \mathrm{d}\vartheta)$. For any $\psi \in A^2_\vartheta$ and $z \in \mathbb{D}^n$, we have

$$(T_f\psi)(z) = \langle T_f\psi, K_z\rangle = \langle f\psi, K_z\rangle = \int_{\mathbb{D}^n} f(w)\psi(w)\overline{K}_z(w)\mathrm{d}\vartheta(w).$$

This shows that T_f is an integral operator on A^2_ϑ with kernel $f(w)\overline{K}_z(w), w, z \in \mathbb{D}^n$. If f vanishes outside a compact subset M of $\mathbb{D}^n$, then

$$\begin{aligned}\int_{\mathbb{D}^n}\int_{\mathbb{D}^n} |f(w)\overline{K}_z(w)|^2 \mathrm{d}\vartheta(z)\mathrm{d}\vartheta(w) &= \int_{\mathbb{D}^n}\int_{\mathbb{D}^n} |f(w)|^2 |K_w(z)|^2 \mathrm{d}\vartheta(z)\mathrm{d}\vartheta(w)\\ &= \int_{\mathbb{D}^n} |f(w)|^2 \|K_w\|^2 \mathrm{d}\vartheta(w)\\ &\le \|f\|^2_\infty \int_M \|K_w\|^2 \mathrm{d}\vartheta(w)\\ &\le \|f\|^2_\infty C^2_M.\end{aligned}$$

It follows that T_f is a Hilbert-Schmidt operator on A^2_ϑ, hence it is compact.

2. Compact Toeplitz operators with continuous symbols

It is well known (see [6]) that if f is a continuous function on the closed unit ball $\overline{\mathbb{B}}_n$, then the Toeplitz operator T_f on the Bergman space corresponding to the ordinary Lebesgue measure on $\mathbb{B}_n$ is compact if and only if $f(w) = 0$ for all $w \in \partial\mathbb{B}_n = \mathbb{S}_n$. The usual approach to this result uses Berezin transform and the explicit formulas for the normalized reproducing kernel functions. However, this approach does not seem to work for a general rotation-invariant measure because there are no useful formulas for the kernel functions. Nevertheless, the result was extended to the Bergman space of a general rotation-invariant measure on the unit disk by T. Nakazi and R. Yoneda in [9] with a different approach. An extension of the same result to the unit ball was established by the author in [7]. In this section, we will consider the same problem on the Bergman space A^2_ϑ of $\mathbb{D}^n$.

Let $\partial\mathbb{D}^n$ denote the topological boundary of $\mathbb{D}^n$ as a subset of $\mathbb{C}^n$. Then $\partial\mathbb{D}^n$ is the disjoint union of $2^n - 1$ sets of the form $A_1 \times \cdots \times A_n$, where A_j is either $\mathbb{T}$ or $\mathbb{D}$ and not all of them are $\mathbb{D}$. Suppose $A_1 \times \cdots \times A_n$ is a part of $\partial\mathbb{D}^n$. For each j, we put $\gamma_j = \sigma$ if $A_j = \mathbb{T}$ and $\gamma_j = \nu_j$ if $A_j = \mathbb{D}$. We define the measure γ on $A_1 \times \cdots \times A_n$ to be the product measure $\gamma_1 \times \cdots \times \gamma_n$. Then γ is a regular measure on the Borel sets of $\partial\mathbb{D}^n$ (The topology on $\partial\mathbb{D}^n$ is the usual topology as a subset of $\mathbb{C}^n$).

Suppose f is a bounded Borel function on $\mathbb{D}^n$ and $w = (w_1, \ldots, w_n)$ is a point in $\partial\mathbb{D}^n$. There are indexes $1 \le i_1 < \cdots < i_s \le n$ (for some $1 \le s \le n$) such that $|w_{i_1}| = \cdots = |w_{i_s}| = 1$ and $|w_j| < 1$ if $j \notin \{i_1, \ldots, i_s\}$. We write $R\text{–}\lim\limits_{\substack{z\to w\\ z\in\mathbb{D}}} f(z)$ to denote the radial limit $\lim\limits_{(r_1,\ldots,r_s)\to(1,\ldots,1)} f(w_1, \ldots, r_1 w_{i_1}, \ldots, r_s w_{i_s}, \ldots, w_n)$ and $\lim\limits_{\substack{z\to w\\ z\in\mathbb{D}}} f(z)$ to denote the limit of f as z approaches w from $\mathbb{D}^n$ in the usual sense. It is clear that if $\lim\limits_{\substack{z\to w\\ z\in\mathbb{D}}} f(z)$ exists, then $R\text{–}\lim\limits_{\substack{z\to w\\ z\in\mathbb{D}}} f(z)$ exists and these two limits are the same.

Lemma 2.1. *Suppose f is a bounded Borel function on $\mathbb{D}^n$ such that the limit $F(w) = \lim\limits_{\substack{z\to w\\ z\in\mathbb{D}^n}} f(z)$ exists for all $w \in \partial\mathbb{D}^n$. If $F(w) = 0$ for γ-a.e. w in $\partial\mathbb{D}^n$, then there is a Borel function g on $\mathbb{D}^n$ such that $f(z) = g(z)$ for ϑ-a.e. z in $\mathbb{D}^n$ and $\lim\limits_{\substack{z\to w\\ z\in\mathbb{D}^n}} g(z) = 0$ for all $w \in \partial\mathbb{D}^n$.*

Proof. For each $1 \le j \le n$, let V_j be the biggest open (possibly empty) subset of $\mathbb{D}$ such that $\nu_j(V_j) = 0$. The existence of V_j follows from the regularity of ν_j on $\mathbb{D}$. Let $G = (\mathbb{D}\backslash V_1) \times \cdots \times (\mathbb{D}\backslash V_n)$. Then $\vartheta(\mathbb{D}^n\backslash G) = 0$ and hence, $f(z) = f(z)\chi_G(z)$ for ϑ-a.e. z in $\mathbb{D}^n$. We will show that $\lim\limits_{\substack{z\to w\\ z\in\mathbb{D}^n}} f(z)\chi_G(z) = 0$ for all $w \in \partial\mathbb{D}^n$.

Let $w = (w_1, \ldots, w_n) \in \partial\mathbb{D}^n$. By permuting the coordinates of w if necessary, we may assume that $w_1, \ldots, w_s$ are in $\mathbb{T}$ and $w_{s+1}, \ldots, w_n$ are in $\mathbb{D}$, for some $1 \le s \le n$. If $F(w) = 0$, then since $|f(z)\chi_G(z)| \le |f(z)|$, we obtain $\lim\limits_{\substack{z\to w\\ z\in\mathbb{D}^n}} f(z)\chi_G(z) = 0$.

Now suppose $F(w) \ne 0$. Then there is a $\delta > 0$ so that $|f(z)| > |F(w)|/2$ for all $z \in \mathbb{D}^n \cap D(w_1, \delta) \times \cdots \times D(w_n, \delta)$. (Here $D(w_j, \delta)$ denotes the disk in $\mathbb{C}$ of radius δ, centered at w_j.) This implies that $|F(u)| \ge |F(w)|/2$ for all u in $U = (\mathbb{T} \cap D(w_1, \delta)) \times \cdots \times (\mathbb{T} \cap D(w_s, \delta)) \times (\mathbb{D} \cap D(w_{s+1}, \delta)) \times \cdots \times (\mathbb{D} \cap D(w_n, \delta))$ (Note that U is a subset of $\partial\mathbb{D}^n$.) Since $F(u) = 0$ for γ-a.e. u on $\partial\mathbb{D}^n$, it must be true that $\gamma(U) = 0$. This implies $1 \le s < n$ and $\nu_l(\mathbb{D} \cap D(w_l, \delta)) = 0$ for some $s + 1 \le l \le n$. By our choice of V_l, $\mathbb{D} \cap D(w_l, \delta) \subset V_l$. This shows that for $z = (z_1, \ldots, z_n) \in \mathbb{D}^n \cap D(w_1, \delta) \times \cdots \times D(w_n, \delta)$, we have $z_l \in V_l$ and hence, $\chi_G(z) = 0$. Therefore, $\lim\limits_{\substack{z\to w\\ z\in\mathbb{D}^n}} f(z)\chi_G(z) = 0$. $\square$

Proposition 2.2. *Suppose f is a bounded Borel function on $\mathbb{D}^n$ such that $F(w) = \lim\limits_{\substack{z\to w\\ z\in\mathbb{D}^n}} f(z)$ exists for all $w \in \partial\mathbb{D}^n$. If $F(w) = 0$ for $\gamma-$a.e. $w \in \partial\mathbb{D}^n$, then T_f is a compact operator.*

Proof. By redefining f on a ϑ-null set using Lemma 2.1 if necessary, we may assume that $F(w) = 0$ for all $w \in \partial\mathbb{D}^n$. By the compactness of $\partial\mathbb{D}^n$, for any $\epsilon > 0$, there is an $r \in (0, 1)$ such that $|f(z)| < \epsilon$ for all $z \in \mathbb{D}^n\backslash\mathbb{D}^n_r$. This shows that $\|T_f - T_{f\chi_{\mathbb{D}^n_r}}\| \le \epsilon$. Since $f\chi_{\mathbb{D}^n_r}$ is supported in a compact subset of $\mathbb{D}^n$, $T_{f\chi_{\mathbb{D}^n_r}}$ is a compact operator. So T_f, being the limit of a net of compact operators, is also a compact operator. $\square$

Remark 2.3. Care must be taken in the hypothesis of Proposition 2.2. It is not enough to assume that $\lim\limits_{\substack{z\to w\\ z\in\mathbb{D}^n}} f(z) = 0$ for $\gamma-$a.e. $w \in \partial\mathbb{D}^n$. In fact, there is a bounded function f on $\mathbb{D}$ such that $\lim\limits_{\substack{z\to w\\ z\in\mathbb{D}}} f(z) = 0$ for all $w \in \mathbb{T}\backslash\{1\}$ and T_f is not compact on the Bergman space $A^2 = A^2(\mathbb{D}, \mathrm{d}A)$, where $\mathrm{d}A$ is the normalized Lebesgue measure on $\mathbb{D}$. To construct such a function f, we make use of the pseudo-hyperbolic metric on $\mathbb{D}$ and the reproducing kernels of A^2.

For each $z \in \mathbb{D}$, let φ_z be the Mobius map that interchanges 0 and z. The formula $\rho(z, w) = |\varphi_z(w)|$, $z, w \in \mathbb{D}$, defines a metric on $\mathbb{D}$, called the pseudo-hyperbolic metric. For $a \in \mathbb{D}$ and $0 < \epsilon < 1$, we use $E(a, \epsilon)$ to denote the $\rho-$ball of radius ϵ, centered at a. Since ρ is invariant under the action of the automorphism of $\mathbb{D}$ and $\varphi_z \circ \varphi_z = \mathrm{I}$, it can be showed that $\chi_{E(z,\epsilon)} \circ \varphi_z = \chi_{E(0,\epsilon)}$. See pages 65-66 in [12] for more information about the pseudo-hyperbolic metric.

For $z \in \mathbb{D}$, the formula $U_z h = (h \circ \varphi_z)k_z$, $h \in A^2$ defines a self-adjoint unitary operator on A^2. (Recall here that k_z is the normalized reproducing kernel of A^2 at z.) For any bounded function f on $\mathbb{D}$, we have $U_z^* T_f U_z = T_{f\circ\varphi_z}$. See pages 189-190 in [12] for more details.

Let $\{z_j\}_{j=1}^{\infty}$ be a sequence of points in $\mathbb{D}$ so that $\lim_{j\to\infty}|z_j-1|=0$ and $\epsilon=\frac{1}{2}\inf\{\rho(z_{j_1},z_{j_2}):j_1\neq j_2\}>0$. (One may take $z_j=1-2^{-j}$ for $j=1,2,\ldots$ and in this case, $\epsilon=1/6$.) Let $f=\sum_{j=1}^{\infty}\chi_{E(z_j,\epsilon)}$. Then $\lim\limits_{\substack{z\to w\\ z\in\mathbb{D}}}f(z)=0$ for all $w\in\mathbb{T}\backslash\{1\}$. On the other hand, for any integer $j\geq 1$, since $k_{z_j}=U_{z_j}1$, we have

$$\begin{aligned}\langle T_f k_{z_j},k_{z_j}\rangle &\geq \langle T_{\chi_{E(z_j,\epsilon)}}k_{z_j},k_{z_j}\rangle=\langle T_{\chi_E(z_j,\epsilon)}U_{z_j}1,U_{z_j}1\rangle\\ &=\langle U_{z_j}^*T_{\chi_{E(z_j,\epsilon)}}U_{z_j}1,1\rangle=\langle T_{\chi_{E(z_j,\epsilon)}\circ\varphi_{z_j}}1,1\rangle=\langle T_{\chi_{E(0,\epsilon)}}1,1\rangle>0.\end{aligned}$$

This shows that T_f is not a compact operator.

The following lemma is probably well known but we have not been aware of an appropriate reference so we sketch here a proof.

Lemma 2.4. *Suppose $\mu_1,\ldots,\mu_s$ are measures on $[0,1)$ such that $\mu_j((r,1))>0$ for all $0<r<1$ and $1\leq j\leq s$. Suppose φ is a bounded function on $[0,1)^n$ such that $\lim\limits_{(r_1,\ldots,r_s)\to(1,\ldots,1)}\varphi(r_1,\ldots,r_s)=\alpha$. Then we have*

$$\lim_{(m_1,\ldots,m_s)\to(\infty,\ldots,\infty)}\frac{\int_{[0,1)^n}\varphi(r_1,\ldots,r_s)r_1^{m_1}\cdots r_s^{m_s}\mathrm{d}\mu_1(r_1)\cdots\mathrm{d}\mu_s(r_s)}{\int_{[0,1)^n}r_1^{m_1}\cdots r_s^{m_s}\mathrm{d}\mu_1(r_1)\cdots\mathrm{d}\mu_s(r_s)}=\alpha.$$

Proof. Let M be an upper bound for $|\varphi|$ on $[0,1)^n$. Let $\epsilon>0$ be given. There is a number $0<\delta<1$ so that $|\varphi(r_1,\ldots,r_n)-\alpha|<\epsilon$ if $(r_1,\ldots,r_s)\in[\delta,1)^n$. Since

$$[0,1)^n=[\delta,1)^n\cup[0,\delta)\times[0,1)^{n-1}\cup[0,1)\times[0,\delta)\times[0,1)^{n-2}\cup\cdots\cup[0,1)^{n-1}\times[0,\delta),$$

and $|\varphi-\alpha|\leq M+\alpha$, we have

$$\begin{aligned}&\Big|\frac{\int_{[0,1)^n}\varphi(r_1,\ldots,r_s)r_1^{m_1}\cdots r_s^{m_s}\mathrm{d}\mu_1(r_1)\cdots\mathrm{d}\mu_s(r_s)}{\int_{[0,1)^n}r_1^{m_1}\cdots r_s^{m_s}\mathrm{d}\mu_1(r_1)\cdots\mathrm{d}\mu_s(r_s)}-\alpha\Big|\\ &\leq\frac{\int_{[0,1)^n}|\varphi(r_1,\ldots,r_s)-\alpha|r_1^{m_1}\cdots r_s^{m_s}\mathrm{d}\mu_1(r_1)\cdots\mathrm{d}\mu_s(r_s)}{\int_{[0,1)^n}r_1^{m_1}\cdots r_s^{m_s}\mathrm{d}\mu_1(r_1)\cdots\mathrm{d}\mu_s(r_s)}\\ &\leq\epsilon+\sum_{j=1}^{s}(M+\alpha)\frac{\int_{[0,\delta)}r_j^{m_j}\mathrm{d}\mu_j(r_j)}{\int_{[0,1)}r_j^{m_j}\mathrm{d}\mu_j(r_j)}.\end{aligned}$$

For each j, $\dfrac{\int_{[0,\delta)}r_j^{m_j}\mathrm{d}\mu_j(r_j)}{\int_{[0,1)}r_j^{m_j}\mathrm{d}\mu_j(r_j)}\to 0$ as $m_j\to\infty$ (see, for example, Lemma 2 in [9]). By letting $(m_1,\ldots,m_s)\to(\infty,\ldots,\infty)$ and using the fact that $\epsilon>0$ was arbitrary, we obtain the required identity. $\square$

The following theorem is the main result of this section. It implies, in particular, that the converse of Proposition 2.2 holds.

Theorem 2.5. *Suppose f is a bounded Borel function on $\mathbb{D}^n$ such that for γ-a.e. $w\in\partial\mathbb{D}^n$, $F(w)=R\text{–}\lim\limits_{\substack{z\to w\\ z\in\mathbb{D}^n}}f(z)$ exists. If T_f is a compact operator, then $F(w)=0$ for γ-a.e. w in $\partial\mathbb{D}^n$.*

Proof. Suppose T_f is a compact operator. Let $A_1\times\cdots\times A_n$ be a part of $\partial\mathbb{D}^n$ as in the decomposition we discussed at the beginning of the section. We need to show that $F(w)=0$ for γ-a.e. w in $A_1\times\cdots\times A_n$. It suffices to consider the case $A_1 = \cdots = A_s = \mathbb{T}$ and $A_{s+1} = \cdots = A_n = \mathbb{D}$ for some $1 \leq s \leq n$. Let $m_{s+1}, \ldots, m_n \in \mathbb{N}$ be given. Consider the function

$$\begin{aligned}\varphi(r_1, \ldots, r_s) &= \int_{\mathbb{T}^s} \int_{\mathbb{D}^{n-s}} f(r_1\zeta_1, \ldots, r_s\zeta_s, z_{s+1}, \ldots, z_n) \\ &\qquad \times \Big(\prod_{j=s+1}^{n} |z_j|^{2m_j} \Big) \mathrm{d}\nu_{s+1}(z_{s+1}) \cdots \mathrm{d}\nu_n(z_n) \mathrm{d}\sigma(\zeta_1) \cdots \mathrm{d}\sigma(\zeta_s) \\ &= \int_{\mathbb{T}^s \times \mathbb{D}^{n-s}} f(r_1\zeta_1, \ldots, r_s\zeta_s, z_{s+1}, \ldots, z_n) \Big(\prod_{j=s+1}^{n} |z_j|^{2m_j} \Big) \mathrm{d}\gamma,\end{aligned}$$

where $r_1, \ldots, r_s$ are in the interval $[0, 1)$. By the Lebesgue Dominated Convergence Theorem and the assumption about f, we have

$$\begin{aligned}&\lim_{(r_1,\ldots,r_s)\to(1,\ldots,1)} \varphi(r_1, \ldots, r_s) \\ &\qquad = \int_{\mathbb{T}^s \times \mathbb{D}^{n-s}} F(\zeta_1, \ldots, \zeta_s, z_{s+1}, \ldots, z_n) \Big(\prod_{j=s+1}^{n} |z_j|^{2m_j} \Big) \mathrm{d}\gamma.\end{aligned} \tag{2.1}$$

Now for $m_1, \ldots, m_s \in \mathbb{N}$ and $m = (m_1, \ldots, m_n)$, integration in polar coordinates gives

$$\begin{aligned}&\langle T_f e_m, e_m \rangle \\ &= \frac{1}{\alpha_m} \int_{\mathbb{D}^n} f(z) z^m \bar{z}^m \mathrm{d}\vartheta(z) \\ &= \frac{1}{\alpha_m} \int_{[0,1)^s} \varphi(r_1, \ldots, r_s) r_1^{2m_1} \cdots r_s^{2m_s} \mathrm{d}\mu_1(r_1) \cdots \mathrm{d}\mu_s(r_s) \\ &= \frac{\int_{[0,1)^s} \varphi(r_1, \ldots, r_s) r_1^{2m_1} \cdots r_s^{2m_s} \mathrm{d}\mu_1(r_1) \cdots \mathrm{d}\mu_s(r_s)}{\int_{[0,1)^s} \Big(\int_{[0,1)^{n-s}} \prod_{j=s+1}^{n} r_j^{2m_j} \mathrm{d}\mu_j(r_j) \Big) r_1^{2m_1} \cdots r_s^{2m_s} \mathrm{d}\mu_1(r_1) \cdots \mathrm{d}\mu_s(r_s)}.\end{aligned}$$

Taking limits as $(m_1, \ldots, m_s) \to (\infty, \ldots, \infty)$ and using Lemma 2.4 together with (2.1) and the fact that T_f is a compact operator, we obtain

$$0 = \int_{\mathbb{T}^s \times \mathbb{D}^{n-s}} F(\zeta_1, \ldots, \zeta_s, z_{s+1}, \ldots, z_n) \Big(\prod_{j=s+1}^{n} |z_j|^{2m_j} \Big) \mathrm{d}\gamma.$$

For any $k, l \in \mathbb{N}^n$, since $T_{\bar{z}^k f(z) z^l} = T_{\bar{z}^k} T_f T_{z^l}$ is also a compact operator, the above argument gives

$$\int_{\mathbb{T}^s \times \mathbb{D}^{n-s}} F(\zeta_1, \ldots, \zeta_s, z_{s+1}, \ldots, z_n) \prod_{j=1}^{s} \zeta_j^{l_j - k_j} \prod_{j=s+1}^{n} z_j^{m_j + l_j} \bar{z}_j^{m_j + k_j} \mathrm{d}\gamma = 0.$$

Since the span of the set $\Big\{ \prod_{j=1}^{s} \zeta_j^{l_j - k_j} \prod_{j=s+1}^{n} z_j^{m_j + l_j} \bar{z}_j^{m_j + k_j} \ : \ m, k, l \in \mathbb{N}^n \Big\}$ is dense in $C(\mathbb{T}^s \times \overline{\mathbb{D}}^{n-s})$, we conclude that $F(w) = 0$ for γ-a.e. w in $\mathbb{T}^s \times \mathbb{D}^{n-s}$. $\square$

Remark 2.6. It follows from Proposition 2.2 and Theorem 2.5 that if f belongs to $C(\overline{\mathbb{D}}^n)$, then T_f is compact if and only if $f(w) = 0$ for γ-a.e. w in $\partial\mathbb{D}^n$.

3. The commuting problem

The problem of commuting Toeplitz operators has attracted attention of many mathematicians. On the Hardy space of the unit circle, a well known theorem of A. Brown and P.R. Halmos asserts that if f, g are bounded functions then T_f and T_g commute if and only if one of the following statements holds true:

(a) Both f and g are holomorphic.
(b) Both $\bar{f}$ and $\bar{g}$ are holomorphic.
(c) There are constants a, b not both zero so that $af+bg$ is a constant function.

The situation on Bergman spaces turns out to be more complicated. The above Brown-Halmos's result fails. In fact, if f and g are bounded radial functions on $\mathbb{D}$ (i.e. $f(z) = f(|z|)$ and $g(z) = g(|z|)$ for $z \in \mathbb{D}$) then since T_f and T_g are diagonal operators with respect to the standard orthonormal basis, they commute. On the other hand, if both functions are assumed to be harmonic, then S. Axler and Ž. Čučković showed in [1] that Brown-Halmos's result remains valid. This result has been generalized to Toeplitz operators with pluriharmonic symbols on Bergman spaces of the unit ball by various authors. See [5],[8] and [11] for more details.

A different direction in studying the above commuting problem is to put conditions on only g and allow f to be an arbitrary bounded function. One may want to assume only harmonicity (or pluriharmonicity in higher dimensions) of g but as far as we know, there has been no progress on this direction, even on the Bergman space of the unit disk. In fact, it is not known what functions f give rise to operators T_f that commute with $T_{z+\bar{z}}$ on the Bergman space $A^2(\mathbb{D})$. On the other hand, some results have been known if one assume that g is holomorphic. (One may wish to consider the case $\bar{g}$ is holomorphic as well but by taking adjoints, we only need to work with the first case.) The following result was proved by Axler, Čučković and Rao in [2] when ν is the ordinary Lebesgue measure on the unit disk but their argument works also for any rotation-invariant measure ν.

Theorem 3.1. *Let ν be a rotation-invariant regular Borel probability measure on $\mathbb{D}$ such that $\nu(\mathbb{D}\backslash\mathbb{D}_r) > 0$ for any $0 < r < 1$. Suppose g is a non-constant bounded holomorphic function and f is an arbitrary bounded Borel function on $\mathbb{D}$. If T_f and T_g commute on $A^2_\nu(\mathbb{D})$, then there is a holomorphic function h on $\mathbb{D}$ so that $f(z) = h(z)$ for ν-a.e. z in $\mathbb{D}$.*

This result has recently been generalized to operators on Bergman spaces (corresponding to ordinary Lebesgue measure) of pseudoconvex domains in $\mathbb{C}^n$ by G. Cao [3]. The situation becomes different when one considers domains which are not pseudoconvex. The conclusion of Theorem 3.1 fails for Toeplitz operators on the Bergman space of the unit polydisk. In fact, S.H. Sun and D. Zheng [10] showed that if g and $\bar{f}$ are holomorphic on $\mathbb{D}^n$, then the following statements are equivalent:

(a) T_f and T_g commute (on $A^2_\vartheta(\mathbb{D}^n)$, where ϑ is the normalized Lebesgue measure on $\mathbb{D}^n$).
(b) T_f and T_g essentially commute, i.e. $T_fT_g - T_gT_f$ is a compact operator.
(c) $\partial_j\bar{f} = 0$ or $\partial_j g = 0$ for all $1 \leq j \leq n$.

In [4], B.R. Choe, H. Koo and Y.J. Lee studied the commuting problem with the assumption that f and g are pluriharmonic functions on $\mathbb{D}^n$. Their result generalized Sun and Zheng's result.

In the following proposition, we give a sufficient condition for the commutativity of T_g and T_f when g is holomorphic. This proposition shows that the implication $(c) \Rightarrow (a)$ in Sun and Zheng's result still holds true for arbitrary f.

Proposition 3.2. *Suppose g is a bounded holomorphic function and f is an arbitrary bounded function on $\mathbb{D}$. Assume that for each $1 \leq j \leq n$, if $\partial_j g$ is not identically zero, then f is holomorphic in the jth variable. Then T_f and T_g commute on A^2_ϑ.*

Proof. By symmetry, we may assume that $\partial_1 g, \ldots, \partial_s g$ are not identically zero and $\partial_{s+1} g = \cdots = \partial_n g$ are zero functions. So g is independent of $z_{s+1}, \ldots, z_n$. By assumption, f is holomorphic in the first s variables. Here, $1 \leq s \leq n$. For any holomorphic polynomial ψ and any z in $\mathbb{D}^n$, we have

$$
\begin{aligned}
&(T_f T_g \psi)(z)\\
&\quad = T_f(g\psi)(z)\\
&\quad = \int_{\mathbb{D}^n} f(w)g(w)\psi(w)\overline{K}_z(w)\mathrm{d}\vartheta(w)\\
&\quad = \int_{\mathbb{D}^s} \Big\{ \int_{\mathbb{D}^{n-s}} f(w)g(w)\psi(w) \prod_{j=s+1}^{n} \overline{K}_{z_j}(w_j)\mathrm{d}\nu_{s+1}(w_{s+1})\cdots \mathrm{d}\nu_n(w_n)\Big\}\\
&\qquad \times \prod_{j=1}^{s} \overline{K}_{z_j}(w_j)\mathrm{d}\nu_1(w_1)\cdots\mathrm{d}\nu_s(w_s)\\
&\quad = \int_{\mathbb{D}^{n-s}} f(z_1,\ldots,z_s,w_{s+1},\ldots,w_n)g(z_1,\ldots,z_s,w_{s+1},\ldots,w_n)\\
&\qquad \times \psi(z_1,\ldots,z_s,w_{s+1},\ldots,w_n) \prod_{j=s+1}^{n} \overline{K}_{z_j}(w_j)\mathrm{d}\nu_{s+1}(w_{s+1})\cdots\mathrm{d}\nu_n(w_n)\\
&\quad = g(z)\int_{\mathbb{D}^{n-s}} f(z_1,\ldots,z_s,w_{s+1},\ldots,w_n)\psi(z_1,\ldots,z_s,w_{s+1},\ldots,w_n)\\
&\qquad \times \prod_{j=s+1}^{n} \overline{K}_{z_j}(w_j)\mathrm{d}\nu_{s+1}(w_{s+1})\cdots\mathrm{d}\nu_n(w_n)\\
&\qquad (\text{since } g(z_1,\ldots,z_s,w_{s+1},\ldots,w_n) = g(z))\\
&\quad = g(z)\int_{\mathbb{D}^n} f(w)\psi(w)\overline{K}_z(w)\mathrm{d}\vartheta(w)\\
&\quad = (T_g T_f \psi)(z).
\end{aligned}
$$

Since the identity $T_f T_g(\psi) = T_g T_f(\psi)$ holds for ψ in a dense subset of A^2_ϑ, we conclude that $T_f T_g = T_g T_f$. □

We suspect that the converse of Proposition 3.2 is true but we have not been able to prove it. In the following theorem, we obtain a partial converse, when the holomorphic function g depends only on one variable.

Theorem 3.3. *Suppose g is a non-constant bounded holomorphic function on $\mathbb{D}$ so that $\partial_2 g = \cdots = \partial_n g = 0$. Suppose f is bounded on $\mathbb{D}^n$. If T_f commutes with T_g, then there is a function h on $\mathbb{D}^n$ such that $z_1 \mapsto h(z_1, z_2, \ldots, z_n)$ is holomorphic on $\mathbb{D}$ for all $(z_2, \ldots, z_n) \in \mathbb{D}^{n-1}$ and $f(z) = h(z)$ for ϑ-a.e. z in $\mathbb{D}^n$.*

Proof. For $z_1 \in \mathbb{D}$, put $\varphi(z_1) = g(z_1, 0, \ldots, 0)$. Then φ is a non-constant bounded holomorphic function on $\mathbb{D}$ and $\varphi(z_1) = g(z_1, \ldots, z_n)$ for all $z = (z_1, \ldots, z_n) \in \mathbb{D}^n$. Suppose ψ_1 is a bounded holomorphic function on $\mathbb{D}$ and ψ_2 is a bounded holomorphic function on $\mathbb{D}^{n-1}$. Put $\psi(w) = \psi_1(w_1)\psi_2(\tilde{w})$ for $w = (w_1, \tilde{w})$ with $w_1 \in \mathbb{D}$ and $\tilde{w} \in \mathbb{D}^{n-1}$. For any $z = (z_1, \tilde{z}) \in \mathbb{D} \times \mathbb{D}^{n-1} = \mathbb{D}^n$, we have

$$\begin{aligned}
&(T_f T_g \psi)(z)\\
&= \int_{\mathbb{D}^n} f(w) g(w) \psi(w) \overline{K}_z(w) \mathrm{d}\vartheta(w)\\
&= \int_{\mathbb{D}^n} f(w) \varphi(w_1) \psi(w) \overline{K}_z(w) \mathrm{d}\vartheta(w)\\
&= \int_{\mathbb{D}} \Big\{ \int_{\mathbb{D}^{n-1}} f(w_1, \tilde{w}) \psi_2(\tilde{w}) \overline{K}_{\tilde{z}}(\tilde{w}) \mathrm{d}\tilde{\vartheta}(\tilde{w}) \Big\} \varphi(w_1) \psi_1(w_1) \overline{K}_{z_1}(w_1) \mathrm{d}\nu_1(w_1)\\
&\quad (\text{here } \mathrm{d}\tilde{\vartheta}(\tilde{w}) = \mathrm{d}\nu_{s+1}(w_{s+1}) \cdots \mathrm{d}\nu_n(w_n))\\
&= \int_{\mathbb{D}} \Phi(w_1, \tilde{z}) \varphi(w_1) \psi_1(w_1) \overline{K}_{z_1}(w_1) \mathrm{d}\nu_1(w_1),
\end{aligned}$$

where $\Phi(w_1, \tilde{z}) = \int_{\mathbb{D}^{n-1}} f(w_1, \tilde{w}) \psi_2(\tilde{w}) \overline{K}_{\tilde{z}}(\tilde{w}) \mathrm{d}\tilde{\vartheta}(\tilde{w})$ for $(w_1, \tilde{z}) \in \mathbb{D} \times \mathbb{D}^{n-1}$. Also,

$$\begin{aligned}
&(T_g T_f \psi)(z)\\
&= g(z)(T_f \psi)(z)\\
&= \varphi(z_1) \int_{\mathbb{D}} \Big\{ \int_{\mathbb{D}^{n-1}} f(w_1, \tilde{w}) \psi_2(\tilde{w}) \overline{K}_{\tilde{z}}(\tilde{w}) \mathrm{d}\vartheta(\tilde{w}) \Big\} \psi_1(w_1) \overline{K}_{z_1}(w_1) \mathrm{d}\nu_1(w_1)\\
&= \varphi(z_1) \int_{\mathbb{D}} \Phi(w_1, \tilde{z}) \psi_1(w_1) \overline{K}_{z_1}(w_1) \mathrm{d}\nu_1(w_1).
\end{aligned}$$

Since T_f and T_g commute, for any $z_1 \in \mathbb{D}$ and $\tilde{z} \in \mathbb{D}^{n-1}$, we have

$$\begin{aligned}
&\int_{\mathbb{D}} \Phi(w_1, \tilde{z}) \varphi(w_1) \psi_1(w_1) \overline{K}_{z_1}(w_1) \mathrm{d}\nu_1(w_1)\\
&\qquad\qquad = \varphi(z_1) \int_{\mathbb{D}} \Phi(w_1, \tilde{z}) \psi_1(w_1) \overline{K}_{z_1}(w_1) \mathrm{d}\nu_1(w_1).
\end{aligned}$$

Since the above identity holds for all bounded holomorphic functions ψ_1 on $\mathbb{D}$, we conclude that T_φ and $T_{\Phi(\cdot, \tilde{z})}$ commute as operators on $A^2_{\nu_1}(\mathbb{D})$. By Theorem 3.1, for each $\tilde{z} \in \mathbb{D}^{n-1}$ the function $z_1 \mapsto \Phi(z_1, \tilde{z})$ is equal to a holomorphic function ν_1-a.e. on $\mathbb{D}$. So we have, for each $\tilde{z} \in \mathbb{D}^{n-1}$,

$$\Phi(z_1, \tilde{z}) = \int_{\mathbb{D}} \Phi(w_1, \tilde{z}) \overline{K}_{z_1}(w_1) \mathrm{d}\nu_1(w_1) \quad \nu_1\text{-a.e. } z_1 \in \mathbb{D}.$$

This is equivalent to

$$\begin{aligned}
&\int_{\mathbb{D}^{n-1}} f(z_1, \tilde{w}) \psi_2(\tilde{w}) \overline{K}_{\tilde{z}}(\tilde{w}) \mathrm{d}\vartheta(\tilde{w})\\
&\quad = \int_{\mathbb{D}} \int_{\mathbb{D}^{n-1}} f(w_1, \tilde{w}) \psi_2(\tilde{w}) \overline{K}_{\tilde{z}}(\tilde{w}) \mathrm{d}\tilde{\vartheta}(\tilde{w}) \overline{K}_{z_1}(w_1) \mathrm{d}\nu_1(w_1)\\
&\quad = \int_{\mathbb{D}^{n-1}} \Big\{ \int_{\mathbb{D}} f(w_1, \tilde{w}) \overline{K}_{z_1}(w_1) \mathrm{d}\nu_1(w_1) \Big\} \psi_2(\tilde{w}) \overline{K}_{\tilde{z}}(\tilde{w}) \mathrm{d}\tilde{\vartheta}(\tilde{w}).
\end{aligned}$$

Therefore, for each $\tilde{z} \in \mathbb{D}^{n-1}$, for ν_1-a.e. $z_1 \in \mathbb{D}$,

$$\int_{\mathbb{D}^{n-1}} \Big\{ f(z_1, \tilde{w}) - \int_{\mathbb{D}} f(w_1, \tilde{w}) \overline{K}_{z_1}(w_1) \mathrm{d}\nu_1(w_1) \Big\} \psi_2(\tilde{w}) \overline{K}_{\tilde{z}}(\tilde{w}) \mathrm{d}\tilde{\vartheta}(\tilde{w}) = 0.$$

Since the linear span of $\{K_{\tilde{z}} : \tilde{z} \in \mathbb{D}^{n-1}\}$ is dense in $A^2_{\tilde{\vartheta}}(\mathbb{D}^{n-1})$, we conclude that, for ν_1-a.e. $z_1 \in \mathbb{D}$,

$$\int_{\mathbb{D}^{n-1}} \Big\{ f(z_1, \tilde{w}) - \int_{\mathbb{D}} f(w_1, \tilde{w}) \overline{K}_{z_1}(w_1) \mathrm{d}\nu_1(w_1) \Big\} \psi_2(\tilde{w}) \bar{\eta}(\tilde{w}) \mathrm{d}\tilde{\vartheta}(\tilde{w}) = 0,$$

for any $\eta \in A^2_{\tilde{\vartheta}}(\mathbb{D}^{n-1})$. Now, since the linear span of the set

$$\big\{ \psi_2 \overline{\eta} : \psi_2, \eta \text{ are bounded holomorphic functions on } \mathbb{D}^{n-1} \big\}$$

contains all polynomials in $\tilde{w}$ and $\overline{\tilde{w}}$, it is dense in $L^1(\mathbb{D}^{n-1}, \mathrm{d}\tilde{\vartheta})$. It then follows that

$$f(z_1, \tilde{w}) = \int_{\mathbb{D}} f(w_1, \tilde{w}) \overline{K}_{z_1}(w_1) \mathrm{d}\nu_1(w_1),$$

for ϑ-a.e. $(z_1, \tilde{w}) \in \mathbb{D}^n$. The right-hand side of the above identity is a holomorphic function of z_1 for each $\tilde{w} \in \mathbb{D}^{n-1}$, so the conclusion of the theorem follows. □

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