TRACE CLASS TOEPLITZ OPERATORS WITH UNBOUNDED SYMBOLS ON WEIGHTED BERGMAN SPACES

TRIEU LE

Abstract. In this note we give an easy construction of functions on the unit ball that are not essentially bounded near any point on the unit sphere but the corresponding Toeplitz operators are all trace class on the Bergman space.

Let $n \geq 1$ be a fixed integer. As usual we write $\mathbb{B}_n$ (respectively, $\mathbb{S}_n$) for the unit ball (respectively, unit sphere) in $\mathbb{C}^n$.

Let μ be a regular Borel probability measure on the interval $[0, 1)$ such that $\mu([r, 1)) > 0$ for all $0 < r < 1$ and that μ has no atom. Let $d\nu = d\sigma \times d\mu$, where σ is the unique rotation-invariant probability Borel measure on $\mathbb{S}_n$. Then $d\nu$ is a regular positive Borel measure on $\mathbb{B}_n$ and for any function $f \in L^1(\mathbb{B}_n, d\nu)$ we have

$$\int_{\mathbb{B}_n} f(z)\ d\nu(z) = \int_{[0,1)} \Big(\int_{\mathbb{S}} f(r\zeta)\ d\sigma(\zeta) \Big) d\mu(r).$$

The Bergman space $A^2 = A^2(\mathbb{B}_n, d\nu)$ consists of all holomorphic functions that belong also to $L^2 = L^2(\mathbb{B}_n, d\nu)$. It is well known that A^2 is a closed subspace of L^2. Let P be the orthogonal projection from L^2 onto A^2. For any function $\varphi \in L^2$, the Toeplitz operator T_φ is defined by $T_\varphi h = P(\varphi h)$, for all h for which the function φh belongs to L^2. Since the range of T_φ contains the space of all holomorphic polynomials, T_φ is densely defined on A^2.

It is clear from the definition that if φ is a bounded function, then T_φ is a bounded operator on A^2 and $\|T_\varphi\| \leq \|\varphi\|_\infty$. It is also well known that if φ is compactly supported in $\mathbb{B}_n$, then T_φ extends to a compact operator on A^2. Many authors have constructed functions that are unbounded near the unit sphere to which the corresponding Toeplitz operators are in fact trace class. In [2], J. Cima and Ž. Čučković, among other things, constructed such functions in one dimension. In higher dimensions, G. Cao [1] and S. Huang and Cao [3] produced examples for weighted Lebesgue measure with radial weights.

The purpose of this note is to provide a construction that, I believe, is considerably easier and less explicit than those in the aforementioned papers. Our construction works for a wider class of measures, not just those that are absolutely continuous with respect to Lebesgue measure on $\mathbb{B}_n$.

Recall that a function φ on $\mathbb{B}_n$ is called radial if there is a function η defined on $[0, 1)$ such that $\varphi(z) = \eta(|z|)$ for all $z \in \mathbb{B}_n$.

Theorem 1. *There exists a non-negative radial function $\varphi \in L^2$ such that T_φ is of trace class in A^2 and for any $\zeta \in \mathbb{S}_n$ and $0 < \delta < 1$, $\operatorname{ess\,sup}\{|\varphi(z)| : |z - \zeta| < \delta\} = \infty$.*

In the proof of our main theorem, we make use of the following elementary results on measures.

Lemma 2. *Suppose γ is a positive regular Borel measure with no atom on a locally compact Hausdorff space $\mathcal{X}$. Suppose $\mathcal{M}$ is a subset of $\mathcal{X}$ such that $0 < \gamma(\mathcal{M}) < \infty$. Then for any $\epsilon > 0$, there is a compact subset $\mathcal{K}$ of $\mathcal{M}$ such that $0 < \gamma(\mathcal{K}) < \epsilon$.*

Proof. We recall the proof here for the reader's convenience. Replace $\mathcal{M}$ by a compact subset if necessary (using the regularity of γ), we may assume that $\mathcal{M}$ is compact. By regularity again, we may find a smallest compact subset $\widetilde{\mathcal{M}}$ such that $\gamma(\widetilde{\mathcal{M}}) = \gamma(\mathcal{M})$ (the set $\widetilde{\mathcal{M}}$ is just the support of the restriction of γ on $\mathcal{M}$). Since γ has no atom, $\widetilde{\mathcal{M}}$ does not have any isolated point, in particular, $\widetilde{\mathcal{M}}$ is an infinite set.

Choose a natural number s for which $\epsilon < \gamma(\mathcal{M})/s$. Choose s distinct points in $\widetilde{\mathcal{M}}$. Choose s pairwise disjoint neighborhoods $V_1, \ldots, V_s$ of the above s points. By the choice of $\widetilde{\mathcal{M}}$, $\gamma(\widetilde{\mathcal{M}} \cap V_j) > 0$ for all $j = 1, \ldots, s$. Now choose V to be the set of smallest measure among $\widetilde{\mathcal{M}} \cap V_1, \ldots, \widetilde{\mathcal{M}} \cap V_s$. Then $0 < \gamma(V) \le \gamma(\mathcal{M})/s < \epsilon$.

Using the regularity of γ, we may find a compact subset $\mathcal{K}$ of V such that $0 < \gamma(\mathcal{K}) \le \gamma(V) < \epsilon$. □

Corollary 3. *Let γ be a regular Borel positive measure on $[0, 1)$ that has no atom such that $\gamma([r, 1)) > 0$ and $\gamma([0, r)) < \infty$ for all $0 < r < 1$. Let $\epsilon > 0$ and $0 < a < 1$ be given. Then there exists a compact subset $\mathcal{K}$ of $[a, 1)$ such that $0 < \gamma(\mathcal{K}) < \epsilon$.*

Proof. Since $\gamma([a, 1)) > 0$ and $[a, 1) = \cup_{a<s<1}[a, s)$, there exists a number s such that $a < s < 1$ and $\gamma([a, s)) > 0$. By assumption, $\gamma([a, s))$ is finite. By Lemma 2, there is a compact subset $\mathcal{K}$ of $[a, s)$ such that $0 < \gamma(\mathcal{K}) < \epsilon$. □

Proof of main theorem. Recall that A^2 has a reproducing kernel function $K(z, w)$. Because of the rotation invariant property of ν, we have $K(Uz, w) = K(z, U^{-1}w)$ for any $z, w \in \mathbb{B}_n$ and any $n \times n$ unitary matrix U. It shows that the map $z \mapsto K(z, z)$ is radial. On the other hand, it is well known that $K(z, z)$ is bounded when z belongs to any ball centered at the origin of radius strictly less than 1.

Let $\{e_m : m \ge 0\}$ be any orthonormal basis. As well known,

$$\begin{aligned} K(z, z) = \langle K_z, K_z\rangle &= \sum_{m=0}^{\infty} \langle K_z, e_m\rangle\langle e_m, K_z\rangle \\ &= \sum_{m=0}^{\infty} \bar{e}_m(z)e_m(z) = \sum_{m=0}^{\infty} |e_m(z)|^2. \end{aligned}$$

For any positive function φ, since T_φ is a positive operator, we have

$$\begin{aligned} \operatorname{tr}(T_\varphi) = \sum_{m=0}^{\infty} \langle T_\varphi e_m, e_m\rangle &= \sum_{m=0}^{\infty} \int_{\mathbb{B}_n} \varphi |e_m|^2 \, d\nu \\ &= \int_{\mathbb{B}_n} \varphi(z)\Big(\sum_{m=0}^{\infty} |e_m(z)|^2\Big)\, d\nu(z) \\ &= \int_{\mathbb{B}_n} \varphi(z)K(z, z)\, d\nu(z) \\ &= \int_{[0,1)} \Big(\int_{\mathbb{S}_n} \varphi(r\zeta)\, d\sigma(\zeta)\Big)K(r, r)\, d\mu(r). \end{aligned}$$

If φ is radial such that $\varphi(z) = \eta(|z|)$ then we have

$$\mathrm{tr}(T_\varphi) = \int_{[0,1)} \eta(r) K(r,r)\, d\mu(r). \tag{1}$$

We now construct a function η on $[0,1)$ that is μ-essentially unbounded near 1 but is integrable with respect to $d\gamma(r) = K(r,r)d\mu(r)$. Note that since $r \mapsto K(r,r)$ is bounded and non-vanishing on any closed subinterval of $[0,1)$, the measure γ satisfies the hypothesis of Corollary 3.

For each $m \geq 1$, Corollary 3 (applied to the measure $\gamma + \mu$) gives a compact subset $\mathcal{K}_m$ of the interval $[1 - \frac{1}{m}, 1)$ with $0 < \gamma(\mathcal{K}_m) \leq \frac{1}{m^3}$ and $0 < \mu(\mathcal{K}_m) \leq \frac{1}{m^4}$. We may arrange so that these compact sets are not overlapping. Now put

$$\eta = \sum_{m=1}^{\infty} m \chi_{\mathcal{K}_m}.$$

Then η is square integrable with respect to μ on $[0,1]$ and we have

$$\int_{[0,1)} \eta(r) K(r,r)\, d\mu(r) = \int_{[0,1)} \eta(r)\, d\gamma(r) = \sum_{m=1}^{\infty} m \int_{\mathcal{K}_m} d\gamma(r) \leq \sum_{m=1}^{\infty} \frac{1}{m^2} < \infty.$$

Put $\varphi(z) = \eta(|z|)$ for $z \in \mathbb{B}_n$. Then $\varphi \in L^2$ and hence T_φ is densely defined. By (1) we see that T_φ extends to an operator in trace class. On the other hand, for any $\zeta \in \mathbb{S}_n$ and any $\delta > 0$, the set $\{ru : r \in [1 - \delta/2, 1),\ u \in \mathbb{S}_n \text{ with } |u - \zeta| < \delta/2\}$ is contained in the ball of radius δ centered at ζ. Therefore,

$$\begin{aligned} &\text{ess sup}\{|\varphi(z)| : |z - \zeta| < \delta\} \\ &\qquad \geq \text{ess sup}\{|\varphi(ru)| : r \in [1 - \delta/2, 1),\ u \in \mathbb{S}_n \text{ with } |u - \zeta| < \delta/2\} = \infty. \ \square \end{aligned}$$

References

[1] Guangfu Cao, *Toeplitz operators with unbounded symbols of several complex variables*, J. Math. Anal. Appl. **339** (2008), no. 2, 1277–1285. MR 2377084 (2008k:47066)

[2] Joseph A. Cima and Željko Čučković, *Compact Toeplitz operators with unbounded symbols*, J. Operator Theory **53** (2005), no. 2, 431–440. MR 2153157 (2006e:47060)

[3] Sui Huang and Guang Fu Cao, *Trace class Toeplitz operators with unbounded symbols on weighted Bergman spaces*, Acta Math. Sin. (Engl. Ser.) **26** (2010), no. 8, 1567–1574. MR 2661136

Department of Mathematics, Mail Stop 942, University of Toledo, Toledo, OH 43606

E-mail address: `trieu.le2@utoledo.edu`