MAŁGORZATA ANETA MARCINIAK

University of Toledo

1 A Very Short Introduction to Toric Varieties

The theory of toric varieties was introduced in the early 1970s and since that time has progressed far; today it is still active, providing a basis for fresh ideas. Toric varieties give rise to interesting applications with their rich structure and relatively easy combinatorics. However, toric varieties are normal, rational, and not necessarily projective, which makes them good candidates for examples or counter-examples in a wider class of varieties. Examples of their benefits are:

- Toric varieties are trivial from the minimal model theory point of view; however, they offer an excellent means to explain its main ideas.

- Fano toric varieties are easier to handle, but they are still an interesting subclass of Fano varieties.

- A pair of mirror Calabi-Yau threefolds can be constructed using "reflexive" polytopes.

- Other benefits are related to combinatorial geometry, error-correcting codes, Gromov-Witten invariants, Lagrangian torus fibrations, symplectic geometry, etc.

1.1 Affine Toric Varieties

Affine toric varieties play basically the same role for toric varieties as open subsets of $\mathbb{C}^n$ (or $\mathbb{R}^n$) for analytic (real) varieties. An affine toric variety can be associated with a cone.

1.1.1. Lattices and Cones. Consider an r-dimensional lattice N that can be identified with $\mathbb{Z}^r$, and let $M = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ be its dual lattice. We can define scalar extensions of N and M as: $N_{\mathbb{R}} = N \otimes_{\mathbb{R}} \mathbb{R}$ and $M_{\mathbb{R}} = M \otimes_{\mathbb{R}} \mathbb{R}$. Figure 1.1 shows an example of a cone. Now, the definition of a cone can be provided.

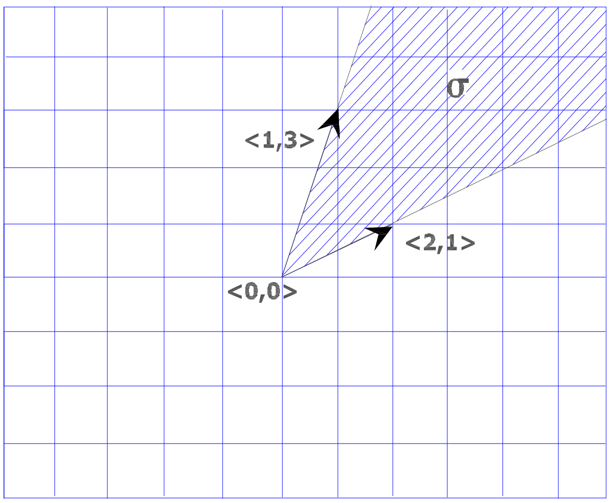

Figure 1.1: The cone $\sigma = (2e_1 + e_2)\mathbb{R}_{\geq 0} + (e_1 + 3e_2)\mathbb{R}_{\geq 0} \subset \mathbb{R}^2$

Definition 1.1.1 ***(Rational polyhedral cones)*** *A subset $\sigma \subset N_{\mathbb{R}}$ is a rational polyhedral cone with apex at the origin* 0 *if there are $a_1, ..., a_s \in N$ such that*

$$\sigma = a_1\mathbb{R}_{\geq 0} + ... + a_s\mathbb{R}_{\geq 0} = \{a_1 t_1 + ... + a_s t_s : \forall_{1\leq j\leq s}\, t_j \in \mathbb{R}_{\geq 0}\},$$

where $\mathbb{R}_{\geq 0}$ is a set of nonnegative real numbers. A cone σ is strictly convex if it is convex as a subset of $N_{\mathbb{R}}$ and does not contain a straight line.

If a point $p \in \sigma$ has a representation $p = a_1 t_1 + \ldots + a_s t_s$ and all $t_j > 0$, for $1 \leq j \leq s$, then p belongs to the relative interior of σ.

In further discussions, except where clarification is needed, we will refer to strictly convex rational polyhedral cones simply as cones. It is important to imagine how cones look. The origin $\{0\} \subset N_{\mathbb{R}}$ is a cone. It can be represented as $\sigma = 0\mathbb{R}_{\geq 0}$ and in further discussion the cone $0\mathbb{R}_{\geq 0}$ will be denoted as 0. Since the lattice M is dual to N, there is a dual product denoted as $(,) : N \times M \longrightarrow \mathbb{Z}$.

Definition 1.1.2 ***(Dual cones)*** *For any cone $\sigma \subset N_{\mathbb{R}}$, we can define its dual cone as $\sigma^{\vee} \subset M_{\mathbb{R}}$: $\sigma^{\vee} = \{u \in M_{\mathbb{R}} : \forall_{v \in \sigma}\, (v, u) \geq 0\}$.*

Figure 1.2 shows an example of a cone and its dual.

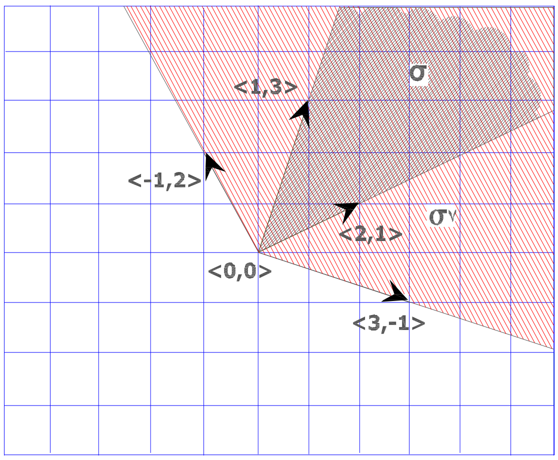

Figure 1.2: A cone and its dual

Example 1.1.1 Consider the cone $\sigma = (2e_1 + e_2)\mathbb{R}_{\geq 0} + (e_1 + 3e_2)\mathbb{R}_{\geq 0} \subset \mathbb{R}^2$. Then its dual is $\sigma^{\vee} = (3e_1^* - e_2^*)\mathbb{R}_{\geq 0} + (-e_1^* + 2e_2^*)\mathbb{R}_{\geq 0} \subset \mathbb{R}^2$. ▮

Notice that if $\sigma \subset N_{\mathbb{R}}$ is a strictly convex rational polyhedral cone with apex at the origin then $\sigma^{\vee} \subset M_{\mathbb{R}}$ is a rational polyhedral cone with apex at the origin, but it is not necessarily strictly convex. As an example, consider the zero cone $0 \subset N_{\mathbb{R}}$. Then $(0)^{\vee} = \{u \in M_{\mathbb{R}} : \forall_{v \in 0}\, (v, u) \geq 0\} = \{u \in M_{\mathbb{R}} : (u, 0) \geq 0\} = M_{\mathbb{R}}$.

1.1.2. Semigroups and Gordan's Lemma. The dual cone $\sigma^\vee$ allows us to define a semigroup $S_\sigma = \sigma^\vee \cap M$ associated with cone σ. The semigroup S_σ is, in fact, finitely generated, which is a key condition in the theory of toric varieties. Consider the following lemma:

Lemma 1.1.1 *(Gordan's lemma)* ***([1], Lec. 1, Prop. 5.4)*** *If σ is a rational polyhedral cone, then S_σ is a finitely generated additive semigroup, i.e., there exists $m_1, \ldots, m_t \in S_\sigma$ so that*

$$S_\sigma = m_1\mathbb{Z}_{\geq 0} + \ldots + m_t\mathbb{Z}_{\geq 0}. \blacksquare$$

Figure 1.3 shows the generators of the semigroup.

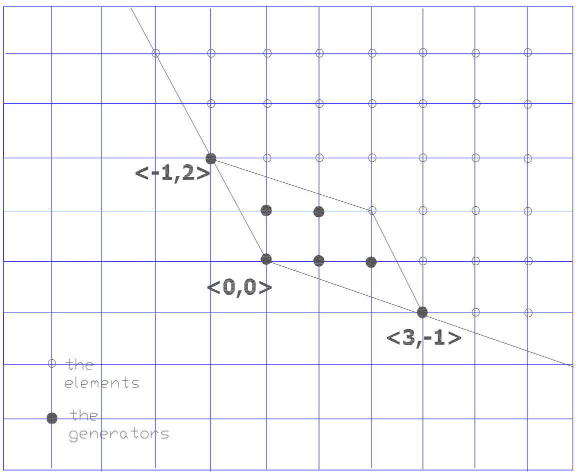

Figure 1.3: The semigroup and its generators

1.1.3. Semigroup Algebras and Toric Ideals. Any finitely generated semigroup S_σ defines $\mathbb{C}$-algebra $\mathbb{C}[S_\sigma]$ as follows. With an element $u \in S_\sigma$ we associate an element $\chi_u \in \mathbb{C}[S_\sigma]$, which we call a character. If $u = u_1 e_1^* + \ldots + u_n e_n^*$, and if $(t_1, \ldots, t_n)$ are local coordinates, then

$$\chi_u(t_1, \ldots, t_n) = t_1^{u_1} \ldots t_n^{u_n}$$

The algebra $\mathbb{C}[S_\sigma]$ is generated by characters $\{\chi_{u_i}\}_{i \in I}$, where $\{u_i\}_{i \in I}$ are generators of S_σ. Any element of $\mathbb{C}[S_\sigma]$ is a finite linear combination of the form $\sum_{i \in I} n_i \chi_{u_i}$, where $n_i \in \mathbb{C}$. Notice that for any $u_1, u_2 \in S_\sigma$, we have $\chi_{u_1} \cdot \chi_{u_2} = \chi_{u_1 + u_2}$. The following examples show some important cones and their algebras.

Example 1.1.2 Consider $0 \in N_\mathbb{R}$, where $\dim N_\mathbb{R} = n$. Then $0^\vee = \{u \in M_\mathbb{R} : (u, 0) \geq 0\} = M_\mathbb{R}$, so $S_0 = M$ and $\mathbb{C}[S_0] = \mathbb{C}[M] = \mathbb{C}[\mathbb{Z}^n] = \mathbb{C}[z_1, \ldots, z_n, \frac{1}{z_1 \ldots z_n}]$. Notice that the algebra $\mathbb{C}[\mathbb{Z}^n]$ can be equivalently written as $\mathbb{C}[z_1, \ldots, z_n, \frac{1}{z_1}, \ldots, \frac{1}{z_n}]$, which depends on a choice of generators of $\mathbb{Z}^n$.

Example 1.1.3 For $\sigma = \mathbb{R}_{\geq 0} \subset \mathbb{R}$, we have $\sigma^\vee = \mathbb{R}_{\geq 0}$ and $\mathbb{C}[S_\sigma] = \mathbb{C}[\mathbb{N}] = \mathbb{C}[z]$. For $\sigma = e_1 \mathbb{R}_{\geq 0} + \ldots + e_n \mathbb{R}_{\geq 0} \subset \mathbb{R}^n$, we have $\sigma^\vee = e_1 \mathbb{R}_{\geq 0} + \ldots + e_n \mathbb{R}_{\geq 0} \subset \mathbb{R}^n$ (where $e_1, \ldots, e_n$ is a standard basis of $\mathbb{R}^n$) and $\mathbb{C}[S_\sigma] = \mathbb{C}[\mathbb{N}^n] = \mathbb{C}[z_1, \ldots, z_n]$.▮

With any algebra $\mathbb{C}[S_\sigma]$ defined by a cone (or with any cone $\sigma \subset N = \mathbb{Z}^n$), we can associate a toric ideal $\mathcal{I}_\sigma$. As noted above, $\mathbb{C}[S_\sigma]$ is generated by characters $\{\chi^{u_i}\}_{i\in I}$, where $\{u_i\}_{i\in I}$ are generators of S_σ. Therefore, the ideal $\mathcal{I}_\sigma$ expresses relations between generators of $\mathbb{C}[S_\sigma]$. Notice that linear relations between elements from S_σ: $\sum a_i u_i = \sum b_j u_j$, where $a_i, b_j \in \mathbb{Z}_{>0}$, turn into multiplicative relations between elements of $\mathbb{C}[S_\sigma]$: $\prod \chi_{u_i}^{a_i} = \prod \chi_{u_j}^{b_j}$. On the other hand, a toric ideal $\mathcal{I}_\sigma$ is a kernel of the homomorphism $\mathbb{C}[\mathbb{N}^k] \to \mathbb{C}[S_\sigma]$, where k is a number of generators of S_σ. The next example shows how to obtain $\mathcal{I}_\sigma$ as a kernel and specifically, how to obtain it from linear relations, which are, in fact, the same thing.

Example 1.1.4 Let $\sigma^\vee = (3e_1 - 1e_2)\mathbb{R}_{\geq 0} + (-1e_1 + 2e_2)\mathbb{R}_{\geq 0} \subset \mathbb{R}^2$. Then $\mathbb{C}[S_\sigma] = \mathbb{C}[x, y, \frac{x^3}{y}, \frac{y^2}{x}]$, and the kernel of the homomorphism $\mathbb{C}[a,b,c,d] \xrightarrow{\phi} \mathbb{C}[x, y, \frac{x^3}{y}, \frac{y^2}{x}]$, which sends $a \mapsto x$, $b \mapsto y$, $c \mapsto \frac{x^3}{y}$, $d \mapsto \frac{y^2}{x}$, is generated by $cb - a^3$ and $da - b^2$. Thus $\mathcal{I}_\sigma = (cb - a^3, da - b^2)$. For linear relations from S_σ, its generators can be chosen as: $e_1^*, e_2^*, 3e_1^* - e_2^*, -e_1^* + 2e_2^*$ with relations: $(3e_1^* - e_2^*) + e_2^* = 3e_1^*$ and $(-e_1^* + 2e_2^*) + e_1^* = 2e_2^*$. Using notation $\chi_{e_1^*} = a$, $\chi_{e_2^*} = b$, $\chi_{3e_1^* - e_2^*} = c$, $\chi_{-e_1^* + 2e_2^*} = d$, we obtain the multiplicative relations $cb = a^3$ and $da = b^2$.

1.1.4. Affine Toric Varieties. From this point an affine toric variety U_σ associated with a cone σ can be defined in many equivalent ways. Most convenient in the present context is to define U_σ as a set of zeros of generators of a toric ideal $\mathcal{I}_\sigma$. This approach treats U_σ is treated as an algebraic set in $\mathbb{C}^{n_\sigma}$, where n_σ is the number of generators of the semigroup S_σ. Equivalently, points of U_σ could be identified with homomorphisms $\mathbb{C}[S_\sigma] \to \mathbb{C}$ or with maximal ideals of algebra $\mathbb{C}[S_\sigma]$. Our final object not only consists of an affine toric variety U_σ, but it is a pair $(U_\sigma, \mathbb{C}[S_\sigma])$ of an affine toric variety U_σ and its algebra of regular functions.

Definition 1.1.3 ***(Algebraic variety associated with a cone)*** *Let $\sigma \subset N_\mathbb{R}$ be a cone. The algebraic variety U_σ associated with σ is defined as a set of zeros of polynomials of the form*

$$\left\{\prod \chi_{u_i}^{a_i} - \prod \chi_{u_j}^{b_j}\right\}, \qquad \textit{where} \qquad \left\{\sum a_i u_i = \sum b_j u_j,\ a_i, b_j \in \mathbb{Z}_{>0}\right\}$$

are relations between the generators of the semigroup $S_\sigma = \sigma^\vee \cap M$.

Example 1.1.5 For $0 \subset N_{\mathbb{R}}$, where $\dim N_{\mathbb{R}} = n$, we have $\mathbb{C}[S_0] = \mathbb{C}[z_1, \ldots, z_n, \frac{1}{z_1 \ldots z_n}]$. Thus, $\mathcal{I}_0 = (z_1 \ldots z_{n+1} - 1)$ and $U_0 = (\mathbb{C}^*)^n$. Because 0 is a special cone and its affine toric variety $U_0 = (\mathbb{C}^*)^n$ plays a crucial role in the theory of toric varieties, we will use notation $U_0 = (\mathbb{C}^*)^n = T^n$ and call T^n an algebraic torus of dimension n.

Example 1.1.6 Consider $\sigma = e_1\mathbb{R}_{\geq 0} + \ldots + e_n\mathbb{R}_{\geq 0} \subset N_{\mathbb{R}}$, where $e_1, ..., e_n$ is a basis of $N_{\mathbb{R}} = \mathbb{R}^n$. Then $\sigma^{\vee} = \{u \in M_{\mathbb{R}} : \forall_{v \in \sigma} (u, v) \geq 0\} = e_1^*\mathbb{R}_{\geq 0} + \ldots + e_n^*\mathbb{R}_{\geq 0} = M_{\mathbb{R}}$, where $e_1^*, ..., e_n^*$ is a dual basis in $M_{\mathbb{R}} = \mathbb{R}^n$. Thus, $S_\sigma = \mathbb{N}^n$ and $\mathbb{C}[S_\sigma] = \mathbb{C}[\mathbb{N}^n] = \mathbb{C}[z_1, ..., z_n]$. The ideal is $\mathcal{I}_\sigma = (0)$, and we finally obtain $U_\sigma = \mathbb{C}^n$.

Example 1.1.7 Let $\sigma = e_1\mathbb{R}_{\geq 0} + ... + e_d\mathbb{R}_{\geq 0} \subset N_{\mathbb{R}}$, where $e_1, ..., e_d$ is a part of a basis of $N_{\mathbb{R}} = \mathbb{R}^n$ and $d < n$. Then $\sigma^{\vee} = \{u \in M_{\mathbb{R}} : \forall_{v \in \sigma} (u, v) \geq 0\} = e_1^*\mathbb{R}_{\geq 0} + ... + e_d^*\mathbb{R}_{\geq 0} + e_{d+1}^*\mathbb{R}_{\geq 0} + (-e_{d+1}^*)\mathbb{R}_{\geq 0} + \ldots + e_n^*\mathbb{R}_{\geq 0} + (-e_n^*)\mathbb{R}_{\geq 0} \subset M_{\mathbb{R}}$; therefore, $S_\sigma = \mathbb{N}^d \times \mathbb{Z}^{n-d}$ and $\mathbb{C}[S_\sigma] = \mathbb{C}[\mathbb{N}^d \times \mathbb{Z}^{n-d}] = \mathbb{C}[z_1, \ldots, z_n, \frac{1}{z_{d+1} \cdots z_n}]$. Then $\mathcal{I}_\sigma = (z_{d+1} \ldots z_{n+1} - 1)$ and $U_\sigma = \mathbb{C}^d \times (\mathbb{C}^*)^{n-d}$.∎

1.2 GLUING AFFINE TORIC VARIETIES

This Section explains how to glue affine toric varieties along open and dense subsets which are, in fact, affine toric varieties. These subvarieties are related to faces of a cone.

1.2.1. Faces. Any face of a cone is determined by a hyperplane and a half-space. First, therefore, we recall their definitions. For $0 \neq u \in M_{\mathbb{R}}$, we define a hyperplane $H_u = \{v \in N_{\mathbb{R}} : (u, v) = 0\}$ and the half-space $H_u^+ = \{v \in N_{\mathbb{R}} : (u, v) \geq 0\}$.

Definition 1.2.1 ***(Face of a cone)*** *A subset $\tau \subset \sigma$ is a face of σ if $\tau = H_u \cap \sigma$ for some $0 \neq u \in M_{\mathbb{R}}$ and $\sigma \subset H_u^+$. We will use notation $\tau \prec \sigma$ for faces of σ.*

In the following theorems the cone σ is considered its own face. Hyperplanes and half-spaces define not only faces of a cone, but the whole cone. Obviously, the finite intersection of (closed) half-spaces is a convex polyhedral cone, but there is a much stronger result, which claims that any n-dimensional cone is an intersection of half-spaces determined by its $(n-1)$-dimensional faces:

Theorem 1.2.1 ***([1], Lec. 1, Prop. 3.4)*** *Let σ be a convex n-dimensional cone and let τ_i, $i = 1, ..., k$ be its $(n-1)$-dimensional faces, such that $\tau_i = \sigma \cap H_{u_i}$ for some collection of $u_i \in M_{\mathbb{R}}$. Then $\sigma = \bigcap_{i=1}^{k} H_{u_i}^{+}$.* ▮

Of course, any face is a cone itself; and σ_0 is a face of any cone. The natural questions are: Which affine toric varieties are associated with faces? And how are they related to the affine variety defined by the cone? First, we define a dual to the face τ.

Definition 1.2.2 *(**Dual faces**) Let $\tau \prec \sigma$; then the dual to τ is: $\tau^* = \{u \in \sigma^\vee : \forall_{v \in \tau} (u, v) = 0\}$.*

Proposition 1.2.1 ***([1], Lec. 1, Prop. 3.6)*** *If τ^* is a face of $\sigma^\vee$, then the correspondence $\tau \to \tau^*$ between faces of σ and faces of $\sigma^\vee$ is 1-1.* ▮

1.2.2. Fans and Toric Varieties. This subsection provides the definition of a fan, which is a set of cones. This definition allows us to glue affine toric varieties. Notice that the cone $\{0\}$ belongs to any fan. Figure 1.4 shows an example of a fan.

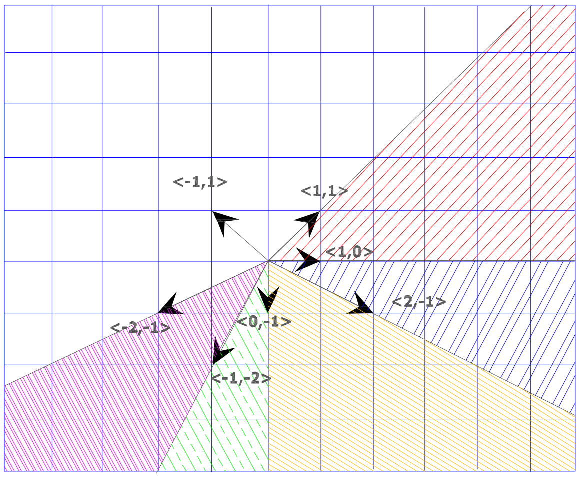

Figure 1.4: Example of a fan in $\mathbb{R}^2$

Definition 1.2.3 ***(Fan)*** *Let N be a lattice. A fan (Σ, N) is a finite, nonempty set of strictly convex rational polyhedral cones in $N_{\mathbb{R}}$ satisfying the following conditions:*

1. *If $\sigma \in \Sigma$ and $\tau \prec \sigma$, then $\tau \in \Sigma$.*

2. *If $\sigma_1, \sigma_2 \in \Sigma$, then $\sigma_1 \cap \sigma_2 \prec \sigma_1$ and $\sigma_1 \cap \sigma_2 \prec \sigma_2$.*

We say that Π is a subfan of a fan Σ if Π is a fan and $\Pi \subset \Sigma$.

The following two propositions prepare us to glue affine toric varieties along common affine toric subvarieties.

Proposition 1.2.2 ***([1], Lec. 1, Prop. 5.6)*** *If* $\tau_1, \tau_2 \prec \sigma$ *are faces such that* $\tau_1 \cap \tau_2 \prec \sigma$*, then* $S_{\tau_1 \cap \tau_2} = S_{\tau_1} + S_{\tau_2}$*.*▌

Here, the notation $(U_{\sigma_1})_{\chi_u}$ is used to describe the subset of U_{σ_1}, where the character χ_u does not vanish. Similarly, $(U_{\sigma_2})_{\chi_{-u}}$ describes the subset of U_{σ_2}, where the character χ_{-u} does not vanish.

Proposition 1.2.3 ***([12], Prop. 1.3)*** *If* $\tau \prec \sigma_1$ *and* $\tau \prec \sigma_2$*, then both* $U_\tau \hookrightarrow U_{\sigma_2}$ *and* $U_\tau \hookrightarrow U_{\sigma_1}$ *are open embeddings, and*

$$\tau = H_u \cap \sigma_1 \quad \textit{for} \quad u \in S_{\sigma_1} \quad \textit{and} \quad \tau = H_{-u} \cap \sigma_2 \quad \textit{for} \quad -u \in S_{\sigma_2};$$

therefore,

$$U_\tau = (U_{\sigma_1})_{\chi_u} \subset U_{\sigma_1} \quad \textit{and} \quad U_\tau = (U_{\sigma_2})_{\chi_{-u}} \subset U_{\sigma_2}.$$▌

Using analytic language, the propositions state that if $\varphi_1 : U_\tau \to U_{\sigma_1}$ and $\varphi_2 : U_\tau \to U_{\sigma_2}$ are open embeddings, then the images of points from U_τ can be identified. Therefore, the map is determined by $\varphi_2 \circ \varphi_1^{-1} : \varphi_1(U_\tau) \to \varphi_2(U_\tau)$, where $\varphi_2 \circ \varphi_1^{-1}$ is an n-tuple of Laurent monomials (i.e. $\varphi_2 \circ \varphi_1^{-1}(z_1, \ldots, z_n) = \left(z_1^{\alpha_{1,1}} \ldots z_n^{\alpha_{1,n}}, \ldots, z_1^{\alpha_{n,1}} \ldots z_n^{\alpha_{n,n}}\right)$ with $\alpha_{i,j} \in \mathbb{Z}$ for $i, j = 1, \ldots, n$ and $\det(\alpha_{i,j}) = \pm 1$).

The following definition ([12], Theorem 1.4) clarifies this idea.

Definition 1.2.4 ***(Toric variety)*** *Let (Σ, N) be a fan. Then the toric variety X_Σ associated with Σ is defined as follows. For any cone $\sigma \in \Sigma$, take an affine toric variety U_σ with its algebra of regular functions $\mathbb{C}[S_\sigma]$. And for such a collection $\{U_\sigma, \mathbb{C}[S_\sigma]\}_{\sigma\in\Sigma}$, notice that conditions described above imply that affine toric varieties can be glued along affine toric varieties associated with their common faces. This construction gives the toric variety associated with the fan Σ.*

Toric varieties are Hausdorff complex analytic spaces as described in [12], Theorem 1.4. Moreover, nonsingularity conditions of a toric variety can be expressed in terms of the fan. In the next theorem, $\mathbb{Z}$-basis means a basis with coefficients in $\mathbb{Z}$ that is also invertible over $\mathbb{Z}$.

Theorem 1.2.2 ***([12], Theorem 1.10)*** *The toric variety X_Σ associated with a fan Σ in N is nonsingular, i.e., a complex manifold, if and only if for each $\sigma \in \Sigma$ there exists a $\mathbb{Z}$-basis $\{n_1, \ldots, n_r\}$ of N and $s \leq r$ such that $\sigma = n_1\mathbb{R}_{\geq 0} + \ldots + n_s\mathbb{R}_{\geq 0}$.* ▮

1.2.3. Torus Action and Orbit Decomposition. The cone $\{0\} \in \mathbb{R}^n$ is a face of any cone and belongs to any fan. Thus, any toric variety contains an algebraic torus $T^n = U_{\{0\}} = (\mathbb{C}^*)^n$ as an open and dense subset ([7], Part 2, Section VI, Lemma 3.4). The algebraic torus T^n admits a structure of a multiplicative group and acts on itself by transitions. For $t = (t_1, \dots, t_n) \in T^n$ and $z = (z_1, \dots, z_n) \in T^n$, the multiplication $t \cdot z$ is defined as:

$$t \cdot z = (t_1 z_1, \dots, t_n z_n) \in T^n$$

Moreover, the action can be extended naturally and continuously to the whole toric variety X_Σ as described in [7], Part 2, Section VI, Theorem 5.2 and 5.3.

Definition 1.2.5 ***(An orbit)*** *Let G be a group that acts on a set X. An orbit O_p of a point $p \in X$ is defined as follows:*

$$O_p = \{x \in X : x = g \cdot p \quad \text{for some} \quad g \in G\}$$

where $g \cdot p$ describes an action of $g \in G$ on $p \in X$.

Since the torus T^n itself is an open orbit, other orbits are contained in its closure. (See [7], Part 2, Section VI, Theorem 5.3.) On toric varieties, the orbits are described by the cones and their faces. Let O_τ be an orbit defined by a cone $\tau \in \Sigma$. Then the orbit defined by τ is a torus as well, but of lower dimension:

Lemma 1.2.1 ***([2], Lecture 5, Lemma 1.2)*** *For $\tau \in \Sigma \subset N$ with* $\dim N = n$, $\dim O_\tau + \dim\tau = n$ *and* $O_\tau \simeq \mathbb{C}^{n-\dim\tau}$.▮

There are no orbits in X_Σ other than those defined by the cones $\tau \in \Sigma$:

Lemma 1.2.2 ***([2], Lecture 5, Lemma 1.3)*** *Every orbit of the torus action on X_Σ is of the form O_τ for some $\tau \in \Sigma$.*▮

Notice that the closures of orbits $V(\tau) = \overline{O}_\tau$ consist of tori of lower dimension than $\dim O_\tau$ and are invariant subsets of X_Σ. Particularly, the closure of the open orbit T^n is the whole toric variety X_Σ.

Theorem 1.2.3 ***([2], Lecture 5, Theorem 1.9)*** *The orbits O_τ, the orbits closures $V(\tau)$, and the affine open subset U_σ of a toric variety X_Σ are related as follows:*

(i) $U_\sigma = \bigcup_{\tau \prec \sigma} O_\tau$;

(ii) $V(\tau) = \bigcup_{\tau \prec \gamma} O_\gamma$;

(iii) $O_\tau = V(\tau) \setminus \bigcup_{\tau \prec \gamma} V(\gamma)$. ▮

Consider the following example of a smooth 2-dimensional toric variety E_k with a projective curve defined by the cone $e_2\mathbb{R}_{\geq 0}$. The fan of E_2 is shown in Figure 1.5.

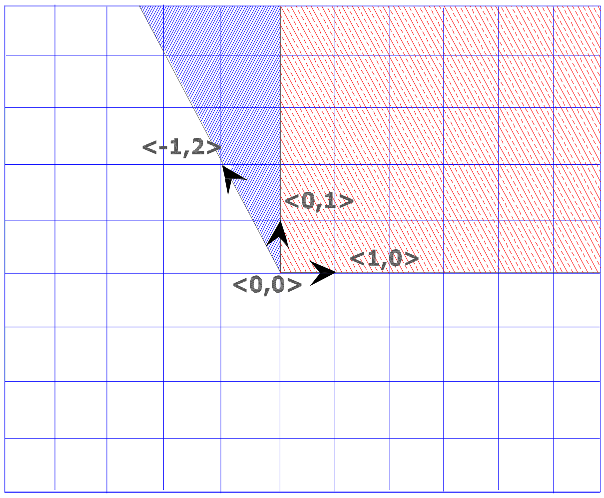

Figure 1.5: The fan of the toric variety E_2

Example 1.2.1 The toric variety E_k for $k \in \mathbb{Z}$ is described by the fan:

$$\Sigma = \{0, e_1\mathbb{R}_{\geq 0}, e_2\mathbb{R}_{\geq 0}, (-1e_1 + ke_2)\mathbb{R}_{\geq 0}, e_1\mathbb{R}_{\geq 0} + e_2\mathbb{R}_{\geq 0}, e_2\mathbb{R}_{\geq 0} + (-1e_1 + ke_2)\mathbb{R}_{\geq 0}\}.$$

The variety E_k consists of two patches X_0 and X_1, associated respectively with 2-dimensional cones $\sigma_0 = e_1\mathbb{R}_{\geq 0} + e_2\mathbb{R}_{\geq 0}$ and $\sigma_1 = e_2\mathbb{R}_{\geq 0} + (-1e_1 + ke_2)\mathbb{R}_{\geq 0}$. The coordinates $(z, w) \in X_0 \simeq \mathbb{C}^2$ and $(z_1, w_1) \in X_1 \simeq \mathbb{C}^2$ are related on $X_0 \cap X_1 \simeq \mathbb{C}^* \times \mathbb{C}^1$ according to the rule:

$$z_1 = \frac{1}{z} \quad \text{and} \quad w_1 = z^k w$$

E_k contains a projective curve, which is the orbit closure of O_τ with $\tau = e_2\mathbb{R}_{\geq 0}$. Since τ is a face of the cones $\sigma_0 = e_1\mathbb{R}_{\geq 0} + e_2\mathbb{R}_{\geq 0}$ and $\sigma_1 = e_2\mathbb{R}_{\geq 0} + (-1e_1 + ke_2)\mathbb{R}_{\geq 0}$, we obtain $V(\tau) \simeq \mathbb{P}^1$.

1.2.4. Mappings Between Toric Varieties. For a complete view on toric varieties, we must define mappings between them and maps between the associated fans.

Definition 1.2.6 ***(Map of fans)*** *$\varphi : (\Delta_1, N_1) \to (\Delta_2, N_2)$ is a map of fans if it is a $\mathbb{Z}$ linear homomorphism $\varphi : N_1 \to N_2$ that satisfies the property that for any $\sigma \in \Delta_1$ there exists $\tau \in \Delta_2$ such that $\varphi(\sigma) \subset \tau$.*

A map between fans allows us to define a map between toric varieties in a covariant way. The algebraic torus T, if considered in different lattices, needs a subscript. The next theorem uses the notation: $T_{N_i} = N_i \otimes_{\mathbb{Z}} \mathbb{C}^* \simeq (\mathbb{C}^*)^{\dim N_i}$ for $i = 1, 2$.

Theorem 1.2.4 ***([12], Theorem 1.13)*** *A map of fans $\varphi : (\Sigma_1, N_1) \to (\Sigma_2, N_2)$ gives rise to a holomorphic map $\varphi_* : X(\Sigma_1) \to X(\Sigma_2)$ whose restriction to the open subset T_{N_1} coincides with the homomorphism of algebraic tori $\varphi \otimes 1 : T_{N_1} \to T_{N_2}$ arising from φ. Through this homomorphism, φ_* is equivariant with respect to the actions of T_{N_1} and T_{N_2} on the toric varieties. Conversely, suppose $f : T_{N_1} \to T_{N_2}$ is a homomorphism of algebraic tori, and $\varphi_* : X_{\Sigma_1} \to X_{\Sigma_2}$ is a holomorphic map equivariant with respect to f. Then there exists a unique $\mathbb{Z}$ linear homomorphism $\varphi : N_1 \to N_2$, which gives rise to a map of fans $\varphi : (\Sigma_1, N_1) \to (\Sigma_2, N_2)$ such that $f = \varphi_*$.* ▮

It is worth noting at this point that, particularly if Σ_1 is a subfan of Σ_2, then the embedding $\varphi : \Sigma_1 \to \Sigma_2$ induces an embedding of toric varieties $\varphi_* : X_{\Sigma_1} \to X_{\Sigma_2}$.

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