6. (10 pts) (Problem 16 from Sec 3.1) $2y''(t) - 3y'(t) + y = 0$. The characteristic equation of $2y''(t) - 3y'(t) + y = 0$ is $2r^2 - 3r + 1 = (2r - 1)(r - 1) = 0$. We have $r = \frac{1}{2}$ or $r = 1$. Thus the general solution is $y(t) = c_1e^{\frac{1}{2}t} + c_2e^t$.

2. (15 pts) (Problem 10 from Sec 3.1) The characteristic equation of $y''(t) + 4y'(t) + 3y(t) = 0$ is $r^2 + 4r + 3 = (r + 1)(r + 3) = 0$. We have $r = -1$ or $r = -3$. Thus the general solution is $y(t) = c_1e^{-t} + c_2e^{-3t}$.

Computing $y'(t)$, we get $y'(t) = -c_1e^{-t} - 3c_2e^{-3t}$.

Using $y(0) = 2$ and $y'(0) = -1$, we have $c_1e^0 + c_2e^0 = 2$ and $-c_1e^0 - 3c_2e^0 = -1$, i.e. $c_1 + c_2 = 2$ and $-c_1 - 3c_2 = -1$. We have $c_1 = \frac{5}{2}$ and $c_2 = -\frac{1}{2}$. So $y(t) = \frac{5}{2}e^{-t} - \frac{1}{2}e^{-3t}$.

3. (15 pts) (Problem 21 from Sec 3.1) The characteristic equation of $y''(t) - y'(t) - 2y(t) = 0$ is $r^2 - r - 2 = (r - 1)(r + 2) = 0$. We have $r = -1$ or $r = 2$. Thus the general solution is $y(t) = c_1e^{-t} + c_2e^{2t}$.

Computing $y'(t)$, we get $y'(t) = -c_1e^{-t} + 2c_2e^{2t}$.

Using $y(0) = 0$ and $y'(0) = 2$, we have $c_1e^0 + c_2e^0 = 0$ and $-c_1e^0 + 2c_2e^0 = 2$, i.e. $c_1 + c_2 = 0$ and $c_1 - 2c_2 = 2$. We have $c_1 = \frac{2a-2}{3}$ and $c_2 = \frac{a+2}{3}$. So $y(t) = \frac{2a-2}{3}e^{-t} + \frac{a+2}{3}e^{2t}$. Since $\lim_{t \to -\infty} e^{-2t} = 0$ and $\lim_{t \to -\infty} e^{2t} = \infty$, the solution $y(t) = \frac{2a-2}{3}e^{-t} + \frac{a+2}{3}e^{2t}$ approaches 0 only if $\frac{a+2}{3} = 0$, i.e. $a = -2$.

4. (15 pts) (Problem 24 from Sec 3.1) The characteristic equation of $y''(t) + (3 - \alpha)y'(t) - 2(\alpha - 1)y(t) = 0$ is $r^2 + (3 - \alpha)r - 2(\alpha - 1) = (r - \alpha + 1)(r + 2) = 0$. We have $r = \alpha - 1$ or $r = -2$. Thus the general solution is $y(t) = c_1e^{(\alpha - 1)t} + c_2e^{-2t}$. If $\alpha < 1$, then $\lim_{t \to -\infty} y(t) = 0$.

Since $\lim_{t \to -\infty} e^{-2t} = 0$, there is no $\alpha$ such that $y(t)$ is unbounded.

5. (15 pts) (Problem 9 from Sec 3.2) Rewrite the equation $t(t - 4)y''(t) + 3ty'(t) + 4y(t) = 2$ as $y'' + \frac{3}{t - 4}y' + \frac{4}{t(t - 4)}y = \frac{2}{t(t - 4)}$. So $p(t) = \frac{3}{t - 4}$, $q(t) = \frac{4}{t(t - 4)}$ and $g(t) = \frac{2}{t(t - 4)}$. Hence $p(t)$ is continuous if $t \in (-\infty, 4) \cup (4, \infty)$, $q(t)$ is continuous if $t \in (-\infty, 0) \cup (0, 4) \cup (4, \infty)$ and $g(t)$ is continuous if $t \in (-\infty, 0) \cup (0, 4) \cup (4, \infty)$. Therefore $p(t)$, $q(t)$ and $g(t)$ are continuous if $t \in (-\infty, 0) \cup (0, 4) \cup (4, \infty)$. The initial conditions are $y(3) = 0$ and $y'(3) = -1$. We have $3 \in (0, 4)$. Thus the solution exists on the interval $(0, 4)$.

6. (10 pts) (Problem 16 from Sec 3.2)

Since $y(t) = \sin(t^2)$, we have $y(0) = 0$, $y'(0) = 2t \cos(t^2)$ and $y''(0) = 0$. By the uniqueness of the solution of homogeneous equation, we must have $y(t) = 0$. This means that $y(t) = \sin(t^2)$ can’t be a solution of $y'' + p(t)y' + q(t)y = 0$.

7. (15 pts) (Sec 3.2 Problem 25) Solution: Since $y_1(x) = x$ and $y_2(x) = xe^x$, we have $y'_1(x) = 1$, $y'_2(x) = 0$, $y'_2(x) = e^x + xe^x$ and $y''_2(x) = e^x + e^x + xe^x = 2e^x + xe^x$. So $x^2y'_2 - x(x + 2)y'_2 + (x + 2)y_2 = 0 - x(x + 2) + (x + 2)x = 0$ and $x^2y' - x(x + 2)y'_2 + (x + 2)y_2 = x^2(2e^x + xe^x) - x(x + 2)(e^x + xe^x) + (x + 2)xe^x = 0$. Therefore $y_1(x) = x$ is a solution of the differential equation.
$2x^2e^x + x^3e^x - x^2e^x - x^3e^x - 2xe^x - 2x^2e^x + x^2e^x + 2xe^x = 0$. So $y_1$ and $y_2$ are solutions of $x^2y'' - x(x + 2)y' + (x + 2)y = 0$. The Wronskain of $y_1$ and $y_2$ is $W(y_1, y_2)(x) = y_1y'_2 - y_2y'_1 = x \cdot (e^x + xe^x) - xe^x \cdot 1 = x^2e^x > 0$ if $x > 0$. Hence $y_1$ and $y_2$ form a set of fundamental solutions.