Solution to HW 9

1. (Sec 4.1 Problem 6) (15 pts) We rewrite the equation \( (x^2 - 4)y''(t) + x^2y'''(t) + 9y = 0 \) as \( y''(t) + \frac{x^2}{x^2 - 4}y'''(t) + \frac{9}{x^2 - 4}y = 0 \). Now the function \( \frac{x^2}{x^2 - 4} \) and \( \frac{9}{x^2 - 4} \) are continuous on \((-\infty, -2) \cup (-2, 2) \cup (2, \infty)\). So the solution exists on \((-\infty, -2) \cup (-2, 2) \cup (2, \infty)\).

2. (Sec 4.2 Problem 12) (15 pts) \( y''(t) - 3y'(t) + 3y(t) - y = 0 \).
   The characteristic equation of \( y''(t) - 3y'(t) + 3y(t) - y = 0 \) is \( r^3 - 3r^2 + 3r - 1 = (r^3 - 1) - 3r(r - 1) = (r - 1)(r^2 + r + 1) - 3r(r - 1) = (r - 1)(r^2 + r + 1 - 3r) = (r - 1)(r^2 - 2r + 1) = (r - 1)^3 = 0 \). So \( r = 3 \) is a root of characteristic equation of order 3. The general solution is \( y(t) = c_1e^t + c_2te^t + c_3t^2e^t \).

3. (Sec 4.2 Problem 15) (25 pts) The characteristic equation of \( y''(t) + y(t) = 0 \) is \( r^2 + 1 = 0 \). So \( r = \pm i \) is \( k \) is an integer. Now \( r = e^{i(\pi + 2k\pi)} \) for \( k = 0, 1, 2, 3, 4 \) and 5. So \( r = e^{i(\pi/2)} = \cos(\pi/2) + i \sin(\pi/2) = \sqrt{2} + i \), \( r = e^{i(3\pi/2)} = \cos(3\pi/2) + i \sin(3\pi/2) = i \), \( r = e^{i(5\pi/2)} = \cos(5\pi/2) + i \sin(5\pi/2) = -\sqrt{2} + i \), \( r = e^{i(7\pi/2)} = \cos(7\pi/2) + i \sin(7\pi/2) = -i \), \( r = e^{i(9\pi/2)} = \cos(9\pi/2) + i \sin(9\pi/2) = \sqrt{2} - i \). So \( r = \pm i \) and \( r = \mp i \). Thus the general solution is \( y(t) = c_1e^{\sqrt{2}t} \cos(t/2) + c_2e^{-\sqrt{2}t} \sin(t/2) + c_3 \cos(t) + c_4 \sin(t) + c_5 e^{-\sqrt{2}t} \cos(\pi/2) + c_6 e^{-\sqrt{2}t} \sin(\pi/2) \).

4. (Sec 4.2 Problem 22) (20 pts) \( y''(t) + 2y'(t) + y = 0 \).
   The characteristic equation of \( y''(t) + 2y'(t) + y = 0 \) is \( r^2 + 2r + 1 = (r + 1)^2 \). Its roots are \( r = \pm i \) with multiplicity 2. The general solution is \( y(t) = c_1 \cos(t) + c_2 \sin(t) + c_3 \cos(t) + c_4 \sin(t) \).

5. (Sec 4.2 Problem 29) (25 pts) \( y''(t) + y'(t) = 0 \) with \( y(0) = 0 \), \( y'(0) = 1 \) and \( y''(0) = 2 \).
   The characteristic equation of \( y''(t) + y'(t) = 0 \) is \( r^3 + r = r(r^2 + 1) \). Its roots are \( r = \pm i \) and \( r = 0 \). The general solution is \( y(t) = c_1 \cos(t) + c_2 \sin(t) + c_3 \). So \( y'(t) = -c_1 \sin(t) + c_2 \cos(t) \) and \( y''(t) = -c_1 \cos(t) - c_2 \sin(t) \). Using \( y(0) = 0 \), \( y'(0) = 1 \), \( y''(0) = 2 \), \( \cos(0) = 1 \) and \( \sin(0) = 0 \), we have \( c_1 + c_3 = 0 \), \( c_2 = 1 \) and \( -c_1 = 2 \). Hence \( c_1 = -2 \), \( c_2 = 1 \) and \( c_3 = -c_1 = 2 \). The solution to the IVP is \( y(t) = -2 \cos(t) + \sin(t) + 2 \).