Instructions:

1. If you think that there is a mistake ask the proctor. If the proctor’s explanation is not satisfactory, interpret the problem as you see fit, but not in such a way that the answer is trivial.

2. From each part solve 3 of 6 problems.

3. If you solve more that three problems from a part indicate the problems that you wish to have graded.

Part A: ODE Questions

1. Consider the autonomous system $\dot{x} = f(x)$ where $f : \mathbb{R}^n \to \mathbb{R}^n$ is Lipshitz continuous. Suppose that $f(x)$ satisfies $\langle x, f(x) \rangle \geq ||x||^3$. Show that the solution to the initial value problem $x(t_0) = x_0 \neq 0$ cannot extend to $[t_0, \infty)$.  

2. Suppose that $g(v)$ is a continuously differentiable function of a single variable satisfying $0 < g'(v) \leq 2M$ and $0 < g(v) < Mv$ for some positive constant $M$ and $v > 0$. Consider the initial value problem

$$\dot{v} = h + g(v); \quad v(0) = 1$$

where $h$ is a constant. Show that the solution $v(t)$ satisfies

$$|v(t) - (ht + 1)| \leq \frac{1}{2} \left( \frac{h}{M} + 1 \right) e^{2Mt}.$$  

3. Find the fundamental solution of $X(t)$ with $X(0) = I$ to the system

$$\frac{d}{dt} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -x_2 + x_3 \\ x_1 + x_4 \\ -x_4 \\ x_3 \end{pmatrix}.$$  

4. Consider the system $\dot{x} = h(t)Ax$ where $x : \mathbb{R} \to \mathbb{R}^n$, $A$ is a constant $n \times n$ matrix, and $h(t)$ is strictly positive and continuous. Show that $0$ is
asymptotically stable if all the eigenvalues of $A$ have negative real parts and $\int_0^\infty h(t)dt$ diverges. By example show that the stability may not hold if the integral converges.

5. Let $L$ be a periodic solution of a Lipschitz continuous planar autonomous system $\dot{x} = f(x)$ with flow $\phi_t : \mathbb{R}^2 \to \mathbb{R}^2$. Recall that an $\epsilon$- neighborhood $N_\epsilon$ of $L$ is small if $N_\epsilon$ contains no singular points and for any $q_1, q_2 \in N_\epsilon$ with $|q_1 - q_2| < 2\epsilon$,

$$< f(q_1), f(q_2) > \frac{1}{||f(q_1)|| ||f(q_2)||} > \frac{1}{\sqrt{2}}.$$ 

Suppose that $L^1 \subset N_\epsilon$ is a second periodic solution and that for some $p \in L$ and tranverse segment $\overline{n^1 p^n}$ to $L$ there is a $p^1 \in L^1$ with $p^1 \in \overline{n^1 p^n}$. Let $T$ be the smallest parameter value so that $\phi_T(p^1) \in \overline{n^1 p^n}$. Show that $T$ is the period of $L^1$.

6. Suppose $p(t), q(t)$ are continuous functions on an interval $I$. Let $y_1(t), y_2(t)$ be two linearly independent solutions of

$$y'' + p(t)y' + q(t)y = 0.$$ 

Prove that between any two roots of $y_1(t)$ there is at least one root of $y_2(t)$.

**Part B: PDE Questions**

1. Consider the quasi-linear system for an unknown function $u : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ given by

$$\frac{\partial u}{\partial t} + \tilde{a}(\bar{x}, t) \cdot \frac{\partial u}{\partial \bar{x}} = b(\bar{x}, t, u)$$

where $t \in \mathbb{R}$ and $\bar{x} \in \mathbb{R}^n$. Show that if $\tilde{a}, b$ are bounded and Lipshitz continuous, then the Cauchy problem with Cauchy data defined on the hyperplane $H = \{(\bar{x}, t) | t = 0\}$ has global solutions.

2. Determine the canonical form and the general solution to the second order euqation

$$4x^2u_{xx} + 4xu_{xy} - 8y^2u_{yy} + 4xu_x - 8u_y = 0.$$ 

3. Consider the Cauchy-Kowalewski system

$$\frac{\partial}{\partial t} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & u \\ v & 0 \end{pmatrix} \frac{\partial}{\partial x} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} v & 0 \\ 0 & -u \end{pmatrix} \frac{\partial}{\partial y} \begin{pmatrix} u \\ v \end{pmatrix}.$$
Determine the Taylor expansion of the solution to second order with Cauchy data given by \( u(x, y, 0) = x, v(x, y, 0) = y \).

4. Suppose that on a bounded domain \( \Omega \) a function \( u \in C^2(\Omega) \cap C^0(\overline{\Omega}) \) satisfies \( \Delta u + \lambda u^3 = 0 \) in \( \Omega \) with \( \lambda > 0 \) and \( u = 0 \) on \( \partial \Omega \). Suppose that \( u \) is non-negative in \( \Omega \). Show that \( u \) is strictly positive in \( \Omega \).

5. Let \( u \) be a harmonic function on a bounded domain \( \Omega \). Suppose \( \phi : R \rightarrow R \) is a convex \( C^2 \) function. Show that \( \phi \circ u \) is subharmonic.

6. Suppose \( w \) is a harmonic function on \( \mathbb{R}^n \) and satisfies

\[
\int_{\mathbb{R}^n} w^2(x) dx < \infty.
\]

Prove that \( w \equiv 0 \).