Fall 2003 Ph.D Qualifying Exam in Real Analysis

Time 3 hours, closed book, no notes. Answer three questions from each of the parts A & B.

Part A

1. (The Ascoli-Arzela Theorem) Let \( X \) be a compact metric space and let \( C(X) \) be the normed linear space of real-valued continuous functions on \( X \) with the supremum norm.

   (a) Define what it means for a subset \( F \subset C(X) \) to be an equi-continuous family.

   (b) Prove that \( F \) is totally bounded in \( C(X) \) if and only if

   i. \( F \) is bounded and
   ii. \( F \) is an equi-continuous family.

2. (a) Consider the power series \( f(x) = \sum_{k=0}^{\infty} a_k x^k \) and \( g(x) = \sum_{k=1}^{\infty} k a_k x^{k-1} \) where the sequence \( \{a_k\} \) is bounded. Show

   i. Both series converge uniformly on \([-\rho, \rho]\) for any \( \rho \), \( 0 < \rho < 1 \),

   ii. \( f'(x) = g(x) \) for \(-1 < x < 1\).

   (b) Suppose \( 0 \leq k \leq |a_k| \leq k^2 \) for all \( k \). What is the radius of convergence of the power series \( \sum_{k=0}^{\infty} a_k x^k \)?

3. Suppose \((X, d)\) is a compact metric space and that \( f \) is an isometry of \( X \) into itself (i.e., \( f : X \to X \) with \( d(f(x_1), f(x_2)) = d(x_1, x_2) \) for \( x_1, x_2 \) in \( X \)). Clearly \( f \) is a 1–1 map (an injection). Show that \( f \) is an onto map (a surjection). [Hint: Take a point \( p \) outside \( f(X) \) if possible and look at the sequence of iterates \( p, f(p), f(f(p)), \ldots \).]

4. Let \( f(x) \) be continuous on \([0, 1]\). Show

   (a) \( \lim_{n \to \infty} \int_0^1 x^n f(x) dx = 0 \),

   (b) \( \lim_{n \to \infty} \int_0^1 nx^n f(x) dx = f(1) \).

5. Prove Egorov’s Theorem namely: Suppose \((X, \mathcal{M}, \mu)\) is a measure space and \( \mu(X) < \infty \). Let \( \{f_n\} \) be a sequence of measurable functions such that \( f_n \to f \) (a.e.). Show that for any \( \epsilon > 0 \), there is a measurable set \( E \) with \( \mu(E) < \epsilon \) such that \( f_n \to f \) uniformly on \( X \setminus E \).

6. Let \( \phi_n(x) = \sqrt{n} e^{-n^2|x|} \) and let \( f(x) = \sum_{n=0}^{\infty} \phi_n(x - r_n) \) where \( \{r_n\} \) is an enumeration of rational numbers in \( \mathbb{R} \). Show:

   (a) \( f(x) \in L^1(\mathbb{R}) \) and compute \( \int_{\mathbb{R}} f(x) dx \)

   (b) \( f(x) \) is unbounded in any open interval \((a, b)\).
Part B

1. Let \( f(x) = 0 \) if \( x \in [0, 1] \) is irrational and \( f(x) = \frac{1}{q} \) if \( x = \frac{p}{q} \) in lowest terms. Show: \( f(x) \) is continuous at every irrational point and discontinuous at every rational point in \([0, 1]\).

2. Let \( f_n(x) = (1 + x^n)^{1/n}, \) \( 0 \leq x \leq 2, \ n \in \mathbb{N}. \) Show that \( f_n(x) \to f(x) \) uniformly on \([0, 2] \) as \( n \to \infty \) where \( f(x) = 1 \) for \( x \in [0, 1] \) and \( x \) for \( x \in [1, 2]. \)

3. (a) State carefully the Stone-Weierstrass Theorem.

(b) Prove that if \( f(x) \) is continuous in \([0, 1]\) and if \( \int_0^1 x^{4n} f(x) \, dx = 0 \) for \( n = 0, 1, 2, \ldots, \) then \( f(x) \equiv 0. \)

(c) Suppose \( \int_0^1 x^k f(x) \, dx = 0 \) for \( k \) odd. What can you say about \( f(x) \)?

4. Show that

\[
\lim_{n \to \infty} \int_0^n \left(1 - \frac{x}{n}\right)^n \, dx = \lim_{n \to \infty} \int_0^n \left(1 + \frac{x}{n}\right)^n e^{-2x} \, dx = 1.
\]

5. Suppose that \( f, \) a function defined on an open interval \((a, b), \) satisfies the intermediate value theorem i.e., if \( f \) assumes the values \( y_1, y_2, \) it assumes all values between \( y_1, y_2. \) Show that if \( f \) is not continuous, it assumes some value infinitely often.

6. State the definition of Riemann integrability and prove directly that any continuous function on a closed interval \([a, b]\) is Riemann integrable.