Part A

1. Let $E$ be a subset of $\mathbb{R}^n$ with the property that every continuous function on $E$ is bounded. Prove or disprove: $E$ is compact.

2. (a) Give a careful statement of the Stone-Weierstrass Theorem for the case of continuous complex-valued functions on a compact metric space.

(b) Let $X$, $Y$ be compact metric spaces. Show that continuous complex-valued functions of the form

$$F(x, y) = \sum_{i=1}^{n} f_i(x)g_i(y), \quad x \in X, \ y \in Y$$

where $f_i \in \mathcal{C}(X)$ and $g_i \in \mathcal{C}(Y)$, are dense in the supnorm topology on $\mathcal{C}(X \times Y)$ where

$$||G||_\infty \equiv \sup\{|G(x, y)| : (x, y) \in X \times Y\}, \ G \in \mathcal{C}(X \times Y).$$

3. (a) Let $I$ be a closed interval on the real line with Lebesgue measure, $m$. For $0 < p < q$ show that the space $L^p$ is contained in $L^q$ and that

$$||f||_p \leq ||f||_q[m(I)]^{\frac{1}{p} - \frac{1}{q}}.$$ 

(b) If $f \in L^1(\mathbb{R})$ with respect to Lebesgue measure and $a \in \mathbb{R}$, prove that

$$\int_{-\infty}^{\infty} f(x + a)dx = \int_{-\infty}^{\infty} f(x)dx.$$ 

4. Define $f(x) = \left[\int_{0}^{x} e^{-t^2}dt\right]^2$ and $g(x) = \int_{0}^{1} \frac{e^{-x^2(t^2+1)}}{t^2 + 1} dt$.

(a) Show that $f'(x) + g'(x) = 0$ and deduce that $f(x) + g(x) = \frac{\pi}{4}$.

(b) Use (a) to prove that

$$\lim_{x \to +\infty} \int_{0}^{x} e^{-t^2}dt = \frac{\sqrt{\pi}}{2}.$$
5. (a) Let \( \{f_n\} \) be a sequence of continuous real-valued functions on \([0, 1]\) and assume that \( f_n \to f \) uniformly on \([0, 1]\). Prove or disprove:

\[
\lim_{n \to \infty} \int_0^1 f_n(x) \, dx = \int_0^1 f(x) \, dx.
\]

(b) Let \( f_n(x) = \frac{1}{n} e^{-n^2 x^2}, \ x \in \mathbb{R}, \ (n = 1, 2, 3 \ldots) \).

Show that \( f_n \to 0 \) uniformly on \( \mathbb{R} \), that its derivative \( f'_n \to 0 \) pointwise on \( \mathbb{R} \) but that the convergence of \( \{f'_n\} \) is not uniform on any interval containing the origin.

6. (a) What is the formula, in terms of \( a_n \), of the radius of convergence of the power series \( \sum_{n=0}^{\infty} a_n z^n \)? Prove that at any point inside the circle of convergence the power series converges absolutely.

(b) Suppose that \( \sum_{n=0}^{\infty} a_n z^n \) has radius of convergence, 2. Given that \( k \) is a fixed positive integer, find the radii of convergence of the following series.

i. \( \sum_{n=0}^{\infty} a_n^k z^n \),  
ii. \( \sum_{n=0}^{\infty} a_n z^{kn} \),  
iii. \( \sum_{n=0}^{\infty} a_n z^{n^2} \)
Part B

1. Let $f : \mathbb{R} \to \mathbb{R}$. For each $x \in \mathbb{R}$, define

$$\omega(x) = \inf \{ \delta(f(U)) : U \text{ a neighborhood of } x \}$$

where, if $E \subset \mathbb{R}$, $\delta(E) \equiv \sup \{|x - y| : x, y \in E \}$.

Prove the following:

(a) The function $f$ is continuous at $x$ if and only if $\omega(x) = 0$.
(b) For each $\alpha \in \mathbb{R}$ the set $\{ x \in \mathbb{R} : \omega(x) < \alpha \}$ is open.
(c) The set $\{ x \in \mathbb{R} : f(x) \text{ is continuous} \}$ is a $G_\delta$ set.
(d) There is no real-valued function, $f$, on $\mathbb{R}$ such that $\{ x \in \mathbb{R} : f(x) \text{ is continuous} \} = \mathbb{Q}$, the rational numbers in $\mathbb{R}$.
   (Hint: For (d) use the Baire Category Theorem to show that $\mathbb{Q}$ cannot be a $G_\delta$ set in $\mathbb{R}$).

2. Let $(X, d)$ be a compact metric space. A function $f : X \to \mathbb{R}$ is said to be Lipschitz continuous if

$$||f||_d \equiv \sup \left\{ \frac{|f(x) - f(y)|}{d(x, y)} : x \in X, y \in Y, x \neq y \right\} < \infty.$$  

Denote by Lip$(X, d)$ the collection of all Lipschitz continuous functions on $X$.

(a) Prove that Lip$(X, d)$ is a Banach space under the norm

$$||f|| = ||f||_\infty + ||f||_d$$

where $||f||_\infty = \max \{|f(x)| : x \in X \}$. Show that the multiplicative inequality

$$||fg|| \leq ||f|| \cdot ||g||, \quad f, g \in \text{Lip}(X, d)$$

is also true.

(b) Let $\{f_n\}_{n=1}^\infty$ be a sequence of functions in Lip$(X, d)$ with $||f_n|| \leq 1$. Show that there is a subsequence $\{f_{n_k}\}$ and $f \in$ Lip$(X, d)$ such that $f_{n_k} \Rightarrow f$ uniformly on $X$. 
3. (a) Let $f$ be a Lebesgue integrable function on the real line. Show that given $\varepsilon > 0$ there is a $\delta > 0$ such that whenever $A$ is a Lebesgue measurable subset of $\mathbb{R}$ then

$$m(A) < \delta \Rightarrow \int_A |f| \, dx < \varepsilon.$$ 

(b) If $f \in C[0, 1]$, show that $\lim_{n \to \infty} ||f||_n$ exists and compute this limit where

$$||f||_n = \left[ \int_0^1 |f|^n \, dx \right]^\frac{1}{n} \quad (n = 1, 2, \ldots).$$

4. (a) Suppose $f, f_n (n = 1, 2, 3, \ldots)$ are real-valued Lebesgue measurable functions on $\mathbb{R}$. Define what is meant by saying $f_n \to f$ in $m$-measure. (Here $m$ is Lebesgue measure on $\mathbb{R}$).

(b) If we identify Lebesgue measurable functions on $\mathbb{R}$ that agree almost everywhere $[m]$, show that

$$d(f, g) \equiv \int_{\mathbb{R}} \frac{|f - g|}{1 + |f - g|} \, dm$$

is a metric on the space of Lebesgue measurable functions on $\mathbb{R}$.

(c) Show that $f_n \to f$ in $m$-measure if and only if $\lim_{n \to \infty} d(f_n, f) = 0$.

5. (a) Give careful statements of the Lebesgue Monotone Convergence Theorem and the Lebesgue Dominated Convergence Theorem.

(b) Use these theorems to establish the following:

i. $\lim_{n \to \infty} \int_1^n (1 - \frac{x}{n})^n \ln x \, dx = \int_1^\infty e^{-x} \ln x \, dx$.

ii. $\lim_{n \to \infty} \int_0^1 (1 - \frac{x}{n})^n \ln x \, dx = \int_0^1 e^{-x} \ln x \, dx$. 