1. Let \( \{r_n\} \) be a fixed sequentialization of the rational numbers in the interval \([0, 1]\) and define \( \{f_n\} \) by the formula \( f_n(x) = 0 \) for \( x \in \{r_1, r_2, \ldots, r_n\} \) and equals 1 otherwise.

(a) Show that for each \( n \) the function \( f_n \) is Riemann integrable and that

\[
\int_0^1 f_n(x) \, dx = 1.
\]

(b) Let \( f(x) = \lim_{n \to \infty} f_n(x) \) for \( x \in [0, 1] \). Show that \( f \) is not Riemann integrable but that it is Lebesgue integrable.

2. Let \( h \) be a continuous function defined on \([a, b]\) and suppose that

\[
\int_a^b h(x)u(x) \, dx = 0
\]

for all Lebesgue integrable functions \( u \) on \([a, b]\) satisfying

\[
\int_a^b u(x) \, dx = 0.
\]

Show that \( h \) is a constant function.

3. Let \( f : [a, b] \to (0, +\infty) \) be a continuous function. Show that

\[
\int_a^b F(t) \, dt \cdot \int_a^b \frac{1}{f(t)} \, dt \geq (b - a)^2.
\]

4. Let \( f \) be a real-valued continuous function on \([0, 1]\). Prove that

\[
\lim_{n \to +\infty} \int_0^1 f(t^n) \, dt = f(0).
\]

5. Show that for any finite set of complex numbers \( z_1, z_2, \ldots, z_n \),

\[
|z_1 + z_2 + \ldots + z_n| \leq |z_1| + |z_2| + \ldots + |z_n|
\]

and equality occurs when and only when there exists a non-zero complex number \( z \) such that \( \frac{z_1}{z}, \frac{z_2}{z}, \ldots, \frac{z_n}{z} \) are all real and non-negative. You must prove the triangle inequality before using it.

6. Let \( X \) be a complete metric space and \( X = \bigcup F_n \), countable union of closed sets \( F_n \). Then at least one of \( F_n \) has non-empty interior.
7. Assume that $a_n > 0$ and the series $\sum a_n$ diverges. Show that

$$\sum \frac{a_n}{1 + a_n}$$

diverges. What can be said about

$$\sum \frac{a_n}{1 + na_n}, \sum \frac{a_n}{1 + n^2a_n}?$$

8. Let $n_k$ be an increasing sequence of positive integers and let $E$ be the set of all $x$ for which $\{\sin n_kx\}$ converges. Show that the measure of $E$ is zero.